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On the orbits of an orthogonal group action

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Let G be the Lie group $\mathrm{SO}(n, \mathbb{R}) \times \mathrm{SO}(n, \mathbb{R})$ and let V be the vector space of $n \times n$ real matrices. An action of G on V is given by

$$(g, h).v := g^{-1}vh, \quad (g, h) \in G, \quad v \in V.$$

We consider the orbits of this group action and demonstrate a cross-section to the orbits. We then determine the stabilizer for a typical element in this cross-section and completely describe the fundamental group of an orbit of maximal dimension.

1. Introduction

Let G be the Lie group $\mathrm{SO}(n, \mathbb{R}) \times \mathrm{SO}(n, \mathbb{R})$ and let V be the vector space of $n \times n$ real matrices. An action of G on V is given by

$$(g, h).v := g^t vh = g^{-1}vh, \quad (g, h) \in G, \quad v \in V,$$

where g^t denotes the matrix transpose of g and where the operation on the right is matrix multiplication. This action is obviously smooth (having continuous derivatives of all orders) since the matrix entries in $(g, h).v$ are polynomial functions of the matrix entries of g, h and v.

For each $v \in V$ we define the *orbit of* v, denoted by $G.v \subseteq V$, as the set

$$G.v := \{(g, h).v \mid (g, h) \in G\}.$$

For $v, w \in V$ the relation

$$v \sim w \text{ if } v \text{ and } w \text{ are in the same } G\text{-orbit}$$

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is an equivalence relation and so V is partitioned into G-orbits. We also define G_υ, the *stabilizer of* υ, to be the those elements in G that fix υ:

$$G_\upsilon =: \{(g,h) \in G \mid (g,h).\upsilon = \upsilon\}.$$

For each $\upsilon \in V$, G_υ is a closed (usually not normal) subgroup of G, and so is a Lie group.

Let G/G_υ denote the set of left cosets of G_υ in G. Since G_υ is a closed subgroup of G, G/G_υ is a differentiable manifold and $dim G/G_\upsilon = \dim G - \dim G_\upsilon$, where dim indicates the dimension. Furthermore, G/G_υ is diffeomorphic to the orbit $G.\upsilon$. If G_υ is normal in G, then G/G_υ is a Lie group [Bröcker and tom Dieck 1985, Section1.4].

A subset D of V is a *cross-section* to the orbits if every G-orbit intersects D. That is, for each $\upsilon \in V$ there is an element $(g,h) \in G$ and an element $d \in D$ such that $(g,h).\upsilon = d$. Some definitions of a cross-section are more restrictive, requiring that each orbit intersect the cross-section exactly once.

In this paper we consider the orbits of this group action. In Section 2 we demonstrate a cross-section of the orbits, and in Section 3 we determine the stabilizer for a typical element in this cross-section. In Section 4 we discuss the orbits for the case $n = 2$ and introduce generic orbits — those of maximal dimension — for arbitrary n. Section 5 reviews some useful information about fundamental groups, covering spaces, and the covering group $\mathrm{Spin}(n)$. Our main result is in Section 6 where we connect these ideas in order to completely describe the fundamental group of a generic orbit, and in Section 7 we work through an example that further exposes the anatomy. We close with a few remarks in Section 8 regarding those orbits that do not have maximal dimension.

2. Cross section to the orbits

In this section we show that the diagonal matrices with non-negative entries constitute a cross-section to the group action.

Proposition 2.1. *Let $G = \mathrm{SO}(n) \times \mathrm{SO}(n)$ and let V be the vector space of $n \times n$ real matrices. Let G act on V via $(g,h).\upsilon = g^t \upsilon h$. Then for each $\upsilon \in V$ there is a $(k_1, k) \in G$ such that $(k_1,k).\upsilon = \mathrm{diagonal}(d_1,\dots,d_n)$, with $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$.*

Proof. Let $\upsilon \in GL(n,\mathbb{R})$ where $GL(n,\mathbb{R})$ is the (dense, open) subset of invertible $n \times n$ matrices in V. Then $\upsilon^t \upsilon$ is a symmetric matrix with positive eigenvalues, and hence is diagonalizable via conjugation by an element in $\mathrm{SO}(n,\mathbb{R})$. That is, there is a k in $\mathrm{SO}(n,\mathbb{R})$ such that

$$k^t \upsilon^t \upsilon k = a,$$

where $a = \mathrm{diagonal}(a_1,\dots,a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n > 0$.

Now let $a^{-1/2} = \text{diagonal}(1/\sqrt{a_1}, \ldots, 1/\sqrt{a_n})$. If $\mathcal{I}_n$ is the $n \times n$ identity matrix we have

$$\mathcal{I}_n = a^{-1/2}\, a\, a^{-1/2} = a^{-1/2}\, [k^t v^t v k]\, a^{-1/2} = (vka^{-1/2})^t vka^{-1/2}.$$

It follows that $vka^{-1/2}$ is in $O(n, \mathbb{R})$. Let $a^{1/2} = \text{diagonal}(\sqrt{a_1}, \ldots, \sqrt{a_n})$. Then

$$a^{1/2} = \mathcal{I}_n a^{1/2} = [a^{-1/2} k^t v^t v k a^{-1/2}]\, a^{1/2} = a^{-1/2} k^t v^t v k.$$

Thus, if $k_1 = vka^{-1/2}$, we can write this as

$$(k_1)^t vk = (k_1, k).v = a^{1/2},$$

where $k_1 \in O(n, \mathbb{R})$ and $k \in \mathrm{SO}(n, \mathbb{R})$. If k_1 happens to be in $\mathrm{SO}(n, \mathbb{R})$ we are done. If not, we can change the sign of one of the entries in $a^{-1/2}$ so that k_1 *is* in $\mathrm{SO}(n, \mathbb{R})$, proving the result for any V in the dense subset of invertible $n \times n$ matrices. Since our group action is continuous, the result holds for all $v \in V$. We could also modify the above proof slightly to account for those eigenvalues of $v^t v$ that are equal to zero. □

3. The stabilizer of a representative element

Let Γ be an arbitrary group acting on a set X. If x and y are in the same Γ-orbit, then $x = \gamma . y$ for some $\gamma \in \Gamma$. It is a standard result that $\gamma^{-1} \Gamma_x \gamma = \Gamma_y$, that is, the stabilizers are isomorphic via conjugation. Therefore, it is sufficient to determine the stabilizers of those elements that are in the cross section.

