involve
a journal of mathematics

The Gram determinant for plane curves

Józef H. Przytycki and Xiaoqi Zhu

mathematical sciences publishers

2010 vol. 3, no. 2

The Gram determinant for plane curves

Józef H. Przytycki and Xiaoqi Zhu

(Communicated by Kenneth S. Berenhaut)

We investigate the Gram determinant of the pairing arising from curves in a planar surface, with a focus on the disk with two holes. We prove that the determinant based on $n-1$ curves divides the determinant based on n curves. Motivated by the work on Gram determinants based on curves in a disk and curves in an annulus (Temperley–Lieb algebra of type A and B, respectively), we calculate several examples of the Gram determinant based on curves in a disk with two holes, and advance conjectures on the complete factorization of Gram determinants.

1. Introduction

Gram matrices and Gram determinants. Let B be a finite set and R a commutative ring. A *pairing* over B is a map $B \times B \to R$, denoted by $\langle \cdot , \cdot \rangle$. A very simple case is the Kronecker delta,

$$\langle i, j\rangle = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \qquad \text{for } i, j \in B.$$

Let $b_1, \ldots, b_n$ be a list of the elements of B, with $b_i \neq b_j$ if $i \neq j$. The *Gram matrix* of the pairing $\langle \cdot , \cdot \rangle$ is the $n \times n$ matrix

$$G = [\langle b_i, b_j\rangle]_{1\leq i,j\leq n},$$

and the *Gram determinant* is the determinant of this matrix.

The name is derived from the classical case where R is a field, $B = \{b_1, \ldots, b_n\}$ is a set of points in a vector space V over R, and the pairing is given by an inner product $\langle \cdot , \cdot \rangle$ on V. This situation is familiar; for instance, B is an orthonormal basis of V if and only if V has dimension n and the pairing coincides with the Kronecker delta described above.

MSC2000: 05A99, 57M99.

Keywords: Gram determinants, planar curves, noncrossing partitions, chromatic joins, Temperley–Lieb.

The Gram determinant plays a significant role in the classical case; for example, a set of vectors $B = \{b_1, \dots, b_n\} \subset V$ is linearly independent if and only if the Gram determinant of B is nonzero.

In our situation B will be a certain set of equivalence classes arising from sets of curves on a disk with holes. The ring R is a polynomial ring in many variables, and the pairing describes the interaction between the sets of curves when two copies of the disk are glued along their outer boundaries.[1]

The Gram matrix for a system of plane curves. Let F_0^n be a unit disk with $2n$ points on its boundary. Let $\boldsymbol{B}_0^n$ be the set of all possible diagrams, up to deformation, in F_0^n with n noncrossing chords connecting these $2n$ points. It is known that $|\boldsymbol{B}_0^n|$ is equal to the n-th Catalan number $C_n := \binom{2n}{n}/(n+1)$; see [Stanley 1999], for example. Accordingly, we will call $\boldsymbol{B}_0^n$ the set of *Catalan states*.

Consider the following generalized setup. Let $F_k \subset D^2$ be a plane surface with $k+1$ boundary components, which are given distinct labels. In particular, $F_0 = D^2$, and for $k \geq 1$, F_k is equal to D^2 with k holes. Let F_k^n be F_k with $2n$ points, $a_0, \dots, a_{2n-1}$, arranged counterclockwise along the outer boundary; see Figure 1, left. Throughout this paper, we use a_k and a_{k-1} to denote two adjacent points along the outer boundary, where k is taken modulo $2n$.

Let $\boldsymbol{B}_k^n$ be the set of all possible diagrams, up to equivalence, in F_k^n with n noncrossing chords connecting these $2n$ points, where equivalence is defined as follows: for each diagram $b \in \boldsymbol{B}_k^n$, there is a corresponding diagram $\gamma(b) \in \boldsymbol{B}_0^n$ obtained by filling the k holes in b. We call $\gamma(b)$ the underlying Catalan state of b (see Figure 1, right).

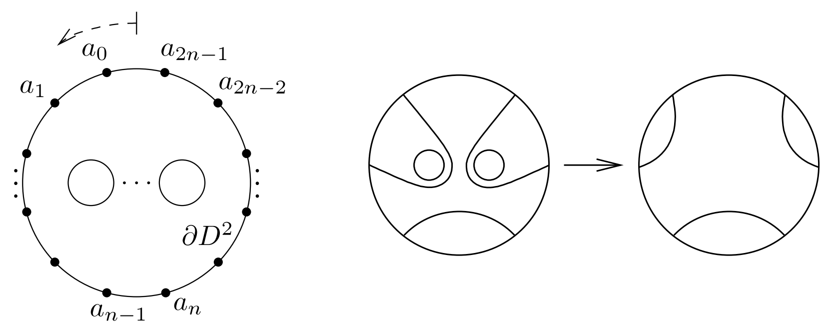


Figure 1. Left: notational conventions for F_k^n. Right: action of γ.

[1] A pairing over B extends to a bilinear form on the free R-module over B; this form is similar to the inner product on a vector space. Note, however, that the inner product over a *complex* vector space is linear in the first variable only, and conjugate-linear in the second; additionally, it is positive definite and conjugate-symmetric (skew-symmetric). The corresponding bilinear form in the more general setting need not be positive definite, symmetric, or conjugate-symmetric, although we will see echos of these properties in our situation.

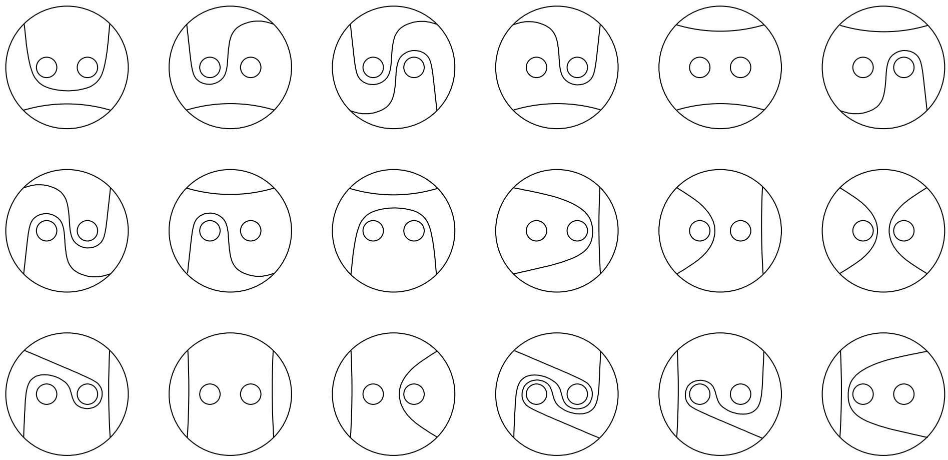


Figure 2. List of diagrams in $\boldsymbol{B}^2 = \boldsymbol{B}_2^2$.

A given diagram in F_k^n partitions F_k into $n+1$ regions. Two diagrams are equivalent if and only if they have the same underlying Catalan state and the labeled holes are distributed in the same manner across regions. Accordingly, $\boldsymbol{B}_k^n$ has elements $(n+1)^{k-1}\binom{2n}{n}$. See Figure 2 for the 18 diagrams in $\boldsymbol{B}_2^2$.

We remark that if $k = 0$ or $k = 1$, two diagrams are equivalent only if they are homotopic, but for $k > 2$, this need not be true; see Figure 3 for a counterexample.

The study of noncrossing partitions of n points has a long history in enumerative combinatorics. Beyond purely combinatorial questions, noncrossing partitions arise in the study of a number of problems lying at the intersection of combinatorics and topology. Lickorish examines the matrix of a bilinear form defined on noncrossing planar diagrams in a disk, motivated by the theory of 3-manifold invariants. Motivated by the work of Birkhoff and Lewis [1946] on the four color conjecture, Tutte [1991] introduced the matrix of chromatic joins.

In this paper, we define a pairing over $\boldsymbol{B}_k^n$ and investigate the Gram matrix of the pairing. This concept is a generalization of a problem posed by W. B. R. Lickorish [1991; 1997] for type A Gram determinants — those based on a disk, i.e., $k = 0$ —

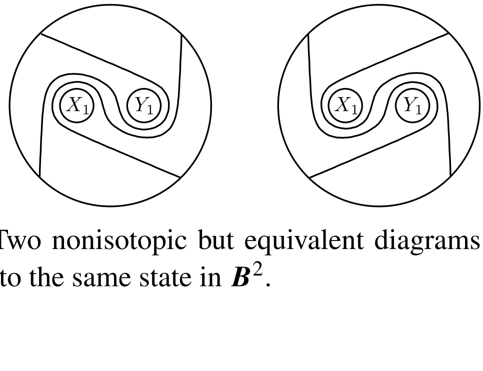


Figure 3. Two nonisotopic but equivalent diagrams in F_2^2; they correspond to the same state in $\boldsymbol{B}^2$.

and by Rodica Simion for type B Gram determinants [Schmidt 2004; Simion 2000] (these are related to $k = 1$ and the Kauffman bracket skein module of an annulus; see [Przytycki 1999]). Simion was motivated by Tutte's work [1991; 1993] on chromatic joins; see also [Chen and Przytycki 2008].

