A complete classification of $\mathbb{Z}_p$-sequences corresponding to a polynomial

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Let p be a prime number and set $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$. A $\mathbb{Z}_p$-sequence is a function $S : \mathbb{Z} \to \mathbb{Z}_p$. Let $\mathcal{R}$ be the set $\{P \in \mathbb{R}[X] \mid P(\mathbb{Z}) \subseteq \mathbb{Z}\}$. We prove that the set of sequences of the form $(P(n) \pmod p)_{n\in\mathbb{Z}}$, where $P \in \mathcal{R}$, is precisely the set of periodic $\mathbb{Z}_p$-sequences with period equal to a p-power. Given a $\mathbb{Z}_p$-sequence, we will also determine all $P \in \mathcal{R}$ that correspond to the sequence according to the manner above.

1. Preliminaries

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Definition 1. Define the sequence $(P_i)_{i\in\mathbb{N}_0}$ of polynomials in $\mathbb{R}[X]$ as follows:

$$P_0 = 1 \quad \text{and} \quad \text{for all } i \in \mathbb{N} : P_i = \binom{X}{i} = \frac{\prod_{j=0}^{i-1}(X - j)}{i!}.$$

Lemma 2 [Niven et al. 1991, pp. 42–43, Problems 11, 14, 15]. *We have*

$$\mathcal{R} = \Big\{\sum_{i=0}^{m} c_i P_i \mid m \in \mathbb{N}_0, c_0, \ldots, c_m \in \mathbb{Z}\Big\}.$$

Proof. Clearly, $\mathcal{R} \supseteq \big\{\sum_{i=0}^{m} c_i P_i \mid m \in \mathbb{N}_0, c_0, \ldots, c_m \in \mathbb{Z}\big\}$, so we only need to prove the reverse inclusion.

Let $P \in \mathcal{R}$ have degree m. If the system of equations

$$P(j) = \sum_{i=0}^{m} c_i P_i(j), \quad j = 0, \ldots, m, \tag{1}$$

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in the unknowns $c_0, \dots, c_m$ has a solution $(c_0, \dots, c_m) \in \mathbb{Z}^{m+1}$, then $P = \sum_{i=0}^{m} c_i P_i$ because P and $\sum_{i=0}^{m} c_i P_i$ are polynomials of degree at most m that agree at the $m+1$ points $0, \dots, m$. However, (1) is equivalent to the system

$$c_j = P(j) - \sum_{i=0}^{j-1} \binom{j}{i} c_i, \quad j = 0, \dots, m,$$

which clearly has a unique solution $(c_0, \dots, c_m) \in \mathbb{Z}^{m+1}$. □

Lemma 3. *Let p be a prime number. For every $k \in \mathbb{N}$,*

$$\binom{p^k}{0} \equiv 1 \pmod{p} \quad \textit{and} \quad \textit{for all } i \in \{1, \dots, p^k - 1\} : \binom{p^k}{i} \equiv 0 \pmod{p}.$$

Proof. The first identity is clearly true. When $i \in \{1, \dots, p^k - 1\}$, we have

$$\frac{p^k}{i} = \frac{\binom{p^k}{i}}{\binom{p^k - 1}{i - 1}}.$$

Write i as $p^l m$, where $l \in \mathbb{N}_0$ and m is a positive integer not divisible by p. From the equation

$$\frac{p^{k-l}}{m} = \frac{p^k}{i} = \frac{\binom{p^k}{i}}{\binom{p^k - 1}{i - 1}},$$

we immediately obtain

$$p^{k-l} \binom{p^k - 1}{i - 1} = m \binom{p^k}{i}.$$

Since $i < p^k$, we have $k - l \geq 1$. Thus p divides $m \binom{p^k}{i}$, and since it does not divide m, it must divide $\binom{p^k}{i}$. This proves that

$$\binom{p^k}{i} \equiv 0 \pmod{p}.$$

As $i \in \{1, \dots, p^k - 1\}$ was arbitrary, Lemma 3 is true. □

Lemma 4. *Let p be a prime number. Then, for every $n \in \mathbb{Z}$, $k \in \mathbb{N}_0$ and $i \in \{0, \dots, p^k - 1\}$,*

$$\binom{n + p^k}{i} \equiv \binom{n}{i} \pmod{p}. \tag{2}$$

Proof. Let $k \in \mathbb{N}_0$. Define a well-ordering $\prec$ on $\{0, \dots, p^k - 1\} \times \mathbb{N}_0$ by setting $(i, n) \prec (i', n')$ if either (i) $i < i'$, or (ii) $i = i'$ and $n < n'$. By the principle of induction, it suffices to prove the following statements:

(A) For every $i \in \{0, \dots, p^k - 1\}$,

$$\binom{0+p^k}{i} \equiv \binom{0}{i} \pmod{p}.$$

(B) Given $(i^*, n^*) \in \{0, \dots, p^k - 1\} \times \mathbb{N}_0$, if

$$\text{for every } (i,n) \preceq (i^*, n^*): \quad \binom{n+p^k}{i} \equiv \binom{n}{i} \pmod{p}, \tag{3}$$

and

$$\text{for every } (i,n) \preceq (i^*, n^*): \quad \binom{-n+p^k}{i} \equiv \binom{-n}{i} \pmod{p}, \tag{4}$$

then, respectively,

$$\binom{n^*+1+p^k}{i^*} \equiv \binom{n^*+1}{i^*} \pmod{p} \tag{5}$$

and

$$\binom{-(n^*+1)+p^k}{i^*} \equiv \binom{-(n^*+1)}{i^*} \pmod{p}. \tag{6}$$

Statement (A) holds by Lemma 3. For Statement (B), we consider two cases: (i) $i^* = 0$ and (ii) $i^* > 0$. In Case (i), (B) is vacuously true. In Case (ii), we deduce (5) from (3) by applying Pascal's Rule:

$$\begin{aligned}\binom{n^*+1+p^k}{i^*} &= \binom{n^*+p^k}{i^*-1} + \binom{n^*+p^k}{i^*} && \text{(by Pascal's Rule)}\\ &\equiv \binom{n^*}{i^*-1} + \binom{n^*}{i^*} && \text{(from (3))}\\ &\equiv \binom{n^*+1}{i^*} \pmod{p} && \text{(by Pascal's Rule again).}\end{aligned}$$

In a similar fashion, we deduce (6) from (4):

$$\begin{aligned}\binom{-(n^*+1)+p^k}{i^*} &= \binom{-n^*+p^k}{i^*} - \binom{-(n^*+1)+p^k}{i^*-1}\\ &\equiv \binom{-n^*}{i^*} - \binom{-(n^*+1)}{i^*-1} \equiv \binom{-(n^*+1)}{i^*} \pmod{p}.\end{aligned}$$

Therefore (B) is true in Case (ii). Since $k \in \mathbb{N}_0$ was arbitrary, Lemma 4 is true. □

Corollary 5. *For every $i \in \mathbb{N}_0$, the sequence $\left(\binom{n}{i} \pmod{p}\right)_{n \in \mathbb{Z}}$ is periodic with period equal to a p-power.*

Proof. Choose $k \in \mathbb{N}_0$ such that $i < p^k$. By Lemma 4, $\binom{n+p^k}{i} \equiv \binom{n}{i} \pmod{p}$ for every $n \in \mathbb{Z}$. This clearly implies the claim. □

Corollary 6. *For every $P \in \mathcal{R}$, the sequence $(P(n) \pmod{p})_{n \in \mathbb{Z}}$ is periodic with period equal to a p-power.*

Proof. Let $P \in \mathfrak{R}$. By Lemma 2, there exist $m \in \mathbb{N}_0$ and $c_0, \ldots, c_m \in \mathbb{Z}$ such that $P = \sum_{i=0}^{m} c_i P_i$. Then,

$$(P(n) \pmod{p})_{n \in \mathbb{Z}} = \left(\sum_{i=0}^{m} c_i P_i(n) \pmod{p} \right)_{n \in \mathbb{Z}}.$$

By Corollary 5, each $(P_i(n) \pmod{p})_{n \in \mathbb{Z}}$ is periodic with period equal to a p-power. We conclude that $(P(n) \pmod{p})_{n \in \mathbb{Z}}$ is also periodic with period equal to a p-power. □

2. Main results

Theorem 7. *Let $p\mathfrak{R}$ be the subset of $\mathfrak{R}$ obtained by multiplying every $P \in \mathfrak{R}$ by p. A polynomial $P \in \mathfrak{R}$ lies in $p\mathfrak{R}$ if and only if p divides $P(n)$ for all $n \in \mathbb{Z}$; in symbols,*

$$p\mathfrak{R} = \left\{ P \in \mathfrak{R} \mid (P(n) \pmod{p})_{n \in \mathbb{Z}} = (0)_{n \in \mathbb{Z}} \right\}.$$

Proof. It is clear that every polynomial in $p\mathfrak{R}$ corresponds to $(0)_{n \in \mathbb{Z}}$, so let us suppose that $P \in \mathfrak{R}$ satisfies

$$(P(n) \pmod{p})_{n \in \mathbb{Z}} = (0)_{n \in \mathbb{Z}}.$$

Then, by Lemma 2, there exist $m \in \mathbb{N}_0$ and $c_0, \ldots, c_m \in \mathbb{Z}$ such that $P = \sum_{i=0}^{m} c_i P_i$. We claim that $c_0, \ldots, c_m \equiv 0 \pmod{p}$.