We start with a simple example that demonstrates the general idea for the situation that we are considering. Let $d \in V$ and $(g, h) \in G$ be given by

$$d = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{pmatrix}, \quad \text{where } d_1 > d_2 > 0,$$

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} & g_{1,3} \\ g_{2,1} & g_{2,2} & g_{2,3} \\ g_{3,1} & g_{3,2} & g_{3,3} \end{pmatrix}, \quad h = \begin{pmatrix} h_{1,1} & h_{1,2} & h_{1,3} \\ h_{2,1} & h_{2,2} & h_{2,3} \\ h_{3,1} & h_{3,2} & h_{3,3} \end{pmatrix}.$$

We may assume $d_1 > d_2$ since conjugation by a matrix such as

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in \mathrm{SO}(3)$$

will reorder the entries in d.

If (g, h) stabilizes d then $g^t dh = d$ or equivalently, $dh = gd$, so we have

$$\begin{pmatrix} d_1h_{1,1} & d_1h_{1,2} & d_1h_{1,3} \\ d_1h_{2,1} & d_1h_{2,2} & d_1h_{2,3} \\ d_2h_{3,1} & d_2h_{3,2} & d_2h_{3,3} \end{pmatrix} = \begin{pmatrix} d_1g_{1,1} & d_1g_{1,2} & d_2g_{1,3} \\ d_1g_{2,1} & d_1g_{2,2} & d_2g_{2,3} \\ d_1g_{3,1} & d_1g_{3,2} & d_2g_{3,3} \end{pmatrix}. \tag{3-1}$$

That is, the first entry in d acts on the first row of h, but acts on the first column of g, etc. The rows of g and h are orthonormal (considered as vectors in $\mathbb{R}^3$ with the usual dot product), and we compare the squared length of the first row of dh with the first row of gd in (3-1):

$$(d_1h_{1,1})^2 + (d_1h_{1,2})^2 + (d_1h_{1,3})^2 = (d_1g_{1,1})^2 + (d_1g_{1,2})^2 + (d_2g_{1,3})^2.$$

Since first rows of both h and g have length 1, we have

$$\begin{aligned} \Rightarrow (d_1)^2 &= (d_1)^2[(h_{1,1})^2 + (h_{1,2})^2 + (h_{1,3})^2] \\ &= (d_1g_{1,1})^2 + (d_1g_{1,2})^2 + (d_2g_{1,3})^2 < (d_1)^2, \end{aligned}$$

since $d_1 > d_2$. But this is impossible unless $g_{1,3} = 0$, and hence $h_{1,3} = 0$. Comparing the lengths of the second rows shows that $g_{2,3} = h_{2,3} = 0$, and applying this same reasoning to the columns gives $h_{3,1} = g_{3,1} = 0$ and $h_{3,2} = g_{3,2} = 0$.

We now have

$$\begin{pmatrix} d_1h_{1,1} & d_1h_{1,2} & 0 \\ d_1h_{2,1} & d_1h_{2,2} & 0 \\ 0 & 0 & d_2h_{3,3} \end{pmatrix} = \begin{pmatrix} d_1g_{1,1} & d_1g_{1,2} & 0 \\ d_1g_{2,1} & d_1g_{2,2} & 0 \\ 0 & 0 & d_2g_{3,3} \end{pmatrix},$$

which immediately implies that $h = g$. The condition that $g^t g = I$ gives us that each of the block submatrices must be orthogonal, and of course g must have determinant 1. Note that if we were to allow $d_2 = 0$ then $g_{3,3}$ and $h_{3,3}$ need not be equal.

An inductive argument on the different eigenvalues of d proves the general case and is not particularly enlightening, so we state the following result.

Proposition 3.1. *Let $G = \mathrm{SO}(n) \times \mathrm{SO}(n)$ and let V be the vector space of $n \times n$ real matrices. Let G act on V via $(g, h).v = g^t vh$. Let*

$$d = \mathrm{diagonal}(\underbrace{d_1, \dots, d_1}_{s_1}, \dots, \underbrace{d_k, \dots, d_k}_{s_k}) \in V$$

with $d_1 > d_2 > \dots > d_k \geq 0$, and let G_d be the stabilizer of d in G. If $d_k > 0$, then $G_d = \{(g, g) : g \in S(O(s_1) \times \cdots \times O(s_k))\}$.

That is, each g consists of block-diagonal matrices where each block is an $s_i \times s_i$ orthogonal matrix and where s_i is the multiplicity of the eigenvalue d_i in d. The "S" indicates that the product of the determinants of the blocks is 1. If $d_k = 0$ then $G_d = (g, h)$ where g and h consist of block-diagonal matrices with each i-th block in $O(s_i)$, and where $g = h$ except for the k-th block.

4. Orbits

A natural question is "What are these orbits like?" From the introduction we know that, for any element $v \in V$, the orbit $G.v$ is diffeomorphic to the coset space G/G_v, with $\dim G.v = \dim G - \dim G_v$. Since any two elements in the same G-orbit have isomorphic stabilizers, it will be sufficient to consider the orbits of those representative elements d in the cross-section D. In particular, the dimension of these orbits is completely determined by the multiplicity of the distinct eigenvalues of d and is independent of their actual values.