Significant research has been completed for the Gram determinants for type A and B. In particular, Di Francesco [1998] and Westbury [1995] gave a closed formula for the type A Gram determinant; a complete factorization of the type B Gram determinant was conjectured by Gefry Barad, and a closed formula (quoted in Theorem 7.4) was proved by Martin and Saleur [1993] and by Chen and Przytycki [2009]. The type A Gram determinant was used by Lickorish to find an elementary construction of Reshetikhin–Turaev–Witten invariants of oriented closed 3-manifolds.

We specifically investigate the Gram determinant G_n of the bilinear form defined over $\boldsymbol{B}_2^n$ and prove that $\det G_{n-1}$ divides $\det G_n$ for $n > 1$. Furthermore, we investigate the diagonal entries of G_n and give a method for computing terms of maximal degree in $\det G_n$. We conclude the paper by briefly discussing generalizations of the Gram determinant and presenting some open questions.

2. Definitions and basic facts for $\boldsymbol{B}_2^n$

Consider F_2^n, a unit disk with two holes, along with $2n$ points along the outer boundary. Denote the holes in F_2^n by X_1 and Y_1. To differentiate between them, we will always place X_1 to the left and Y_1 to the right if labels are not present.

Let

$$\boldsymbol{B}^n := \boldsymbol{B}_2^n =: \left\{b_1, \dots, b_{(n+1)\binom{2n}{n}}\right\}$$

be the set of all possible diagrams with n noncrossing chords connecting these $2n$ points, up to equivalence in F_2^n.

Recall that in complex analysis an *inversion* (in the unit circle) is the involution defined on the sphere $S^2 = \mathbb{C} \cup \infty$ by $z \leftrightarrow z/|z|^2$. Let X_2 and Y_2 be the inversions of X_1 and Y_1, respectively, and let $\mathcal{S} = \{X_1, X_2, Y_1, Y_2\}$. Given $b_i \in \boldsymbol{B}^n$, let b_i^* denote the inversion of b_i. Given $b_i, b_j \in \boldsymbol{B}^n$, we glue b_i with b_j^* along the outer boundary, respecting the labels of the marked points. Since b_i and b_j each contains n noncrossing chords, $b_i \circ b_j^*$ can have at most n closed curves. The resulting diagram, denoted by $b_i \circ b_j^*$, is a set of up to n closed curves in the 2-sphere $S^2 = D^2 \cup (D^2)^*$ with four holes, X_1, X_2, Y_1, Y_2. (Since we glued along ∂D^2, it is no longer a boundary.) Each closed curve partitions the set $\mathcal{S}$ into two sets. Two closed curves are of the same type if they partition $\mathcal{S}$ the same way.

We define a pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ over $\boldsymbol{B}^n$ by associating with $b_i, b_j \in \boldsymbol{B}^n$ a monomial in the variables $d, x_1, x_2, y_1, y_2, z_1, z_2, z_3$, as follows. The exponent of each variable is obtained by counting the number of curves in $b_i \circ b_j^*$ that partition the set $\{X_1, X_2, Y_1, Y_2\}$ in the corresponding way, the correspondence being this:

$x_1:$	$\{X_1\}, \{X_2, Y_1, Y_2\}$	$z_1:$	$\{X_1, X_2\}, \{Y_1, Y_2\}$
$x_2:$	$\{X_2\}, \{X_1, Y_1, Y_2\}$	$z_2:$	$\{X_1, Y_1\}, \{X_2, Y_2\}$
$y_1:$	$\{Y_1\}, \{X_1, X_2, Y_2\}$	$z_3:$	$\{X_1, Y_2\}, \{X_2, Y_1\}$
$y_2:$	$\{Y_2\}, \{X_1, X_2, Y_1\}$	$d:$	$\varnothing, \{X_1, X_2, Y_1, Y_2\}$

Table 1. Indeterminates and partitions. In the monomial $\langle b_i, b_j\rangle$, the exponent of each variable is the number of curves in $b_i \circ b_j^*$ that partition the set $\{X_1, X_2, Y_1, Y_2\}$ in the given way.

Thus $\langle b_i, b_j\rangle$ is a monomial of degree at most n. Some example paired diagrams, with their corresponding monomials, are given in Figure 5.

We can now form the Gram matrix $G_n = [g_{ij}] = [\langle b_i, b_j\rangle]_{1\le i,j\le (n+1)\binom{2n}{n}}$ of this pairing. We write it explicitly for $n = 1$. Order the elements of $\boldsymbol{B}^1$ as in the first

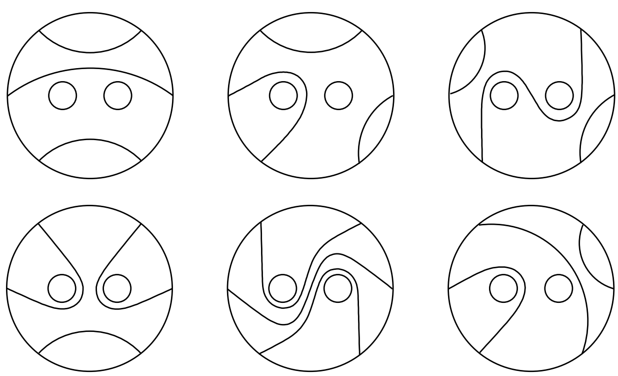
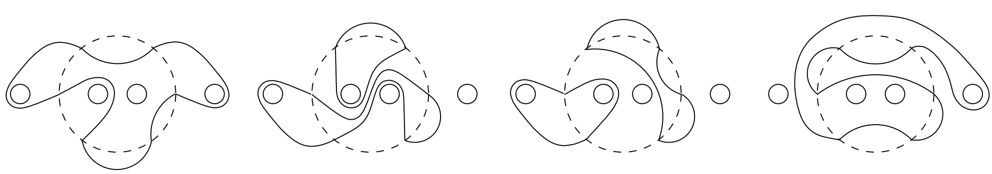


Figure 4. Diagrams of six states $b_1, b_2, b_3, b_4, b_5, b_6 \subset \boldsymbol{B}^3$. The indices are used in the examples, but are not intrinsic.



$\langle b_2, b_4\rangle = x_1$ $\qquad$ $\langle b_5, b_2\rangle = x_1x_2$ $\qquad$ $\langle b_6, b_2\rangle = dz_1$ $\qquad$ $\langle b_1, b_3\rangle = x_2$

Figure 5. Diagrams for $b_i \circ b_j$ on S^2 and the corresponding values of $\langle b_i, b_j\rangle$. Indices are as in Figure 4.

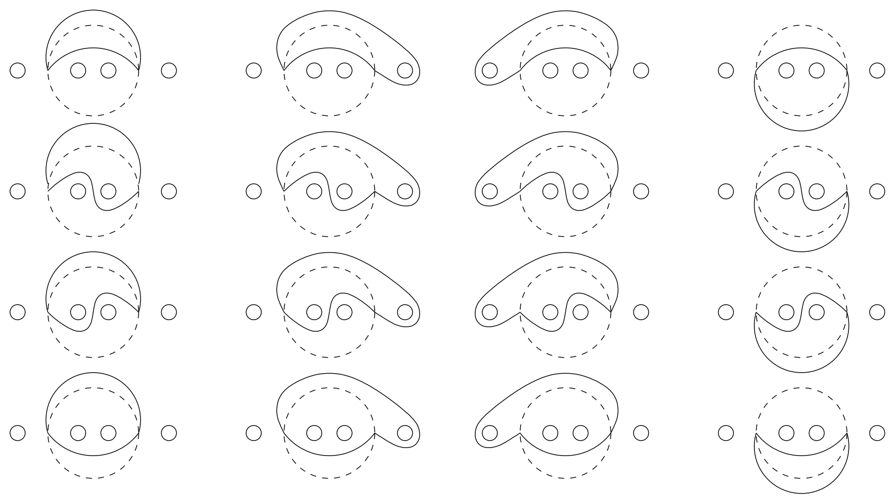


Figure 6. Array of $b_i \circ b_j^*$ for $b_i, b_j \in \boldsymbol{B}^1$.

column of Figure 6 (we're looking at the disk inside the dotted circle). Then we see from the array of diagrams in Figure 6, each of which represents one pair (b_i, b_j), that the Gram matrix of the pairing is

$$G_1 = \begin{bmatrix} d & y_2 & x_2 & z_2 \\ y_1 & z_1 & z_3 & x_1 \\ x_1 & z_3 & z_1 & y_1 \\ z_2 & x_2 & y_2 & d \end{bmatrix}.$$

Therefore the Gram determinant is

$$\begin{aligned}\det G_1 = (dz_1 - z_1z_2 - dz_3 + z_2z_3 - x_1x_2 + x_2y_1 + x_1y_2 - y_1y_2)\\ (dz_1 + z_1z_2 + dz_3 + z_2z_3 - x_1x_2 + x_2y_1 - x_1y_2 + y_1y_2).\end{aligned}$$

This paper is mostly devoted to exploring possible factorizations of $\det G_n$, and is the first step toward computing $\det G_n$ in full generality, which we conjecture to have a nice decomposition.

