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The cardinality of the value sets modulo n of x^2+x^{-2} and x^2+y^2

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Consider the modular circle $\mathscr{C}_{a,n}=\{(x,y):x^2+y^2\equiv a \pmod n,\ 0\le x,y\le n-1\}$ and the modular hyperbola $\mathscr{H}_n=\{(x,y):xy\equiv 1 \pmod n,\ 0\le x,y\le n-1\}$. We provide explicit formulas for the cardinality of the sets

$$\{a \bmod n : \mathscr{C}_{a,n}\cap\mathscr{H}_n\neq\varnothing\} \quad\text{and}\quad \{a \bmod n : \mathscr{C}_{a,n}\neq\varnothing\}.$$

Introduction

Let $\mathscr{H}_n$ denote the *modular hyperbola*

$$\{(x,y):xy=1 \pmod n,\ 0\le x,y\le n-1\}.$$

This simply defined discrete set of points has connections to a variety of other mathematical topics including Kloosterman sums, consecutive Farey fractions, and quasirandomness. These connections have inspired a closer look at the distribution of the points of $\mathscr{H}_n$, and many questions remain open. For a discussion of recent results and open problems on modular hyperbolas, see [Shparlinski 2007].

The propensity of the points on $\mathscr{H}_n$ to collect on lines of slope ± 1 was investigated in [Eichhorn et al. 2009]. In the course of that investigation, formulas for the cardinalities of the sets

$$\{(x-y) \bmod n : (x,y)\in\mathscr{H}_n\} \quad\text{and}\quad \{(x+y) \bmod n : (x,y)\in\mathscr{H}_n\},$$

were derived. The techniques used to determine these formulas are elementary — within the grasp of an undergraduate mathematics major who has had a course in number theory or abstract algebra.

In this article we investigate the intersection of $\mathscr{H}_n$ with the modular circles

$$\mathscr{C}_{a,n}=\{(x,y):x^2+y^2\equiv a \pmod n,\ 0\le x,y\le n-1\},$$

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This work was done as part of Hanrahan's undergraduate honors thesis under Khan's supervision.

and in particular we determine the cardinality of the set

$$\{a \bmod n : \mathscr{C}_{a,n} \cap \mathscr{H}_n \neq \varnothing\} = \{(x^2 + y^2) \bmod n : (x, y) \in \mathscr{H}_n\}.$$

Figure 1 contrasts the modular circle $\mathscr{C}_{1,997}$ with the modular hyperbola $\mathscr{H}_{997}$. Figure 2 shows the two superimposed, and the intersection $\mathscr{C}_{1,997} \cap \mathscr{H}_{997}$.

This short note is a concise version of SH's honors thesis. It is also a natural addendum to [Eichhorn et al. 2009], as we used the formulas found there to prove our results.

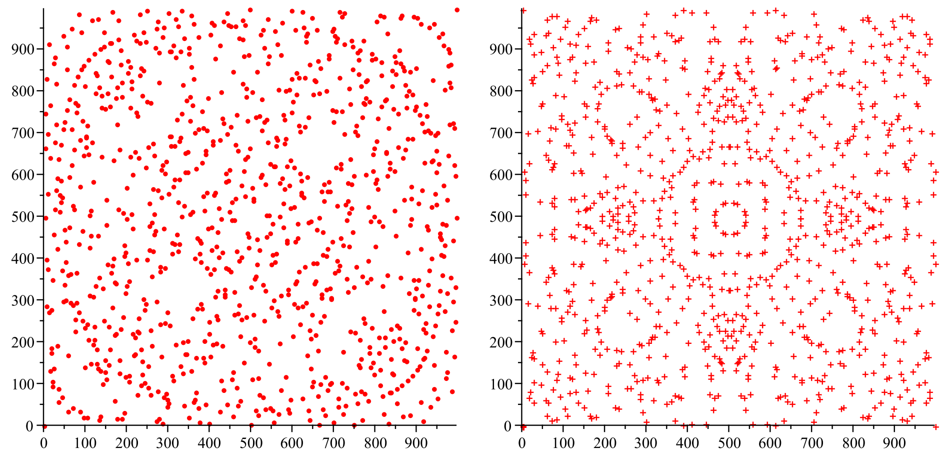

Figure 1. Left: The modular hyperbola $\mathscr{H}_{997}$. Right: The modular circle $\mathscr{C}_{1,997}$.