Example: $n = 2$. In low-dimensional cases we can use computer graphics to get an idea about the nature of these orbits, and we now illustrate this for the two-dimensional Lie group $G = \mathrm{SO}(2) \times \mathrm{SO}(2)$. Figure 1 shows the orbit of $d = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, with a cut-away view on the right. Note that, for $n = 2$, the orbit lies in $\mathrm{Mat}(2, \mathbb{R}) \cong \mathbb{R}^4$, and each figure is a projection of this orbit onto $\mathbb{R}^2$. Since G is abelian, G_d is normal in G and so G/G_d is an abelian Lie group which is compact since the quotient map is continuous. Since $G_d = \{(\mathcal{I}_2, \mathcal{I}_2), (-\mathcal{I}_2, -\mathcal{I}_2)\}$ which is discrete, the orbit $G.d$ has dimension 2. We conclude that this orbit is diffeomorphic to the 2-torus embedded in $\mathbb{R}^4$, since this is the only two-dimensional compact abelian Lie group. Notice that the graphics could be misleading, since we usually picture the 2-torus in $\mathbb{R}^3$ as resembling the surface of a donut.

Note that if an element d in the cross-section D has only one eigenvalue, then the stabilizer G_d is isomorphic to $\mathrm{SO}(2)$ and so the orbit $G.d$ is one-dimensional and is diffeomorphic to $\mathrm{SO}(2)$, that is, a circle.

Generic orbits. We now move on to consider the following special case of *generic* orbits — those with maximal dimension — for arbitrary n. We will reserve the symbol δ for a diagonal matrix in the cross-section D with n distinct eigenvalues.

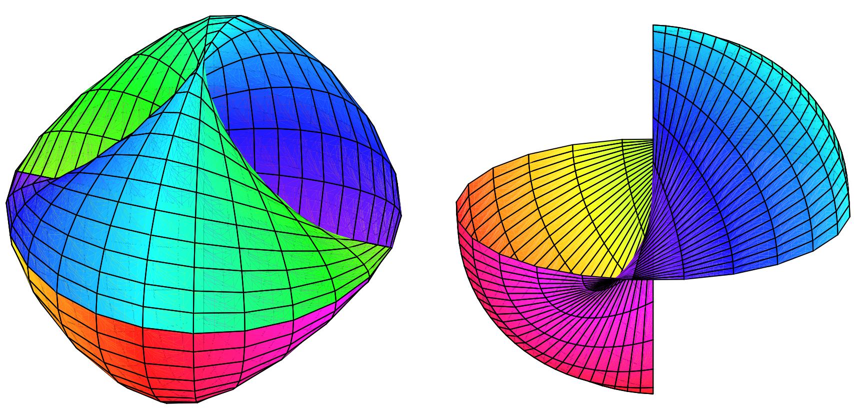

Figure 1. An orbit for $n = 2$ projected onto $\mathbb{R}^2$. Right: cut-away view of same orbit.

That is, $\delta = \text{diagonal}(d_1, \dots, d_n)$ with $d_1 > d_2 > \cdots > d_n \geq 0$. From Proposition 3.1 we have $G_\delta = (g, g)$, where $g = \text{diagonal}(\pm 1, \dots, \pm 1)$ has an even number of entries equal to -1. Since the stabilizer of δ is discrete, the dimension of the G-orbit of δ is equal to the dimension of G.

Proposition 4.1. *Let $G = \mathrm{SO}(n) \times \mathrm{SO}(n)$ and let V be the vector space of $n \times n$ real matrices. Let G act on V via $(g, h).v = g^t v h$. Let*

$$\delta = \text{diagonal}(d_1, d_2, \dots, d_n) \in V$$

with $d_1 > d_2 > \cdots > d_n \geq 0$, and let G_δ be the stabilizer of δ in G. Then $|G_\delta|$, the order of G_δ, is 2^{n-1}.

Proof. From Proposition 3.1, G_δ consists of n copies of $O(1) = \pm 1$ lying in $\mathrm{SO}(n)$, so there must be an even number of entries equal to -1. Thus

$$|G_\delta| = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{k},$$

where $k = n$ if n is even and $k = n - 1$ if n is odd. From the binomial theorem,

$$\begin{aligned} 2^n &= (1+1)^n + (1-1)^n \\ &= \left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n}\right] + \left[\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n}\right] \\ &= 2\left[\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots + \binom{n}{k}\right] = 2\,|G_\delta|. \end{aligned}$$ □

Again, what are these orbits like? Figure 2 shows a (projection of a) two-dimensional slice of the orbit of $\delta = \text{diagonal}(2, 1, 0)$ for the case $n = 3$. Could this be just a torus in disguise, as was the case $n = 2$? One way to determine how interesting the orbits are is to consider their fundamental groups.

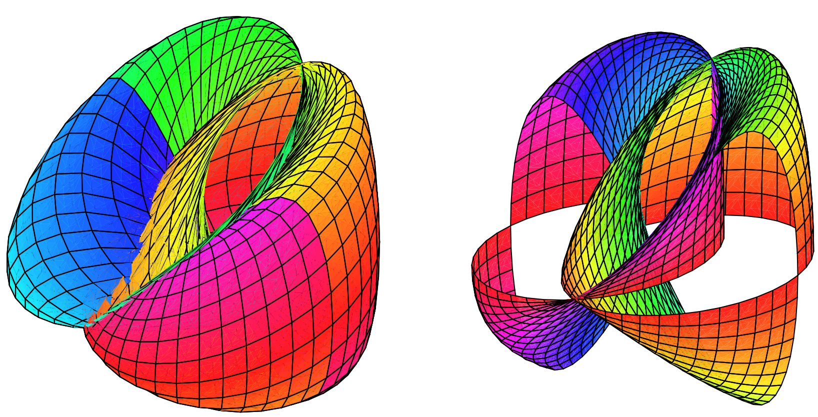

Figure 2. A section of an orbit for $n = 3$. Right: cut-away view of same section.