Though the pairing (and hence the Gram matrix) is not symmetric, it is skew-symmetric with respect to an certain involution of the ring R. (An involution is isomorphism equal to its own inverse.) Specifically, given $b_i, b_j \in \boldsymbol{B}^n$, we can obtain $b_j \circ b_i^*$ from $b_i \circ b_j^*$ by inversion in the unit circle, which interchanges X_1 with X_2 and Y_1 with Y_2. Consequently, $\langle b_j, b_i\rangle$ can be obtained from $\langle b_i, b_j\rangle$ by interchanging x_1 with x_2 and y_1 with y_2, as these interchanges have the same effect in the corresponding partition (see Table 1) as the hole interchange $X_1 \leftrightarrow X_2$, $Y_1 \leftrightarrow Y_2$. Note that z_1, z_2, z_3, and d are mapped to themselves under this variable

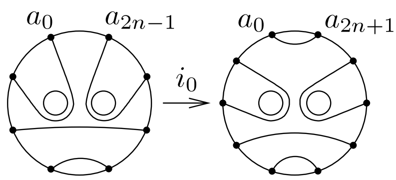


Figure 7. Action of the embedding i_0. Note the relabeling of the boundary points: each a_k on the left becomes a_{k+1} on the right, and the two new points are labeled a_0 and a_{2n+1}.

swap, because their partitions are invariant under the interchange of holes. (See also Theorem 3.3(4) below.)

To summarize, let h_t be the involution of G_n that interchanges x_1 with x_2 and y_1 with y_2. Then

$$\langle b_i, b_j \rangle = h_t(\langle b_j, b_i \rangle),$$

and the transpose of G_n is given by applying h_t to each individual entry of G_n.

Embedding $\boldsymbol{B}^n$ in $\boldsymbol{B}^{n+1}$. Let $i_0 : \boldsymbol{B}^n \to \boldsymbol{B}^{n+1}$ be the *embedding* (injection) defined as follows: for $b_i \in \boldsymbol{B}^n$, the image $i_0(b_i) \in \boldsymbol{B}^{n+1}$ is given by adding to b_i a noncrossing chord close to the outer boundary and joining two points between a_0 and a_{2n-1}, as suggested in Figure 7. The two new points on the edge become the new a_0 and a_{2n+1}, and each of the old points a_k becomes a_{k+1}. This relabeling explicitly makes $i_0(b_i)$ an element of $\boldsymbol{B}^{n+1}$.

Another embedding we will need, denoted by $i_1 : \boldsymbol{B}^n \to \boldsymbol{B}^{n+1}$ and illustrated in Figure 8, is defined by a construction similar to that of i_0, but this time the added chord joins two points between the old a_0 and a_1, rather than between a_0 and a_{2n-1}. These two new points become a_0 and a_1, while the old a_0 becomes a_{2n+1} and each a_k, for $1 < k < 2n$, becomes a_{k+1}.

More formally, we define i_1 in terms of i_0 by using the notion of a *Dehn twist*, borrowed from surface topology and knot theory. Fix an annulus in the complex plane — the region between two concentric circles, say $R' \le |z| \le 1$. Imagine keeping the inner boundary circle fixed, while the outer one is rotated clockwise by an angle α. The stuff in between also gets rotated, by an amount that depends on how far it is from each circle. The resulting homeomorphism of the annulus is

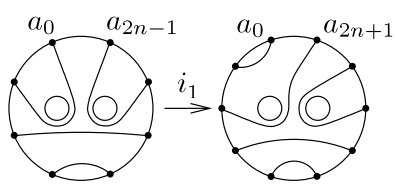


Figure 8. Action of the embedding i_1.

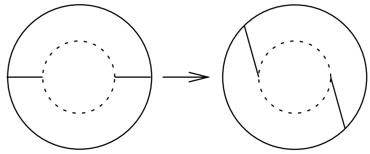


Figure 9. A Dehn twist r_α, with $\alpha = \pi/4$.

called a *Dehn twist* through an angle α. As an explicit formula we can take

$$r_\alpha(z) = z \exp\left(i\alpha \, \frac{|z| - R'}{1 - R'} \right),$$

which says the amount of rotation experienced by a point is proportional to the distance to the inner circle, growing from 0 at $|z| = R'$ to the full angle α at $|z| = 1$. Figure 9 gives a qualitative picture in the case $\alpha = \pi/4$.

Now we get back to the disk with two holes, F_2^n. If we choose R' close enough to 1 that the holes X_1 and Y_1 lie within the circle of radius R', we can extend r_α to a homeomorphism of F_2^n by setting $r_\alpha(z) = z$ for $|z| \le R'$.

Moreover, if $\alpha = \pi/n$, then r_α takes each of the $2n$ marked points a_k on the edge of F_2^n to the next such point a_{k+1}; consequently, it takes a system of noncrossing curves in F_2^n to another such. This defines the action of $r_{\pi/n}$ on $\boldsymbol{B}^n$; it is a permutation because the inverse of a Dehn twist is also a Dehn twist through the opposite angle.[2] The first arrow in Figure 10 illustrates the action of ${r_{\pi/4}}^{-1}$ on a certain element of $\boldsymbol{B}^4$, and the last arrow shows the action of $r_{\pi/5}$ on an element of $\boldsymbol{B}^5$.

We can now express i_1 in terms of i_0 and Dehn twists:

$$i_1 = r_{\pi/(n+1)} \circ i_0 \circ {r_{\pi/n}}^{-1}.$$

This is illustrated in Figure 10. Note that the two Dehn twists are not quite inverse to each other, since their angles differ.

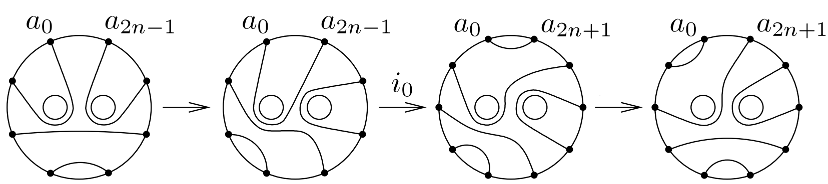


Figure 10. The embedding i_1, illustrated in Figure 8, is obtained from i_0 (Figure 7) by composing with appropriate Dehn twists.

[2]Obviously the repeated application of k Dehn twists through α is a Dehn twist by $k\alpha$, so any $r_{k\pi/n}$ also induces an action on $\boldsymbol{B}^n$. Note that the Dehn twist by a full 2π, though it is not the identity homeomorphism, gives the identity map on $\boldsymbol{B}^n$; an example of its action was shown in Figure 3.

3. More properties of the Gram determinant

Theorem 3.1. $\det G_n \neq 0$ *for all integers* $n \geq 1$.

Lemma 3.2. $\langle b_i, b_j \rangle$ *is a monomial of maximal degree if and only if* $\gamma(b_i) = \gamma(b_j)$.

Proof. Recall that $\langle b_i, b_j \rangle$ has maximal degree if and only if $b_i \circ b_j^*$ has n closed curves; this in turn is equivalent to having each closed curve made of exactly two arcs, one in b_i and one in b_j^*. In this situation, any two points connected by a chord in b_i must also be connected by a chord in b_j, so $\gamma(b_i) = \gamma(b_j)$. □

Proof of Theorem 3.1. Assume $\langle b_i, b_j \rangle$ is a monomial of maximal degree consisting only of the variables d and z_1. Because $\gamma(b_i) = \gamma(b_j)$ by Lemma 3.2, it follows that any two points connected in b_i are also connected in b_j. Each connection in b_i can be drawn in four different ways with respect to X and Y, since there are two ways to position the chord relative to each hole. Because $\langle b_i, b_j \rangle$ is assumed to consist only of the variables d and z_1, it follows that each pair of arcs that form a closed curve in $b_i \circ b_j^*$ either separates $\{X_1, X_2\}$ from $\{Y_1, Y_2\}$ or has $\{X_1, X_2, Y_1, Y_2\}$ on the same side of the curve. One can check each of the four cases to see that this condition implies that any two arcs that form a closed curve in $b_i \circ b_j^*$ must be equal, so $b_i = b_j$. Using Laplacian expansion, this implies that the product of the diagonal of G_n is the unique summand of degree $n(n+1)\binom{2n}{n}$ in $\det G_n$ consisting only of the variables d and z_1. □

We need the following notation for the next theorem: let $f : \alpha_1 \leftrightarrow \alpha_2$ denote a function f which acts on the entries of G_n by interchanging variables α_1 with α_2. We can extend the domain of f to G_n. Let $f(G_n)$ denote the matrix formed by applying f to all the individual entries of G_n.