To prove the claim, we use mathematical induction. By our hypothesis,

$$P(0) = \sum_{i=0}^{m} c_i P_i(0) = c_0 \equiv 0 \pmod{p}.$$

Hence, the claim is true for c_0. Next, suppose that $k \in \mathbb{N}_0$ and that the claim is true for c_j for every $j \le k$. If $j = m$, we are done. If $j < m$, then

$$P(j+1) = \sum_{i=0}^{m} c_i P_i(j+1) \equiv c_{j+1} \equiv 0 \pmod{p}.$$

Hence, the claim is true for c_{j+1} as well. By induction, the claim is true for all $c_0, \ldots, c_m$. This shows that $P \in p\mathfrak{R}$. □

Theorem 8. *The set of sequences of the form $(P(n) \pmod{p})_{n \in \mathbb{Z}}$, where $P \in \mathfrak{R}$, is precisely the set of periodic $\mathbb{Z}_p$-sequences with period equal to a p-power.*

Proof. By virtue of Corollary 6, we only have to prove that every periodic $\mathbb{Z}_p$-sequence with period equal to a p-power corresponds to some $P \in \mathfrak{R}$.

Let $k \in \mathbb{N}_0$. Define A to be the set

$$\left\{\sum_{i=0}^{p^k-1} c_i P_i \;\middle|\; c_1, \dots, c_{p^k-1} \in \{0, \dots, p-1\}\right\},$$

and B to be set of all periodic $\mathbb{Z}_p$-sequences with period equal to p^l, where $0 \le l \le k$. By Lemma 4 and Theorem 7, every polynomial in A corresponds to a unique sequence in B. Since $|A| = |B| = p^{p^k}$, the correspondence is actually one-to-one. Therefore, every periodic $\mathbb{Z}_p$-sequence with period p^k corresponds to a unique polynomial of the form

$$\sum_{i=0}^{p^k-1} c_i P_i,$$

where $c_1, \dots, c_{p^k-1} \in \{0, \dots, p-1\}$. Since k was arbitrary, Theorem 8 is proven.

The theorem, however, would not be of much use unless the coefficients c_i can be determined. Hence, let S be a periodic $\mathbb{Z}_p$-sequence with period p^k, where $k \in \mathbb{N}_0$. By the first part, there exist $c_1, \dots, c_{p^k-1} \in \{0, \dots, p-1\}$ such that

$$S = \left(\sum_{i=0}^{p^k-1} c_i P_i(n) \pmod{p}\right)_{n\in\mathbb{Z}}.$$

From this identity, we obtain the equations

$$S(j) = \sum_{i=0}^{p^k-1} c_i \binom{j}{i}, \qquad j = 0, \dots, p^k - 1.$$

Some algebraic manipulation shows that the c_i's satisfy

$$c_i \equiv \sum_{j=0}^{i} (-1)^j \binom{i}{j} S(i-j) \pmod{p}, \qquad i = 0, \dots, p^k - 1. \qquad \square$$

Corollary 9. *Let S be a periodic $\mathbb{Z}_p$-sequence with period p^k, where $k \in \mathbb{N}_0$. Then, the set of all $P \in \mathfrak{R}$ which correspond to S is*

$$\left(\sum_{i=0}^{p^k-1} c_i P_i\right) + p\mathfrak{R},$$

where c_i is the least positive residue of $\sum_{j=0}^{i} (-1)^j \binom{i}{j} S(i-j) \pmod{p}$ for every $i = 0, \dots, p^k - 1$.

Proof. Let c_i satisfy the hypothesis given in the corollary. By Theorem 8,

$$S - \left(\sum_{i=0}^{p^k - 1} c_i P_i(n) \pmod{p} \right)_{n \in \mathbb{Z}} = (0)_{n \in \mathbb{Z}}.$$

The corollary now follows directly from Theorem 7. □

With both theorems and Corollary 9, we have a complete classification of $\mathbb{Z}_p$-sequences corresponding to a polynomial. Let us now look at some examples.