5. Fundamental groups, covering spaces and spin(n)

In order to make this exposition self-contained and to fix notation we review some background material that will be familiar to many readers.

Review of the fundamental group and covering spaces. Let X be a topological space and let $[0, 1] \subset \mathbb{R}$ be the closed unit interval. A *path in* X is a continuous map $f : [0, 1] \to X$. Two paths f and g from x_1 to x_2 are said to be *homotopic* if one can be continuously deformed into the other. This is obviously an equivalence relation, and we denote the equivalence class of f by $[f]$. Of special interest will be loops, or closed paths that start and end at a distinguished base point $x \in X$, and we can define a multiplication of loops by concatenation. That is, $f \cdot g$ means *first go around* f *and then go around* g. This operation is associative and is well defined when taking equivalence classes: $[f] \cdot [g] = [f \cdot g]$. The constant loop $e_x : [0, 1] \to X$ given by $e_x(t) = x$ serves as the identity element for this operation and the loop f^{-1} is the loop f traversed in the opposite direction. We can then define the *first homotopy group* or the *fundamental group*, denoted $\pi_1(X, x)$, as the group of (equivalence classes of) loops in X that start and end at x, along with this multiplication. If x_1 and x_2 are connected by a path in X, then $\pi_1(X, x_1)$ and $\pi_1(X, x_2)$ are isomorphic. Homeomorphic topological spaces have isomorphic fundamental groups, but the converse need not be true.

We will also require the notion of a covering. Let (X, x), (Y, y) be topological spaces with base points x and y respectively. A map $p : (Y, y) \to (X, x)$ is a *covering map* if

(i) $p(y) = x$;

(ii) p is continuous and surjective;

(iii) for every $x_0 \in X$ there is an open neighborhood $U_{x_0} \subset X$ so that $p^{-1}(U_{x_0})$ is a disjoint union of open sets $\{V_\alpha\}$ and so that for each α, the map p restricted to V_α is a homeomorphism of V_α onto U_{x_0}.

We then say that (Y, y) is a *covering space* of (X, x) and refer the the covering space along with the covering map as a *cover* of (X, x). We will also use the standard results, roughly stated, that the composition of covers is a cover, and that the cover of a product is the product of the respective covers.

Remark 5.1. A topological space with trivial fundamental group is called *simply connected*. A covering space that is simply connected is called a *universal covering space*. It is unique up to homeomorphism.

We will need the notion of *lifting* a path from a space to a covering space.

Let $p : (Y, y) \to (X, x)$ be a covering map. Let $f : [0, 1] \to X$ be a path starting at x. A lifting of f is a path $\tilde{f} : [0, 1] \to Y$ such that $p \circ \tilde{f} = f$. For the cases we are considering, these lifts are unique up to homotopy. That is, let f be a path in X beginning at x, and let $\tilde{f}$ and $\tilde{g}$ be two lifts of f both beginning at y. Then $\tilde{f}$ is homotopic to $\tilde{g}$. In particular, $\tilde{f}$ and $\tilde{g}$ must end at the same point in Y.

Let $p : (Y, y) \to (X, x)$ be a covering map. A homeomorphism $h : Y \to Y$ is called a *deck transformation* or *covering transformation* if $p \circ h = p$. Clearly the collection of all such deck transformations is a group with the operation being composition of maps.

We will use the following fact to determine $\pi_1(G.\delta, \delta)$.

Theorem 5.2. [Massey 1991, Corollary 7.5] *If (Y, y) is a universal covering space of (X, x), the group of deck transformations of (Y, y) is isomorphic to $\pi_1(X, x)$. If $p : (Y, y) :\to (X, x)$ is a covering map, then the order of $\pi_1(X, x)$ is equal to the cardinality of the set $p^{-1}(x)$.*

Now consider the map $p_1 : G \to G.\delta$ given by $g \mapsto g.\delta$. Since $p_1^{-1}(\delta) = \{\gamma \in G \mid \gamma.\delta = \delta\} = G_\delta$ is discrete, Theorem E4 of [Hall 2003] has the following consequence.

Proposition 5.3. *Let $G = \mathrm{SO}(n) \times \mathrm{SO}(n)$ and let $\mathbf{1}$ denote the identity element in G. Let V be the vector space of $n \times n$ real matrices and let G act on V by*

$$(g, h).v := g^t v h, \quad (g, h) \in G, \quad v \in V.$$

If $\delta \in V$ is a diagonal matrix with n distinct eigenvalues, and if $G.\delta$ is the G-orbit of δ, then the map $p_1 : (G, \mathbf{1}) \to (G.\delta, \delta)$ given by $g \mapsto g.\delta$ is a covering map.

Said another way, G is a fiber bundle over the orbit $G.\delta$ with projection map $(g, h) \mapsto (g, h).\delta$ and discrete fiber G_δ.

Spin(n). We now provide a brief review of the construction of the Lie group $\mathrm{Spin}(n)$ and the covering map from $\mathrm{Spin}(n)$ to $\mathrm{SO}(n)$. This abridged description should be sufficient for our purposes, but for a more complete discussion, see [Bröcker and tom Dieck 1985]. The presentation below borrows extensively from the excellent exposition in [Simon 1996].