Define involutions h_1, h_2, h_3, h_t acting on the entries of G_n as follows:

$$\begin{array}{lll} h_1 : & x_1 \leftrightarrow y_1 & z_1 \leftrightarrow z_3 \\ h_2 : & x_2 \leftrightarrow y_2 & z_1 \leftrightarrow z_3 \\ h_3 = h_1 h_2 : & x_1 \leftrightarrow y_1 & x_2 \leftrightarrow y_2 \\ h_t : & x_1 \leftrightarrow x_2 & y_1 \leftrightarrow y_2 \end{array}$$

Theorem 3.3.

(1) $\det h_1(G_1) = -\det G_1$, *and for* $n > 1$, $\det h_1(G_n) = \det G_n$.

(2) $\det h_2(G_1) = -\det G_1$, *and for* $n > 1$, $\det h_2(G_n) = \det G_n$.

(3) $\det h_3(G_n) = \det G_n$.

(4) $\det h_t(G_n) = \det G_n$.

Proof. For assertion (1), note that $h_1(G_n)$ corresponds to exchanging the positions of the holes X_1 and Y_1 for all $b_i \in \boldsymbol{B}^n$. b_j^* is unchanged, so h_1 can be realized by a permutation of rows. For states where X_1 and Y_1 lie in the same region, their corresponding rows are unchanged by h_1. The number of such states is given by $|\boldsymbol{B}^n|/(n+1)$. Thus, the total number of row transpositions is equal to

$$\frac{1}{2}\Big(|\boldsymbol{B}^n|-\frac{|\boldsymbol{B}^n|}{n+1}\Big)=\frac{n}{2}\binom{2n}{n}=\frac{n(n+1)}{2}C_n.$$

It is known that C_n is odd if and only if $n=2^m-1$ for some m; see for instance [Deutsch and Sagan 2006]. Hence, C_n being odd implies that

$$\frac{n(n+1)}{2}=\frac{2^m(2^m-1)}{2}=2^{m-1}(2^m-1),$$

which is even for all $m>1$. Thus, $h_1(G_n)$ can be obtained from G_n by an even permutation of rows for $n>1$, so $\det h_1(G_n)=\det G_n$. Similarly, $h_1(G_1)$ is given by an odd number of row transpositions on G_1, so $\det h_1(G_1)=-\det G_1$.

Assertion (2) can be shown using the same argument, except that h_2 corresponds to interchanging the positions of the holes X_2 and Y_2, rather than X_1 and Y_1.

Since $h_3=h_1h_2$, it follows immediately that $\det h_3(G_n)=\det G_n$ for $n>1$. The sum of two odd permutations is even, so the equality also holds for $n=1$, which proves (3). Assertion (4) follows because $\det h_t(G_n)=\det {}^tG_n=\det G_n$. □

Theorem 3.4. $\det G_n$ *is preserved under the following involutions*:

$$\begin{array}{lllll}
g_1: & x_1\leftrightarrow -x_1 & x_2\leftrightarrow -x_2 & z_2\leftrightarrow -z_2 & z_3\leftrightarrow -z_3\\
g_2: & y_1\leftrightarrow -y_1 & y_2\leftrightarrow -y_2 & z_2\leftrightarrow -z_2 & z_3\leftrightarrow -z_3\\
g_3: & x_1\leftrightarrow -x_1 & y_2\leftrightarrow -y_2 & z_1\leftrightarrow -z_1 & z_2\leftrightarrow -z_2\\
g_1g_2: & x_1\leftrightarrow -x_1 & x_2\leftrightarrow -x_2 & y_1\leftrightarrow -y_1 & y_2\leftrightarrow -y_2\\
g_1g_3: & x_2\leftrightarrow -x_2 & y_2\leftrightarrow -y_2 & z_1\leftrightarrow -z_1 & z_3\leftrightarrow -z_3\\
g_2g_3: & x_1\leftrightarrow -x_1 & y_1\leftrightarrow -y_1 & z_1\leftrightarrow -z_1 & z_3\leftrightarrow -z_3\\
g_1g_2g_3: & x_2\leftrightarrow -x_2 & y_1\leftrightarrow -y_1 & z_1\leftrightarrow -z_1 & z_2\leftrightarrow -z_2
\end{array}$$

Proof. We first show that g_1 can be realized by conjugating the matrix G_n by a diagonal matrix P_n of all diagonal entries equal to ± 1. Define the diagonal entries of P_n by

$$p_{ii}=(-1)^{q(b_i,F_x)},$$

where $q(b_i,F_x)$ is the number of times b_i intersects F_x modulo 2; see Figure 11, where F_x, F_x^*, F_y, F_y^* and $\tilde{F}_x$ are defined. F_x and F_y touch the unit circle between a_0 and a_{2n-1}.

This proves the result about g_1, because curves corresponding to the variables x_1, x_2, z_2 and z_3 intersect $F_x\cup F_x{}^*$ in an odd number of points, whereas curves corresponding to the variables d, z_2, y_1 and y_2 cut it an even number of times. More precisely, for

$$g_{ij}=\langle b_i,b_j\rangle=d^{n_d}{x_1}^{n_{x_1}}{x_2}^{n_{x_2}}{y_1}^{n_{y_1}}{y_2}^{n_{y_2}}{z_1}^{n_{z_1}}{z_2}^{n_{z_2}}{z_3}^{n_{z_3}},$$

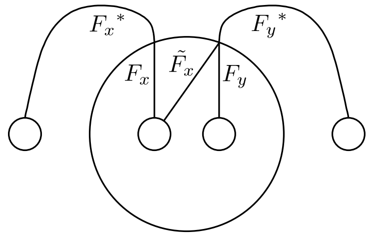


Figure 11. Toward the proof of Theorem 3.4(1).

the entry g'_{ij} of $P_n G_n P_n{}^{-1}$ satisfies

$$\begin{aligned} g'_{ij} &= p_{ii} g_{ij} p_{jj} = p_{ii} p_{jj} g_{ij} = (-1)^{q(b_i,F_x)+q(b_j,F_x)} g_{ij} = (-1)^{n_{x_1}+n_{x_2}+n_{z_2}+n_{z_3}} g_{ij} \\ &= d^{n_d} (-x_1)^{n_{x_1}} (-x_2)^{n_{x_2}} y_1{}^{n_{y_1}} y_2{}^{n_{y_2}} z_1{}^{n_{z_1}} (-z_2)^{n_{z_2}} (-z_3)^{n_{z_3}}. \end{aligned}$$

The results about g_2 and g_3 follow by the same argument, but using F_y and $F_y \cup F_y{}^*$ for g_2 and $\tilde{F}_x$ and $\tilde{F}_x \cup F_y{}^*$ for g_3. The statements about compositions follow directly from the first three. □

4. Terms of maximal degree in det G_n

Theorem 3.1 proves that the product of the diagonal entries of G_n is the unique term of maximal degree, $n(n+1)\binom{2n}{n}$, in $\det G_n$ consisting only of the variables d and z_1. More precisely, the product of the diagonal of G_n is given by

$$\delta(n) = \prod_{b_i \in \boldsymbol{B}^n} \langle b_i, b_i \rangle = d^{\alpha(n)} z_1^{\beta(n)},$$

with $\alpha(n) + \beta(n) = n(n+1)\binom{2n}{n}$. The value of $\delta(n)$ for the first few n are

$$\delta(1) = d^2 z_1^2, \quad \delta(2) = d^{20} z_1^{16}, \quad \delta(3) = d^{144} z_1^{96}, \quad \delta(4) = d^{888} z_1^{512}.$$

Computing the general formula for $\delta(n)$ can be reduced to a purely combinatorial problem. We conjectured that $\beta(n) = (2n)4^{n-1}$ and this was proven by Louis Shapiro (personal communication, 2008) using an involved generating function argument. The result is stated formally below.

Theorem 4.1. $\delta(n) = d^{n(n+1)\binom{2n}{n} - (2n)4^{n-1}} z_1^{(2n)4^{n-1}}$.