3. Examples

Example 10. To determine whether or not a $\mathbb{Z}_p$-sequence corresponds to a polynomial, simply investigate its periodicity. For example, the $\mathbb{Z}_7$-sequence

$$\ldots, \widehat{4}, 0, 6, 4, 0, 6, \ldots$$

(the $\widehat{\ }$ marks the zeroth element of the sequence) does not correspond to any polynomial because, although periodic, it has period 3, which is not a power of 7.

Example 11. The $\mathbb{Z}_3$-sequence

$$\ldots, \widehat{1}, 0, 1, 2, 0, 1, 1, 0, 2, \ldots$$

is periodic with period $9 = 3^2$, so it corresponds to a polynomial. The proof of Theorem 8 says that the sequence corresponds to

$$\binom{X}{0} + 2\binom{X}{1} + 2\binom{X}{2} + \binom{X}{3} + 2\binom{X}{4} + \binom{X}{5} + \binom{X}{6} + 2\binom{X}{7} + 2\binom{X}{8}.$$

4. Conclusion

We can add some algebraic flavor to the classification problem as follows. Let $\mathcal{S}$ denote the set of periodic $\mathbb{Z}_p$-sequences with period equal to a p-power. It is not difficult to see that $\mathcal{S}$ forms an abelian group under component-wise addition. Notice also that $\mathcal{R}$ is a free abelian group generated by the set $\{P_i \mid i \in \mathbb{N}_0\}$ and that the mapping

$$\phi : \mathcal{R} \to \mathcal{S},$$
$$\phi : P \mapsto (P(n) \pmod{p})_{n \in \mathbb{Z}}$$

is a surjective group homomorphism. By the first isomorphism theorem for groups, $\mathcal{R}/\ker(\phi) \cong \mathcal{S}$. However, Theorem 7 says that $\ker(\phi) = p\mathcal{R}$, so we obtain $\mathcal{R}/p\mathcal{R} \cong \mathcal{S}$. This elegant algebraic identity summarizes much of the effort invested in this paper.

Theorem 8 may be generalized so as to obtain a complete classification of all $\mathbb{Z}_m$-sequences corresponding to a polynomial for an arbitrary integer $m \geq 2$. The first step to doing this is to consider the case when m is a prime-power. It would be too good to be true for Lemma 4 to hold if we replace p in (2) by p^a for arbitrary $a \in \mathbb{N}$, and indeed it is.[1] We have the following counterexample:

$$\binom{4+3^2}{3} \equiv 7 \pmod 9 \quad \text{but} \quad \binom{4}{3} \equiv 4 \pmod 9.$$

However, an analogous equation for prime-powers may be obtained from the following proposition (see Theorem 1 of [Granville 1997]):

Proposition 12. *Let p be a prime number. For any positive integer a, define $(a!)_p$ to be the product of those positive integers $\leq a$ which are not divisible by p. Let q, m, n and r be positive integers such that $n = m + r$. Write n in base p as $\sum_{i=0}^{d} n_i p^i$ and let N_j be the least positive residue of $[n/p^j] \pmod{p^q}$ for each $j \geq 0$ (so that $N_j = \sum_{i=0}^{q-1} n_{j+i} p^i$). Also, make the corresponding definitions for m_j, M_j, r_j and R_j. Let e_j denote the number of 'carries', when adding m and r in base p, on and beyond the j-th digit. Then,*

$$\frac{(\pm 1)^{e_{q-1}}}{p^{e_0}} \binom{n}{m} \equiv \left(\prod_{j=0}^{d} \frac{(N_j!)_p}{(M_j!)_p (R_j!)_p} \right) \pmod{p^q},$$

where $(\pm 1) = -1$ except if $p = 2$ and $q \geq 3$.

We will not attempt to generalize Theorem 8 in this paper because it would take us too far afield.

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References

[Granville 1997] A. Granville, "Arithmetic properties of binomial coefficients, I: Binomial coefficients modulo prime powers", pp. 253–276 in *Organic mathematics* (Burnaby, BC, 1995), edited by J. Borwein et al., CMS Conf. Proc. **20**, Amer. Math. Soc., Providence, RI, 1997. MR 99h:11016 Zbl 0903.11005

[1] Note that Theorem 7 still holds if we replace p by p^a, or even by any integer ≥ 2. It is only for Theorem 8 that this method fails, which, in turn, happens because it fails for Lemmas 3 and 4.

[Niven et al. 1991] I. Niven, H. S. Zuckerman, and H. L. Montgomery, *An introduction to the theory of numbers*, 5th ed., Wiley, New York, 1991. MR 91i:11001 Zbl 0742.11001

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