Consider the vector space $\mathbb{R}^n$ with standard basis $\{e_1, \dots, e_n\}$. We form $C(n)$, the *Clifford algebra* on $\mathbb{R}^n$, by declaring that multiplication is associative, distributive over addition, and obeys the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. This is just a fancy way of saying that the basis elements anti-commute and $e_i^2 = -1$. If $I = i_1 i_2 \dots i_k$ is a multiindex with $1 \leq i_1 < \cdots < i_k \leq n$ we set $e_0 = 1$, we set $e_I = e_{i_1} e_{i_2} \dots e_{i_k}$ and we set $|I| = k$. Then $C(n)$ is an algebra with basis $\{e_I\}$ and it follows that the dimension of $C(n)$ is 2^n. We also have the subalgebra of even elements

$$C(n)_{\text{even}} = \{A \in C(n) \mid A \text{ is a linear combination of } e_I \text{ with } |I| \text{ even}\}.$$

Examples. We have canonical isomorphisms:

- $C(0) \cong \mathbb{R}$;
- $C(1) \cong \mathbb{C}$ via the map $e_1 \mapsto i = \sqrt{-1}$;
- $C(2) \cong \mathbb{H}$ (the quaternion algebra) via the map $e_1 \mapsto i$, $e_2 \mapsto j$ and so $e_1 e_2 \mapsto k$. Here, i, j, and k are those elements in $\mathbb{H}$ with $i^2 = j^2 = k^2 = -1$ and $ij = k$;
- we also have $C(3)_{\text{even}} \cong \mathbb{H}$ via the map $e_1 e_2 \mapsto i$, $e_1 e_3 \mapsto j$, so

$$(e_1 e_2)(e_1 e_3) = e_2 e_3 \mapsto k.$$

We can define $\mathrm{Spin}(n)$ to be the invertible elements S of $C(n)_{\text{even}}$ that (among other things) leave the vector space $W = \mathbb{R}^n$ invariant under conjugation:

$$S W S^{-1} \subseteq W.$$

Now consider the quadratic elements

$$q_{ij} = \tfrac{1}{2} e_i e_j,$$

for $1 \le i < j \le n$, and observe that they obey the same commutation relations as the generators L_{ij} of the Lie algebra $\mathfrak{so}(n)$. Therefore these quadratic elements form a Lie algebra isomorphic to $\mathfrak{so}(n)$, and so to get the group $\mathrm{Spin}(n)$ we exponentiate these quadratic elements:

$$\begin{aligned} S_{ij}(t) &:= \exp(t\, q_{ij}) = 1 + (t q_{ij}) + \frac{1}{2!}(t q_{ij})^2 + \frac{1}{3!}(t q_{ij})^3 + \cdots \\ &= \cos(t/2) + \sin(t/2)(2 q_{ij}), \end{aligned}$$

since $q_{ij}^2 = -1$. As t goes from 0 to 4π, $S_{ij}(t)$ gives a copy of $U(1)$ in $\mathrm{Spin}(n)$ which is homeomorphic to a circle in the plane spanned by 1 and $2q_{ij}$.

Now the elements A in $\mathrm{Spin}(n)$ act on $\mathbb{R}^n$ by conjugation and this gives a representation of $\mathrm{Spin}(n)$ on $\mathbb{R}^n$. Consequently, we have a map

$$R : \mathrm{Spin}(n) \to \mathrm{SO}(n, \mathbb{R})$$

defined by

$$A e_i A^{-1} = \sum_{i=1}^{n} R_{ji}(A) e_j. \tag{5-1}$$

We now determine the matrix representation of the group elements

$$S_{ij}(t) := \exp(t q_{ij}) = \cos(t/2) + \sin(t/2)(e_i e_j) \tag{5-2}$$

by determining the action on the basis vectors. First observe that $e_i e_j$ commutes with e_k when k is equal to neither i nor j, so in this case

$$S_{ij}(t) e_k S_{ij}^{-1}(t) = \big(\cos(t/2) + \sin(t/2)(e_i e_j)\big) e_k \big(\cos(t/2) - \sin(t/2)(e_i e_j)\big) = e_k.$$

Now conjugating e_i by $S_{ij}(t)$ we have

$$\begin{aligned} S_{ij}(t)e_i S_{ij}^{-1}(t) &= \big(\cos(t/2)+\sin(t/2)(e_ie_j)\big)e_i\big(\cos(t/2)-\sin(t/2)(e_ie_j)\big) \\ &= \big(\cos(t/2)+\sin(t/2)(e_ie_j)]^2 e_i \\ &= \big(\cos^2(t/2)-\sin^2(t/2)\big)e_i - 2\cos(t/2)\sin(t/2)e_j \\ &= \cos(t)e_i - \sin(t)e_j. \end{aligned}$$

A similar computation applied to e_j gives

$$S_{ij}(t)\, e_j\, S_{ij}^{-1}(t) = \sin(t)e_i + \cos(t)e_j.$$

Therefore, conjugation by $S_{ij}(t) = \exp(tq_{ij})$ induces a rotation by an angle t in the e_i, e_j plane. Since these rotations generate $\mathrm{SO}(n)$, this map is surjective.

The following result is well known (see [Simon 1996, Sections VII.6–VII.7] or [Bröcker and tom Dieck 1985, Section 1.6].

Proposition 5.4. $\mathrm{Spin}(n)$ *is simply connected. If* $A \in \mathrm{Spin}(n)$ *and if* $R(A)$ *is the* $n \times n$ *matrix with entries* $R_{ji}(A)$ *described in* (5-1) *above*, *then the map* $R : (\mathrm{Spin}(n), \mathbf{1}) \to (\mathrm{SO}(n,\mathbb{R}), \mathbf{1})$ *is a twofold universal covering map and a homomorphism of Lie groups. The symbol* $\mathbf{1}$ *denotes the unit elements in the respective groups.*

6. The fundamental group of a generic orbit

We are now ready to determine the fundamental group for a generic orbit of maximum dimension. We will proceed by elaborating on some previously introduced ideas and connecting them together in order to invoke Theorem 5.2.