Let $h(\det G_n)$ denote the truncation of $\det G_n$ to terms of maximal degree, that is, of degree $n(n+1)\binom{2n}{n}$. Each term is a product of $(n+1)\binom{2n}{n}$ entries in G_n, each of which is a monomial of degree n. By Lemma 3.2, $\langle b_i, b_j \rangle$ has degree n if and only if b_i and b_j have the same underlying Catalan state. Divide $\boldsymbol{B}^n$ into subsets corresponding to underlying Catalan states, that is, into subsets $A_1, \ldots, A_{C_n}$, such that for all $b_i, b_j \in A_k$, $\gamma(b_i) = \gamma(b_j)$. Then from Lemma 3.2 we have:

Proposition 4.2. *For $1 \le k \le C_n$, let I_k be the set of indices such that $A_k = \{b_i\}_{i \in I_k}$, and let $\langle A_k, A_k \rangle$ be the submatrix of G_n whose rows and columns are indexed by I_k. Then*

$$h(\det G_n) = \prod_{k=1}^{C_n} \det\langle A_k, A_k \rangle.$$

Note that the $\langle A_k, A_k \rangle$ are simply blocks in G_n, and their determinants can be multiplied together to give the highest terms in $\det G_n$. Finding the terms of maximal degree in $\det G_n$ can give insight into the decomposition of $\det G_n$ for large n.

Example 4.3. $\boldsymbol{B}^1$ corresponds to the single Catalan state in $\boldsymbol{B}_0^1$. Thus, $\det G_1 = h(\det G_1)$, a homogeneous polynomial of degree 4 (given on page 154).

Example 4.4. We can divide $\boldsymbol{B}^2$ into two sets, corresponding to the two Catalan states in $\boldsymbol{B}_0^2$. Thus $h(\det G_2)$ can be found by computing two 9×9 block determinants. The two Catalan states in $\boldsymbol{B}_0^2$ are equivalent up to rotation, so the two block determinants are equal. Specifically, we have:

$$\begin{aligned}
h(\det G_2) = d^6 &(x_1x_2 + x_2y_1 + x_1y_2 + y_1y_2 - dz_1 - z_1z_2 - dz_3 - z_2z_3)^4 \\
&(-x_1x_2 + x_2y_1 + x_1y_2 - y_1y_2 + dz_1 - z_1z_2 - dz_3 + z_2z_3)^4 \\
&(-x_1x_2z_1 - y_1y_2z_1 + dz_1{}^2 + x_2y_1z_3 + x_1y_2z_3 - dz_3{}^2)^2 \\
&(-2x_1x_2y_1y_2 + dx_1x_2z_1 + dy_1y_2z_1 - d^2z_1{}^2 + dx_2y_1z_3 + dx_1y_2z_3 - d^2z_3{}^2)^2 \\
= d^6 &\det G_1{}^4(-x_1x_2z_1 - y_1y_2z_1 + dz_1{}^2 + x_2y_1z_3 + x_1y_2z_3 - dz_3{}^2)^2 \\
&(-2x_1x_2y_1y_2 + dx_1x_2z_1 + dy_1y_2z_1 - d^2z_1{}^2 + dx_2y_1z_3 + dx_1y_2z_3 - d^2z_3{}^2)^2.
\end{aligned}$$

Example 4.5. $\boldsymbol{B}^3$ can be divided into five subsets, corresponding to the five Catalan states in $\boldsymbol{B}_0^3$. We can thus find $h(\det G_3)$ by computing the determinants of five blocks in $\boldsymbol{B}^3$. The determinant of each block gives a homogeneous polynomial of degree $240/5 = 48$. $\boldsymbol{B}_0^3$ forms two equivalence classes up to rotation, so there are only two unique block determinants. The result is

$$\begin{aligned}
&h(\det G_3) \\
&= h(\det G_2)^6 \det G_1{}^{-9} d^{30} w^3 \bar{w}^3 \\
&= d^{66}(-x_1x_2 + x_2y_1 + x_1y_2 - y_1y_2 + dz_1 - z_1z_2 - dz_3 + z_2z_3)^{15} \\
&\quad (-x_1x_2 - x_2y_1 - x_1y_2 - y_1y_2 + dz_1 + z_1z_2 + dz_3 + z_2z_3)^{15} \\
&\quad (-x_1x_2z_1 - y_1y_2z_1 + dz_1^2 + x_2y_1z_3 + x_1y_2z_3 - dz_3{}^2)^{12} \\
&\quad (2x_1x_2y_1y_2 - dx_1x_2z_1 - dy_1y_2z_1 + d^2z_1{}^2 - dx_2y_1z_3 - dx_1y_2z_3 + d^2z_3{}^2)^{12} \\
&\quad (x_1x_2y_1y_2z_1 - dx_1x_2z_1{}^2 - dy_1y_2z_1{}^2 + d^2z_1{}^3 - x_1x_2y_1y_2z_3 + dx_2y_1z_3{}^2 + dx_1y_2z_3{}^2 - d^2z_3{}^3)^3 \\
&\quad (x_1x_2y_1y_2z_1 - dx_1x_2z_1^2 - dy_1y_2z_1^2 + d^2z_1{}^3 + x_1x_2y_1y_2z_3 - dx_2y_1z_3{}^2 - dx_1y_2z_3{}^2 + d^2z_3{}^3)^3.
\end{aligned}$$

5. det G_{n-1} divides det G_n

We defined in Section 2 the embeddings $i_0, i_1 : \boldsymbol{B}^n \to \boldsymbol{B}^{n+1}$. We now introduce inverses of sort for these two maps.

Given $b_i \in \boldsymbol{B}^n$, imagine adding to b_i a noncrossing chord connecting a_0 and a_{2n-1} outside the circle, and then pushing this chord inside the circle, together with the points a_0 and a_{2n-1}; see Figure 12. With the removal of these two points from the boundary, we relabel the remaining ones so the old a_k becomes a_{k-1}, for $0 < k < 2n-1$. So now there are $2n-2$ marked points on the boundary; this establishes a projection $\boldsymbol{B}^n \to \boldsymbol{B}^{n-1}$, with one caveat soon to be discussed. We denote this projection by p_0.

The procedure we've described works fine so long as b_i does not include a chord joining a_0 and a_{2n-1}. Indeed, if a_0 and a_{2n-1} are connected respectively to a_j and a_k in b_i, the added exterior chord ends up, in $p_0(b_i)$, as part of a chord joining a_{j-1} to a_{k-1} (see again Figure 12). However, a problem arises when b_i has a chord from a_0 to a_{2n-1}. In this case, the procedure creates a closed curve inside the disc, coming from the two chords joining the old a_0 to a_{2n-1}, one internal and one external. One could imagine erasing this loop to obtain an element of $\boldsymbol{B}^{n-1}$, but the loop carries information — it may enclose an arbitrary subset of $\{X_1, Y_1\}$. So we keep it at present, and we make p_0 take values in the set $\overline{\boldsymbol{B}}^{n-1}$ of equivalence classes of diagrams in F_2^{n-1} consisting of $n-1$ chords joining marked points on the boundary together with an optional closed loop disjoint from the boundary.

These observations can be summarized as follows:

Lemma 5.1. *An element $b_i \in \boldsymbol{B}^n$ is taken under $p_0 : \boldsymbol{B}^n \to \overline{\boldsymbol{B}}^{n-1}$ to an element of $\boldsymbol{B}^{n-1}$ if and only if b_i contains no chord connecting a_0 and a_{2n-1}.*

A bit of experimentation will persuade the reader of the correctness of the next result — which, incidentally, justifies our decision to expand the range of p_0 to include diagrams with a loop.

Proposition 5.2. *For any $b_i \in \boldsymbol{B}^n$ and $b_j \in \boldsymbol{B}^{n-1}$, we have*

$$b_i \circ i_0(b_j)^* = p_0(b_i) \circ b_j^*,$$

where the equivalence relation implicit in this equality consists of isotopies of the four-holed sphere, not necessarily preserving the unit disk.

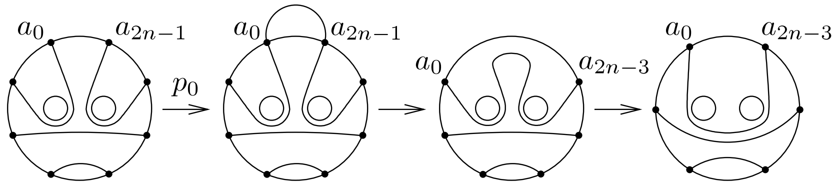


Figure 12. Action of the projection p_0.

We're gearing up toward a demonstration that the Gram determinant for $n-1$ chords divides the Gram determinant for n chords. We need one more lemma.