As before, $\delta \in D$ denotes an element in the cross-section with n distinct eigenvalues. By Proposition 3.1, a typical element in its stabilizer G_δ can be represented by a diagonal matrix with each entry equal to ± 1, and where an even number of entries are equal to -1. From now on, let $I = i_1 i_2 \cdots i_k$ be a multiindex with $1 \le i_1 < \cdots < i_k \le n$, k even and set $l = k/2$. Let ST_I be the element in G_δ with those entries that are equal to -1 indexed by I. For example, if $n = 6$,

$$ST_{1,2,3,5} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using this notation, $G_\delta = \{(ST_I, ST_I) : |I| \text{ is even}\}$.

Let $\tau = (t_1, \ldots, t_l)$ and let $\mathrm{SO}_I(\tau)$ be the matrix consisting of rotations by an angle t_j in the planes indexed pairwise by I. These pairs are of the form i_{2m-1}, i_{2m}.

For example, if $I = 1, 2, 3, 5$ and $\tau = (t_1, t_2)$ then $\mathrm{SO}_I(\tau)$ rotates by an angle t_1 in the $1, 2$ plane and by an angle t_2 in the $3, 5$ plane. For instance, if $n = 6$,

$$\mathrm{SO}_{1,2,3,5}(\tau) = \begin{pmatrix} \cos t_1 & \sin t_1 & 0 & 0 & 0 & 0 \\ -\sin t_1 & \cos t_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos t_2 & 0 & \sin t_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\sin t_2 & 0 & \cos t_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that $\mathrm{SO}_{1,2,3,5}(\tau)$ is equal to the matrix product $\mathrm{SO}_{1,2}(t_1)\,\mathrm{SO}_{3,5}(t_2)$. It should be easy to see that

Lemma 6.1. $ST_I = \mathrm{SO}_I(\pm\pi, \ldots, \pm\pi)$.

We next consider product of elements $S_{ij}(t) \in \mathrm{Spin}(n)$ and relate them to the corresponding elements in $\mathrm{SO}(n)$.

Lemma 6.2. *Let $I = i_1 i_2 \cdots i_k$ be a multiindex with k even and where*

$$i_1 < i_2 < \cdots < i_k.$$

Set $l = k/2$. Let $\tau = (t_1, \ldots, t_l)$ and let $\mathrm{SO}_I(\tau)$ be the matrix consisting of rotations by an angle t_j in the planes indexed pairwise by I. Let $S_{i,j}(t)$ be defined as in (5-2), *and let $S_I(\tau)$ designate the product $S_I(\tau) = S_{i_1 i_2}(t_1) S_{i_3 i_4}(t_2) \cdots S_{i_{k-1} i_k}(t_l)$. Let $R : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be the covering map given by Proposition 5.4. Then $R(S_I(\tau)) = \mathrm{SO}_I(\tau)$.*

Further, $e_I := e_{i_1} e_{i_2} \cdots e_{i_k}$, we have $e_I = S_I(\pi, \ldots, \pi)$.

Proof. Since the entries in the multiindex I are distinct, the designation $\mathrm{SO}_I(\tau) = \mathrm{SO}_{i_1 i_2 \cdots i_k}(\tau) = \mathrm{SO}_{i_1 i_2}(t_1)\,\mathrm{SO}_{i_3 i_4}(t_2) \cdots \mathrm{SO}_{i_{k-1} i_k}(t_l)$ is unambiguous. Since the map R is a representation, we have

$$\begin{aligned} R[S_I(\tau)] &= R[S_{i_1 i_2}(t_1)]\, R[S_{i_3 i_4}(t_2)] \cdots R[S_{i_{k-1} i_k}(t_l)] \\ &= \mathrm{SO}_{i_1 i_2}(t_1)\,\mathrm{SO}_{i_3 i_4}(t_2) \cdots \mathrm{SO}_{i_{k-1} i_k}(t_l) = \mathrm{SO}_I(\tau). \end{aligned}$$

For the last assertion, note that (5-2) gives $e_i e_j = S_{ij}(\pi)$ for any i, j, since $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$. Hence

$$e_I = [e_{i_1} e_{i_2}][e_{i_3} e_{i_4}] \cdots [e_{i_{k-1}} e_{i_k}] = S_{i_1 i_2}(\pi) S_{i_3 i_4}(\pi) \cdots S_{i_{k-1} i_k}(\pi) = S_I(\pi, \ldots, \pi),$$

as required. □

This next result is proven similarly.

Lemma 6.3. *Denote by π^+ an l-tuple $\pi^+ = (\pm\pi, \dots, \pm\pi)$ with an even number of entries equal to $-\pi$ and denote by π^- an l-tuple $\pi^- = (\pm\pi, \dots, \pm\pi)$ with an odd number of entries equal to $-\pi$. Let $S_I(\tau)$ and e_I be as in the previous lemma. Then $S_I(\pi^+) = e_I$ and $S_I(\pi^-) = -e_I$.*

Finally, let $\widetilde{\mathbf{1}}$ denote the unit element in $\widetilde{G} = \mathrm{Spin}(n) \times \mathrm{Spin}(n)$ and let $\mathbf{1}$ denote the unit element in $G = \mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R})$. Then $(\widetilde{G}, \widetilde{\mathbf{1}})$ is the universal covering space (in fact, a covering group) of $(G, \mathbf{1})$ and the map

$$\rho = R \times R : (\widetilde{G}, \widetilde{\mathbf{1}}) \to (G, \mathbf{1})$$

is a fourfold covering map. Now recall the covering map $p_1 : (G, \mathbf{1}) \to (G.\delta, \delta)$ from Proposition 5.3. It follows that the composition

$$P = \rho \circ p_1 : (\widetilde{G}, \widetilde{\mathbf{1}}) \to (G.\delta, \delta)$$

is a covering map and that $\widetilde{G}$ is the universal covering space of the orbit $G.\delta$.

Definition 6.4. $E(n) = \{\pm\, e_I : |I| \text{ is even}\}$.

Observe that $E(n)$ is closed under multiplication since, if $e_I e_J = e_K$ then $|K| = |I| + |J|$ when I and J are distinct indices, and the entries of K contract in pairs when I and J have repeated entries. For example, $e_{1,2}\, e_{2,3} = -e_{1,3}$. Since $(e_I)^{-1} = \pm e_I$, $E(n)$ is a group under multiplication. A computation very similar to that in Proposition 4.1 shows that $|E(n)| = 2^n$.

Definition 6.5. Consider the set $\widetilde{E(n)} = \{(\nu, \pm\nu) \mid \nu \in E(n)\}$. This is a subgroup of $\widetilde{G}$ which is isomorphic to the group $E(n) \times \mathbb{Z}_2$ via the identifications $(\nu, 1) \mapsto (\nu, \nu)$ and $(\nu, -1) \mapsto (\nu, -\nu)$ for $\nu \in E(n)$.

Proposition 6.6. $P^{-1}(\delta) = \widetilde{E(n)}$.

Proof.

$$\begin{aligned}
P[(e_I, e_I)] &= p_1 \circ [R(e_I), R(e_I)],\\
\text{Lemma 6.3} \Rightarrow &= p_1 \circ [R(S_I(\pi^+), R(S_I(\pi^+)],\\
\text{Lemma 6.2} \Rightarrow &= p_1 \circ [\mathrm{SO}_I(\pi^+), \mathrm{SO}_I(\pi^+)],\\
\text{Lemma 6.1} \Rightarrow &= p_1 \circ [ST_I, ST_I],\\
&= \delta.
\end{aligned}$$

The proofs of the other cases such as $P[(e_I, -e_I)] = \delta$ are similar and hence $\widetilde{E(n)} \subseteq P^{-1}(\delta)$.

Now $p_1^{-1}(\delta) = \{(ST_I, ST_I) : |I| \text{ is even}\} \subseteq G$ has order 2^{n-1} (Proposition 4.1) and ρ is a fourfold covering map $\widetilde{G} \to G$. Therefore the set $P^{-1}(\delta)$ has order 2^{n+1} which is equal to the order of $\widetilde{E(n)}$. □

The main result of this paper completely describes the fundamental group of a generic orbit.

Theorem 6.7. *Let $G = \mathrm{SO}(n) \times \mathrm{SO}(n)$ and let V be the vector space of $n \times n$ real matrices. Let G act on V via $(g, h).v = g^t v h$. Let $\delta = \mathrm{diagonal}(d_1, d_2, \dots, d_n) \in V$ with $d_1 > d_2 > \dots > d_n \geq 0$, and let $G.\delta$ be the G-orbit of δ in V. Let $e_1, \dots, e_n$ be the standard basis vectors in $\mathbb{R}^n$ and let $E(n) = \{\pm e_{i_1} \dots e_{i_k} \mid k \text{ is even}\}$ be the group generated by the quadratic units $e_i e_j$, $i < j$ in the Clifford algebra on $\mathbb{R}^n$. Then the fundamental group $\pi_1(G.\delta, \delta)$ is isomorphic to $E(n) \times \mathbb{Z}_2$.*

Proof. We will show that the group of deck transformations $\mathrm{Aut}(\widetilde{G}, P)$ on the universal covering $(\widetilde{G}, \mathbf{1})$ is isomorphic to $\widetilde{E(n)}$ which is isomorphic to $E(n) \times \mathbb{Z}_2$.

For each $\widetilde{\omega} \in \widetilde{E(n)}$ and $\widetilde{s} \in \widetilde{G}$ define the left translation map $\mathscr{L}_{\widetilde{\omega}} : \widetilde{G} \to \widetilde{G}$ by $\mathscr{L}_{\widetilde{\omega}}(\widetilde{s}) = \widetilde{\omega}\widetilde{s}$, the operation on the right-hand side being multiplication in $\widetilde{G}$. It is a standard exercise that the set of all such translations $\mathbb{L} = \{\mathscr{L}_{\widetilde{\omega}} \mid \widetilde{\omega} \in \widetilde{E(n)}\}$ is a group that is isomorphic to $\widetilde{E(n)}$ via the map $\widetilde{\omega} \mapsto \mathscr{L}_{\widetilde{\omega}}$. Since $\widetilde{G}$ is a Lie group, each translation is continuous with a continuous inverse, hence a homeomorphism from $\widetilde{G}$ to $\widetilde{G}$. Furthermore, for each $\widetilde{v} \in \widetilde{E(n)}$, the composition $P \circ \mathscr{L}_{\widetilde{\omega}}(\widetilde{v}) = P(\widetilde{\omega}\widetilde{v}) = \delta$ so each $\mathscr{L}_{\widetilde{\omega}}$ is a deck transformation and therefore $\mathbb{L}$ is a subgroup of $\mathrm{Aut}(\widetilde{G}, P)$. But $\mathrm{Aut}(\widetilde{G}, P)$ has order 2^{n+1} by Theorem 5.2, and since both these groups have the same order, they must be equal. By Theorem 5.2 again we have $\pi_1(G.\delta, \delta) \cong \mathrm{Aut}(\widetilde{G}, P) = \mathbb{L} \cong \widetilde{E(n)} \cong E(n) \times \mathbb{Z}_2$. □

7. An illustration

We conclude with an example for $n = 6$ that further illustrates the previous constructions. The element

$$S_{3,5}(t) = \exp[(t/2)e_3e_5] = \cos(t/2) + \sin(t/2)e_3e_5$$

in Spin(6) defined in (5-2) is homeomorphic to a circle lying in the plane spanned by 1 and e_3e_5 in the Clifford algebra $C(6)$, and which projects onto the rotation $\mathrm{SO}_{3,5}(t)$ in SO(6) via the representation R. Consider the path $\widetilde{f} : [0, 4\pi] \to \widetilde{G}$ given by $t \mapsto (S_{35}(t), S_{35}(t))$.