Lemma 5.3. *Fix $b_i \in \boldsymbol{B}^n$. There exists an element $b_{\alpha(i)} \in \boldsymbol{B}^{n-1}$ and a monomial $q \in \{1, d, x_1, y_1, z_2\}$ such that*

$$\langle p_0(b_i), b_j\rangle = q\,\langle b_{\alpha(i)}, b_j\rangle \quad \textit{for all } b_j \in \boldsymbol{B}^{n-1}.$$

Proof. If $p_0(b_i) \in \boldsymbol{B}^{n-1}$ we can take $b_{\alpha(i)} = p_0(b_i)$ and $q = 1$. Otherwise, it follows from Lemma 5.1 that b_i contains a chord connecting a_0 and a_{2n-1}, and $p_0(b_i)$ is the union of some $b_{\alpha(i)} \in \boldsymbol{B}^{n-1}$ with a loop enclosing a subset of $\{X_1, Y_1\}$. Let q be the variable corresponding to the partition of the holes effected by extra loop, according to Table 1. Then $\langle p_0(b_{\alpha(i)}), b_j\rangle = q\,\langle b_{\alpha(i)}, b_j\rangle$ for any b_j. □

For the remainder of the paper we adopt the following notation: if B and B' are subsets of $\boldsymbol{B}^n$, let

$$\langle B, B'\rangle := \big[\langle b_i, b_j\rangle\big]_{\substack{i\,:\,b_i\in B\\ j\,:\,b_j\in B'}}$$

be the submatrix of G_n whose rows correspond to the elements of B and whose columns correspond to the elements of B'.

Theorem 5.4. *For $n > 1$, $\det G_{n-1}$ divides $\det G_n$.*

Proof. We use the easily checked equality (also proved in detail as Lemma 6.1)

$$\langle i_0(b_i), i_1(b_j)\rangle = \langle i_1(b_i), i_0(b_j)\rangle = \langle b_i, b_j\rangle \quad \text{for all } b_i, b_j \in \boldsymbol{B}^{n-1}.$$

In the notation defined before the theorem, this means that $\langle b_i, \boldsymbol{B}^{n-1}\rangle$ (the i-th row of G_{n-1}) coincides with the row in the submatrix $\langle \boldsymbol{B}^n, i_0(\boldsymbol{B}^{n-1})\rangle$ of G_n given by $\langle i_1(b_i), i_0(\boldsymbol{B}^{n-1})\rangle$.

Reorder the elements of $\boldsymbol{B}^n$ so that $\langle i_0(\boldsymbol{B}^{n-1}), i_0(\boldsymbol{B}^{n-1})\rangle$ forms the upper left block of G_n and $\langle i_1(\boldsymbol{B}^{n-1}), i_0(\boldsymbol{B}^{n-1})\rangle$ forms a block directly underneath it:

$$G_n = \begin{bmatrix} \langle i_0(\boldsymbol{B}^{n-1}), i_0(\boldsymbol{B}^{n-1})\rangle & * & * & * & * & * \\ \langle i_1(\boldsymbol{B}^{n-1}), i_0(\boldsymbol{B}^{n-1})\rangle & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} = \begin{bmatrix} \langle i_0(\boldsymbol{B}^{n-1}), i_0(\boldsymbol{B}^{n-1})\rangle & * & * & * & * & * \\ G_{n-1} & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}.$$

Lemma 5.3 implies that every row of $\langle \boldsymbol{B}^n, i_0(\boldsymbol{B}^{n-1})\rangle$ is a multiple of some row in G_{n-1}. Let $j_1, \dots, j_k$ denote the indices of all rows of $\langle \boldsymbol{B}^n, i_0(\boldsymbol{B}^{n-1})\rangle$ other than those in $\langle i_1(\boldsymbol{B}^{n-1}), i_0(\boldsymbol{B}^{n-1})\rangle$. Let G'_n be the matrix obtained by properly subtracting multiples of rows in $\langle i_1(\boldsymbol{B}^{n-1}), i_0(\boldsymbol{B}^{n-1})\rangle$ from rows $j_1, \dots, j_k$ of G_n

so that the submatrix obtained by restricting G'_n to rows $j_1, \ldots, j_k$ and columns corresponding to states in $i_0(\boldsymbol{B}^{n-1})$ is equal to 0:

$$G'_n = \begin{bmatrix} 0 & * & * & * & * & * \\ G_{n-1} & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix}.$$

Thus, G'_n restricted to the columns corresponding to states in $i_0(\boldsymbol{B}^{n-1})$ contains precisely $n\binom{2n-2}{n-1}$ nonzero rows, each equal to some unique row of G_{n-1}. The determinant of this submatrix is equal to $\det G_{n-1}$. Since $\det G_{n-1}$ divides $\det G'_n$ and $\det G'_n = \det G_n$, this completes the proof. □

6. Further relations between $\det G_{n-1}$ and $\det G_n$

As noted in the previous proof, there is a submatrix of G_n equal to G_{n-1}. We will now focus on identifying multiple nonoverlapping submatrices in G_n equal to multiples of G_{n-1}. This will help in simplifying the computation of $\det G_n$. We start with a detailed justification of the first assertion in the proof of Theorem 5.4:

Lemma 6.1. *For any $b_i, b_j \in \boldsymbol{B}^{n-1}$, $\langle i_0(b_i), i_1(b_j)\rangle = \langle i_1(b_i), i_0(b_j)\rangle = \langle b_i, b_j\rangle$.*

Proof. We begin with the equality $\langle i_1(b_i), i_0(b_j)\rangle = \langle b_i, b_j\rangle$. By Proposition 5.2, $i_1(b_i) \circ i_0(b_j)^* = p_0 i_1(b_i) \circ b_j^*$, so it suffices to prove that

$$p_0 i_1(b_i) = p_0 r_{\pi/n} i_0 {r_{\pi/(n-1)}}^{-1}(b_i) = b_i.$$

This is demonstrated pictorially in Figure 13.

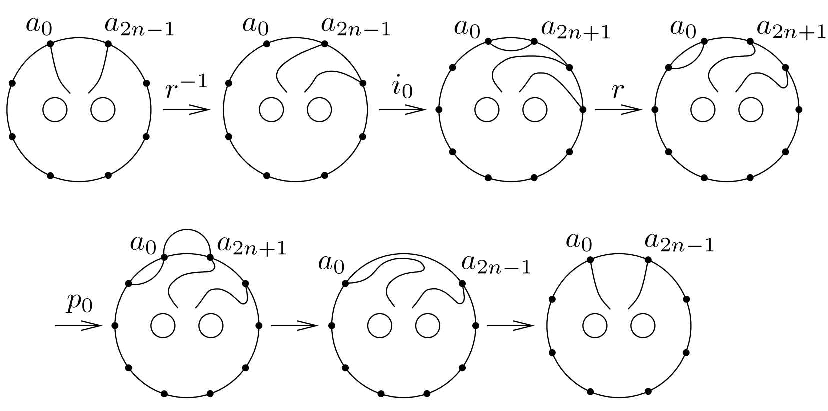


Figure 13. Proof that $p_0 \circ i_1$ is the identity.

Thus, $\langle i_1(b_i), i_0(b_j)\rangle = \langle b_i, b_j\rangle$. Recall that $\langle b_i, b_j\rangle = h_t(\langle b_j, b_i\rangle)$. From this and the previous equality, it follows that

$$\langle i_0(b_i), i_1(b_j)\rangle = h_t(\langle i_1(b_j), i_0(b_i)\rangle) = h_t(\langle b_j, b_i\rangle) = {h_t}^2(\langle b_i, b_j\rangle) = \langle b_i, b_j\rangle. \qquad \square$$

Corollary 6.2. $\langle i_0(\boldsymbol{B}^{n-1}), i_1(\boldsymbol{B}^{n-1})\rangle = \langle i_1(\boldsymbol{B}^{n-1}), i_0(\boldsymbol{B}^{n-1})\rangle = G_{n-1}$.

Lemma 6.3. *For any* $b_i, b_j \in \boldsymbol{B}^{n-1}$,

$$\langle i_0(b_i), i_0(b_j)\rangle = \langle i_1(b_i), i_1(b_j)\rangle = d\langle b_i, b_j\rangle.$$

Proof. $i_0(b_i) \circ i_0(b_j)^*$ is composed of $b_i \circ b_j^*$ in addition to a chord close to the boundary glued with its inverse. These two chords form a trivial loop. Thus, $\langle i_0(b_i), i_0(b_j)\rangle = d\langle b_i, b_j\rangle$ for all $b_i, b_j \in \boldsymbol{B}^{n-1}$.

By symmetry, $\langle i_1(\boldsymbol{B}^{n-1}), i_1(\boldsymbol{B}^{n-1})\rangle = dG_{n-1}$. $\square$

Corollary 6.4. $\langle i_0(\boldsymbol{B}^{n-1}), i_0(\boldsymbol{B}^{n-1})\rangle = \langle i_1(\boldsymbol{B}^{n-1}), i_1(\boldsymbol{B}^{n-1})\rangle = dG_{n-1}$.

Using these facts, we can construct from G_n a $(|B_n|-2|B_{n-1}|)\times(|B_n|-2|B_{n-1}|)$ matrix whose determinant is equal to

$$\frac{\det G_n}{(1-d^2)^{n\binom{2n-2}{n-1}}}(\det G_{n-1})^2.$$

This allows us to compute $\det G_n$ with greater ease, assuming we know $\det G_{n-1}$. This process is shown in the next theorem.

Theorem 6.5. *There is a nonnegative integer*[3] *k such that, for all integers* $n > 1$,

$$\det {G_{n-1}}^2 \ \textit{divides} \ \det G_n(1-d^2)^k.$$

Proof. Order the elements of $\boldsymbol{B}^n$ (or equivalently, the rows and columns of G_n), as shown in Theorem 5.4. Apply the procedure from Theorem 5.4 to construct G'_n, whose form is roughly

$$G'_n = \begin{bmatrix} 0 & (1-d^2)G_{n-1} & * & * & * & * \\ G_{n-1} & dG_{n-1} & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \\ 0 & \boxed{*} & * & * & * & * \end{bmatrix}.$$

Consider the block in G'_n whose columns correspond to states in $i_1(\boldsymbol{B}^{n-1})$ and whose rows correspond to states in neither $i_0(\boldsymbol{B}^{n-1})$ nor $i_1(\boldsymbol{B}^{n-1})$ (boxed above). Every row in this submatrix is a linear combination of two rows from G_{n-1}. More

[3] Clearly this integer is bounded above by $(n+1)\binom{2n}{n}$, or even better, by $|\boldsymbol{B}^n| - 2|\boldsymbol{B}^{n-1}|$. Better bounds are possible, but we do not address them in this paper.

precisely, each row is of the form $a_1 l_1 - a_2 d l_2$, where l_1 and l_2 are two rows, not necessarily distinct, in G_{n-1}, and $a_1, a_2 \in \{1, d, x_1, y_1, z_2\}$. If we assume $1-d^2$ is invertible in our ring (for example, if we consider a ring of rational functions), then each row is a linear combination of two rows from $(1-d^2)G_{n-1}$. We then simplify G'_n as follows.

Let G''_n be the matrix obtained by properly subtracting linear combinations of the first $n\binom{2n-2}{n-1}$ rows of G'_n from the rows which correspond to states in neither $i_0(\boldsymbol{B}^{n-1})$ nor $i_1(\boldsymbol{B}^{n-1})$ so that the submatrix obtained by restricting G''_n to columns corresponding to states in $i_1(\boldsymbol{B}^{n-1})$ and rows corresponding to states in neither $i_0(\boldsymbol{B}^{n-1})$ nor $i_1(\boldsymbol{B}^{n-1})$ is equal to 0:

$$G''_n = \left[\begin{array}{ccc} 0 & (1-d^2)G_{n-1} & * * * * \\ G_{n-1} & dG_{n-1} & * * * * \\ 0 & 0 & \boxed{\begin{array}{c} * * * * \\ * * * * \\ * * * * \\ * * * * \end{array}} \\ 0 & 0 & \\ 0 & 0 & \\ 0 & 0 & \end{array}\right].$$

The block decomposition so far proves that $\det G''_n$ equals $(1-d^2)^{n\binom{2n-2}{n-1}}(\det G_{n-1})^2$ times the determinant of the boxed block, which we denote by $\overline{G}_n$. The latter contains a power of $(1-d^2)^{-1}$, whose degree is unspecified. Thus,

$$\det {G_{n-1}}^2 \ \text{ divides } \ \det G''_n (1-d^2)^k,$$

for some integer $k \geq 0$. We remind the reader that G''_n is obtained from G'_n via determinant-preserving operations, and hence $\det G'_n = \det G_n$. □

Note that if $\det \overline{G}_n$ has fewer than $n\binom{2n-2}{n-1}$ powers of $(1-d^2)^{-1}$, then

$$\det {G_{n-1}}^2 \ \text{ divides } \ \det G_n.$$

It remains an open problem as to whether the former is true. For an example of this decomposition, we mention the equality

$$\det \overline{G}_2 = \frac{\det G_2}{(1-d^2)^4 \det {G_1}^2}.$$

7. Future directions

In this section, we discuss briefly generalizations of the Gram determinant and present a number of open questions and conjectures.

The case of a disk with k holes. We can generalize our setup by considering F_k^n, a unit disk with k holes, in addition to $2n$ points, $a_0, \ldots, a_{2n-1}$, arranged in a similar way to points in F_2^n. For $b_i, b_j \in \boldsymbol{B}_k^n$, let $b_i \circ b_j^*$ be defined in the same way as before. Each paired diagram $b_i \circ b_j^*$ consists of up to n closed curves on the 2-sphere with $2k$ holes. Let $\mathcal{S}$ denote the set of all $2k$ holes. We differentiate between the closed curves based on how they partition $\mathcal{S}$. We define a bilinear form by counting the multiplicities of each type of closed curve in the paired diagram. In the case $k = 2$, we assigned to each paired diagram a corresponding element in a polynomial ring of eight variables, each variable representing a type of closed curve. In the general case, the number of types of closed curves is equal to

$$\frac{2^{|S|}}{2} = \frac{2^{2k}}{2} = 2^{2k-1},$$

so we can define the Gram matrix of the bilinear form for a disk with k holes and $2n$ points with $(n+1)^{k-1}\binom{2n}{n} \times (n+1)^{k-1}\binom{2n}{n}$ entries, each belonging to a polynomial ring of 2^{2k-1} variables. We denote this Gram matrix by $G_n^{F_k}$. For $n = 1$ and $k = 3$, we can easily write this 8×8 Gram matrix. For purposes of notation, let us denote the holes in $F_{0,3}^n$ by ∂_1, ∂_2 and ∂_3, and their inversions by ∂_{-1}, ∂_{-2} and ∂_{-3}, respectively. Hence, each closed curve in the surface encloses some subset of $\mathcal{S} = \{\partial_1, \partial_{-1}, \partial_2, \partial_{-2}, \partial_3, \partial_{-3}\}$. Let x_{a_1,a_2,a_3} denote a curve separating the set of holes $\{\partial_{a_1}, \partial_{a_2}, \partial_{a_3}\}$ from $\mathcal{S} - \{\partial_{a_1}, \partial_{a_2}, \partial_{a_3}\}$. We can similarly define x_{a_1,a_2} and x_{a_1}. The Gram matrix is then

$$G_1^{F_3} = \begin{bmatrix} d & x_{-3} & x_{-2} & x_{-2,-3} & x_{-1} & x_{-1,-3} & x_{-1,-2} & x_{1,2,3} \\ x_3 & x_{3,-3} & x_{-2,3} & x_{1,-1,2} & x_{-1,3} & x_{1,2,-2} & x_{1,2,-3} & x_{1,2} \\ x_2 & x_{2,-3} & x_{2,-2} & x_{1,-1,3} & x_{-1,2} & x_{1,-2,3} & x_{1,3,-3} & x_{1,3} \\ x_{2,3} & x_{1,-1,-2} & x_{1,-1,-3} & x_{1,-1} & x_{1,-2,-3} & x_{1,-2} & x_{1,-3} & x_1 \\ x_1 & x_{1,-3} & x_{1,-2} & x_{1,-2,-3} & x_{1,-1} & x_{1,-1,-3} & x_{1,-1,-2} & x_{2,3} \\ x_{1,3} & x_{1,3,-3} & x_{1,-2,3} & x_{-1,2} & x_{1,-1,3} & x_{2,-2} & x_{2,-3} & x_2 \\ x_{1,2} & x_{1,2,-3} & x_{1,2,-2} & x_{1,-3} & x_{1,-1,2} & x_{-2,3} & x_{3,-3} & x_3 \\ x_{1,2,3} & x_{-1,-2} & x_{-1,-3} & x_{-1} & x_{-2,-3} & x_{-2} & x_{-3} & d \end{bmatrix}.$$

It would be tempting to conjecture that the determinant of the matrix above has a straightforward decomposition of the form $(u+v)(u-v)$. We found that this is the case when any two variables of the form x_{a_1} and x_{a_1,a_2} are replaced by 0; explicitly, we have, with $a_1, a_2 \in \{-3, -2, -1, 1, 2, 3\}$,

$$\begin{aligned} &\det G_1^{F_3}|_{x_{a_1}=x_{a_1,a_2}=0} \\ &\quad = -(d - x_{1,2,3})(d + x_{1,2,3}) \\ &\qquad \times (x_{1,2,-2}x_{1,-1,3}x_{1,-1,-2} + x_{1,3,-3}x_{1,-1,2}x_{1,-1,-3} - x_{1,2,-3}x_{1,-1,3}x_{1,-1,-3} \\ &\qquad\qquad - x_{1,-1,-2}x_{1,-1,-2}x_{1,-2,3} - x_{1,2,-2}x_{1,3,-3}x_{1,-2,-3} + x_{1,2,-3}x_{1,-2,3}x_{1,-2,-3})^2. \end{aligned}$$

In general, however, preliminary calculations suggest that $\det G_n^{F_3}$ may be an irreducible polynomial.