Since $\widetilde{f}$ is homeomorphic to a circle and $\widetilde{G}$ is a simply connected covering group, $[\widetilde{f}]$ is trivial in $\pi_1(\widetilde{G}, \mathbf{1})$. Now as t goes from 0 to π, we get a path $\widetilde{f}_{[0,\pi]}$ from $(1, 1)$ to (e_3e_5, e_3e_5) in $\widetilde{G}$ that projects down via P to a loop $f : [0, \pi] \to G.\delta$ given by $f(t) = (\mathrm{SO}_{3,5}(t), \mathrm{SO}_{3,5}(t)).\delta$. By uniqueness of path lifting, f cannot be homotopic to the trivial loop since $\widetilde{f}_{[0,\pi]}$ is not trivial in $\widetilde{G}$. Similarly, as t goes from π to 2π, we get a path $\widetilde{f}_{[\pi,2\pi]}$ from (e_3e_5, e_3e_5) to $(-1, -1)$ in $\widetilde{G}$ that also projects down to the loop f in the orbit $G.\delta$. Not until t travels the entire distance $[0, 4\pi]$ do we obtain the product f^4 in $G.\delta$ that lifts to the (trivial) loop $\widetilde{f}$ in $\widetilde{G}$.

Thus, $[f]^4$ is trivial in $\pi_1(G.\delta, \delta)$. We chart here the information as the path $\widetilde{f}$ is projected onto G and then $G.\delta$ for the successive landmark values of t.

t	$\widetilde{f}(t))$	$\rho((S_{3,5}(t), S_{3,5}(t)))$	$P(S_{3,5}(t), S_{3,5}(t))$
0	$(1, 1)$	$(\mathscr{I}_6, \mathscr{I}_6)$	δ
π	(e_3e_5, e_3e_5)	$(ST_{3,5}, ST_{3,5})$	δ
2π	$(-1, -1)$	$(\mathscr{I}_6, \mathscr{I}_6)$	δ
3π	$(-e_3e_5, -e_3e_5)$	$(ST_{3,5}, ST_{3,5})$	δ
4π	$(1, 1)$	$(\mathscr{I}_n, \mathscr{I}_n)$	δ

As in the previous discussion regarding deck transformations in the proof of Theorem 6.7, we can translate the loop $\widetilde{f}$ via left multiplication by the element $(e_1e_2, e_1e_2) \in \widetilde{E(n)}$. This gives us the loop $\widetilde{g} : [0, 4\pi] \to \widetilde{G}$ given by $t \mapsto (\nu(t), \nu(t))$ where

$$\nu(t) = e_1e_2[\cos(t/2) + \sin((t/2))e_3e_5] = \cos(t/2)e_1e_2 + \sin(t/2)e_1e_2e_3e_5.$$

This is a loop starting at e_1e_2 which lies in the plane spanned by e_1e_2 and $e_1e_2e_3e_5$ in the Clifford algebra $C(6)$.

We check that

$$\nu^{-1}(t) = [-\cos(t/2)e_1e_2 + \sin(t/2)e_1e_2e_3e_5]$$

and that conjugating the basis vectors $e_i \in \mathbb{R}^6$ by $\nu(t)$ produces the map R which takes $\nu(t)$ to the rotation

$$R(\nu(t)) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos t & 0 & \sin t & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\sin t & 0 & \cos t & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(6).$$

As above, the projection P maps $\widetilde{g}_{[0,\pi]}$ to the loop $g(t) = R(\nu(t)).\delta$ in the orbit $G.\delta$ and $[g]^4$ is trivial. Here is part of this information for the path $\widetilde{g}$:

t	$\widetilde{g}(t)$	$\rho(\widetilde{g}(t)))$	$P(\widetilde{g}(t)))$
0	(e_1e_2, e_1e_2)	$(ST_{1,2}, ST_{1,2})$	δ
π	$(e_1e_2e_3e_5, e_1e_2e_3e_5)$	$(ST_{1,2,3,5}, ST_{1,2,3,5})$	δ
2π	$(-e_1e_2, -e_1e_2)$	$(ST_{1,2}, ST_{1,2})$	δ
3π	$(-e_1e_2e_3e_5, -e_1e_2e_3e_5)$	$(ST_{1,2,3,5}, ST_{1,2,3,5})$	δ

By considering the loops in the orbit $G.\delta$ that lift to the path from

$$(1, 1) \to (e_1e_2, e_1e_2) \to (e_1e_2e_3e_5, e_1e_2e_3e_5)$$

in $\widetilde{G}$ we see that g and f cannot be homotopic, so $[g]$ and $[f]$ are distinct elements in $\pi_1(G.\delta, \delta)$.

8. Final remarks on the general case

Determining the first homotopy group for the orbits in the more general case, when the representative element d in the cross-section contains eigenvalues with multiplicities greater than 1, does not lend itself to such direct construction since the map $G \to G.d$ is not a covering map.

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References

[Bröcker and tom Dieck 1985] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, Grad. Texts in Math. **98**, Springer, New York, 1985. MR 86i:22023 Zbl 0581.22009

[Hall 2003] B. C. Hall, *Lie groups, Lie algebras, and representations*, Grad. Texts in Math. **222**, Springer, New York, 2003. MR 2004i:22001 Zbl 1026.22001

[Massey 1991] W. S. Massey, *A basic course in algebraic topology*, Grad. Texts in Math. **127**, Springer, New York, 1991. MR 92c:55001 Zbl 0725.55001

[Simon 1996] B. Simon, *Representations of finite and compact groups*, Grad. Studies in Math. **10**, American Mathematical Society, Providence, RI, 1996. MR 97c:22001 Zbl 0840.22001

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