Finally, we observe that many of the results we have proved for $\det G_n^{F_2}$ also hold for general $\det G_n^{F_k}$. For example, $\det G_n^{F_k}$ is nonzero and divides $\det G_n^{F_{k+1}}$. In the specific case of $\det G_n^{F_3}$, we conjecture that the diagonal term is of the form $\delta(n) = d^{\alpha(n)}(x_{1,-1}x_{2,-2}x_{3,-3})^{\beta(n)}$, where

$$\alpha(n) + 3\beta(n) = n(n+1)^2\binom{2n}{n} \quad \text{and} \quad \beta(n) = n(n+1)4^{n-1}.$$

***Speculation on the factorization of* $\det G_n$.** Section 5 establishes that

$$\det G_{n-1} \text{ divides } \det G_n,$$

but we conjecture that there are many more powers of $\det G_{n-1}$ in $\det G_n$. Indeed, even in the base case, $\det {G_1}^k$ divides $\det G_2$ for k up to 4. Finding the maximal power of $\det G_{n-1}$ in $\det G_n$ in the general case is an open problem and can be helpful toward computing the full decomposition of $\det G_n$.

Examining the terms of highest degree in $\det G_n$, that is, $h(\det G_n)$ may also yield helpful hints toward the full decomposition. In particular, we note that

$$\det {G_1}^4 \text{ divides } h(\det G_2) \quad \text{and} \quad \frac{h(\det G_2)^6}{\det {G_1}^9} \text{ divides } h(\det G_3).$$

We can conjecture that $(\det {G_2}^6)/(\det {G_1}^9)$ divides $\det G_3$, from which it follows that $\det {G_1}^{15}$ divides $\det G_3$. We therefore offer the following conjecture:

Conjecture 7.1. $\det {G_1}^{\binom{2n}{n-1}}$ divides $\det G_n$ for $n \geq 1$.

The next conjecture is motivated by observations of $\det G_1$ and $\det G_2$.

Conjecture 7.2. Let H_n denote the product of factors of $\det G_n$ not in $\det G_{n-1}$. Then ${H_{n-1}}^{2n}$ divides $\det G_n$.

Conjecture 7.3. Let, as before, $R = \mathbb{Z}[d, x_1, x_2, y_1, y_2, z_1, z_2, z_3]$, and let R_1 be the subgroup of R of elements invariant under h_1, h_2, h_t, and g_1, g_2, g_3. Similarly, let R_2 be the subgroup of R composed of elements $w \in R$ such that

$$h_1(w) = h_2(w) = -w \quad \text{and} \quad h_t(w) = g_1(w) = g_2(w) = g_3(w).$$

Then:

(1) $\det G_n = u^2 - v^2$, where $u \in R_1$ and $v \in R_2$.

(2) $\det G_n = \prod_\alpha (u_\alpha^2 - v_\alpha^2)$, where $u_\alpha \in R_1$ and $v_\alpha \in R_2$, and $u_\alpha - v_\alpha$ and $u_\alpha + v_\alpha$ are irreducible polynomials.

(3) $\det G_n = \prod_{i=1}^n ({u_i}^2 - {v_i}^2)^{\binom{2n}{n-i}}$, where $u_i \in R_1$ and $v_i \in R_2$.

Notice that if $w_1 = u_1^2 - v_1^2$ and $w_2 = u_2^2 - v_2^2$, then

$$w_1 w_2 = (u_1u_2 + v_1v_2)^2 - (u_1v_2 + u_2v_1)^2.$$

We have little confidence in Conjecture 7.3(3). It is closely, maybe too closely, influenced by the case of $\det G_n^{F_1}$, the Gram determinant of type B:

Theorem 7.4 [Martin and Saleur 1993; Chen and Przytycki 2009].

$$\det G_n^{F_1} = \prod_{i=1}^{n} \left(T_i(d)^2 - a^2\right)^{\binom{2n}{n-i}},$$

where $T_i(d)$ is the Chebyshev polynomial of the first kind (recursively defined by $T_0 = 2,\ \ T_1 = d, T_i = d\, T_{i-1} - T_{i-2}$), and d and a correspond to the trivial and the nontrivial curves in the annulus F_1, respectively.

References

[Birkhoff and Lewis 1946] G. D. Birkhoff and D. C. Lewis, "Chromatic polynomials", *Trans. Amer. Math. Soc.* **60** (1946), 355–451. MR 8,284f Zbl 0060.41601

[Chen and Przytycki 2008] Q. Chen and J. H. Przytycki, "The Gram matrix of a Temperley–Lieb algebra is similar to the matrix of chromatic joins", *Commun. Contemp. Math.* **10**:suppl. 1 (2008), 849–855. MR 2009m:05011 Zbl 1158.57012

[Chen and Przytycki 2009] Q. Chen and J. H. Przytycki, "The Gram determinant of the type B Temperley–Lieb algebra", *Adv. in Appl. Math.* **43**:2 (2009), 156–161. MR 2010d:57010 Zbl 1167.57004

[Deutsch and Sagan 2006] E. Deutsch and B. E. Sagan, "Congruences for Catalan and Motzkin numbers and related sequences", *J. Number Theory* **117**:1 (2006), 191–215. MR 2006k:11031 Zbl 1163.11310

[Di Francesco 1998] P. Di Francesco, "Meander determinants", *Comm. Math. Phys.* **191**:3 (1998), 543–583. MR 99e:05007 Zbl 0923.57002

[Lickorish 1991] W. B. R. Lickorish, "Invariants for 3-manifolds from the combinatorics of the Jones polynomial", *Pacific J. Math.* **149**:2 (1991), 337–347. MR 92d:57007 Zbl 0728.57011

[Lickorish 1997] W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics **175**, Springer, New York, 1997. MR 98f:57015 Zbl 0886.57001

[Martin and Saleur 1993] P. Martin and H. Saleur, "On an algebraic approach to higher-dimensional statistical mechanics", *Comm. Math. Phys.* **158**:1 (1993), 155–190. MR 94k:82037 Zbl 0784.05056

[Przytycki 1999] J. H. Przytycki, "Fundamentals of Kauffman bracket skein modules", *Kobe J. Math.* **16**:1 (1999), 45–66. MR 2000i:57015 Zbl 0947.57017

[Schmidt 2004] F. Schmidt, "Problems related to type-A and type-B matrices of chromatic joins", *Adv. in Appl. Math.* **32**:1-2 (2004), 380–390. MR 2004m:06007

[Simion 2000] R. Simion, "Noncrossing partitions", *Discrete Math.* **217**:1-3 (2000), 367–409. MR 2001g:05011 Zbl 0959.05009

[Stanley 1999] R. P. Stanley, *Enumerative combinatorics*, vol. 2, Cambridge Studies in Advanced Mathematics **62**, Cambridge University Press, 1999. MR 2000k:05026 Zbl 0945.05006

[Tutte 1991] W. T. Tutte, "On the Birkhoff–Lewis equations", *Discrete Math.* **92**:1-3 (1991), 417–425. MR 92k:05052 Zbl 0756.05059

[Tutte 1993] W. T. Tutte, "The matrix of chromatic joins", *J. Combin. Theory Ser. B* **57**:2 (1993), 269–288. MR 94a:05144 Zbl 0793.05030

[Westbury 1995] B. W. Westbury, "The representation theory of the Temperley–Lieb algebras", *Math. Z.* **219**:4 (1995), 539–565. MR 96h:20029 Zbl 0840.16008

Received: 2008-11-30 Revised: 2010-04-27 Accepted: 2010-06-03

przytyck@gwu.edu *Department of Mathematics, The George Washington University, Washington, DC 20052, United States*

xzhu@fas.harvard.edu *Department of Applied Mathematics, Harvard University, Cambridge, MA 02138, United States*

See inside back cover or http://pjm.math.berkeley.edu/involve for submission instructions.

The subscription price for 2010 is US $100/year for the electronic version, and $120/year (+$20 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://www.mathscipub.org

A NON-PROFIT CORPORATION

Typeset in LaTeX

involve

2010 vol. 3 no. 2

1944-4176(2010)3:2;1-F