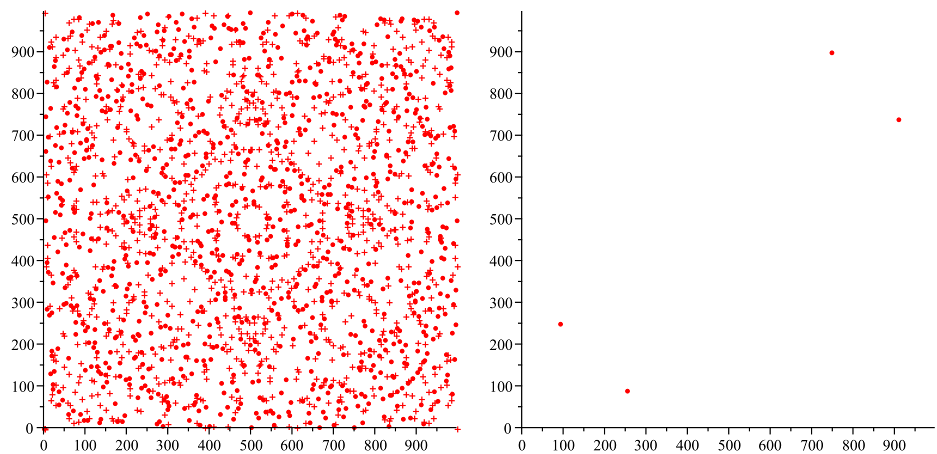

Figure 2. Left: Superposition of the preceding two sets. Points of the modular circle are represented by crosses; those of the modular hyperbola by solid circles. Right: The intersection $\mathscr{C}_{1,997} \cap \mathscr{H}_{997} = \{(91, 252), (252, 91), (745, 906), (906, 745)\}$.

1. Preliminary results

Let $f \in \mathbb{Z}[x_1, \ldots, x_k]$ and let $S \subseteq \mathbb{Z}_n^k$ (where $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ is the set of integers modulo n). Then $I(f, S)$ will denote the set

$$I(f, S) = \{f(x_1, \ldots, x_k) \bmod n : (x_1, \ldots, x_k) \in S\}.$$

We also define two subsets of $I(f, S)$:

$$I'(f, S) = \{a : a \in I(f, S), \gcd(a, n) = 1\},$$
$$I''(f, S) = \{a : a \in I(f, S), \gcd(a, n) \neq 1\}.$$

Our first result is that the quantity $\#I(f, \mathcal{H}_n)$ is a multiplicative function of n. Furthermore, by replacing each occurrence of $\mathcal{H}_n$ with $\mathbb{Z}_n^2$ in the statement and proof of the theorem, we get that $\#I(f, \mathbb{Z}_n^2)$ is also a multiplicative function of n.

Proposition 1. *Let $f \in \mathbb{Z}[x, y]$ and define $f_n : \mathcal{H}_n \to \mathbb{Z}_n$ by*

$$f_n((x, y)) = f(x, y) \bmod n.$$

If $n = a \cdot b$ with $\gcd(a, b) = 1$, *then*

$$\#I(f, \mathcal{H}_n) = \#I(f, \mathcal{H}_a) \cdot \#I(f, \mathcal{H}_b).$$

It follows that if $n = \prod_{i=1}^m p_i^{e_i}$ is the canonical factorization of n, then

$$\#I(f, \mathcal{H}_n) = \prod_{i=1}^{m} \#I(f, \mathcal{H}_{p_i^{e_i}}). \tag{1}$$

Proof. The Chinese remainder theorem says that the map $r : \mathbb{Z}_n \to \mathbb{Z}_a \times \mathbb{Z}_b$ given by

$$r(x) = (x \bmod a, x \bmod b)$$

is an isomorphism of rings. Hence the map $R : \mathcal{H}_n \to \mathcal{H}_a \times \mathcal{H}_b$ defined by

$$R((x, y)) = ((x \bmod a, y \bmod a), (x \bmod b, y \bmod b))$$

is a bijection. The result now follows from the observation that the diagram

$$\begin{array}{ccc} \mathcal{H}_n & \xrightarrow{R} & \mathcal{H}_a \times \mathcal{H}_b \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_a \times f_b} \\ \mathbb{Z}_n & \xrightarrow{r} & \mathbb{Z}_a \times \mathbb{Z}_b. \end{array}$$

commutes. □

Thus we have reduced the problem of determining formulas for $\#I(x^2+y^2, \mathcal{H}_n)$ (or $\#I(x^2+y^2, \mathbb{Z}_n^2)$) to determining them for prime powers. From this point, we shall refer to the set $I(x^2+y^2, \mathcal{H}_n)$ as $I(x^2+x^{-2}, \mathbb{Z}_n)$. All of our formulas were discovered through extensive numerical experimentation with Maple. Maple was the most valuable research tool at our disposal — only in discovering the formulas, but also in the *proving* stage. In the remainder of this section, we list the mathematical results we need to prove these formulas.

It is more convenient to work with the value set $I((x+x^{-1})^2, \mathbb{Z}_n)$ than with $I(x^2+x^{-2}, \mathbb{Z}_n)$. The following lemma justifies the change.

Lemma 2. *For any positive integer n,*

$$\#I(x^2+x^{-2}, \mathbb{Z}_n) = \#I((x+x^{-1})^2, \mathbb{Z}_n). \tag{2}$$

Proof. The map $z \mapsto (z+2) \bmod n$ defines a bijection between $I(x^2+x^{-2}, \mathbb{Z}_n)$ and $I((x+x^{-1})^2, \mathbb{Z}_n)$. □

We next state a basic criterion on the solvability of quadratic congruences modulo prime powers: $x^2 \equiv a \pmod{p^t}$.

Proposition 3 [Ireland and Rosen 1982, Propositions 4.2.3, 4.2.4, p. 46]. *Let p be prime and let a be an integer such that* $\gcd(a, p) = 1$.

(1) *Suppose $p > 2$. If the congruence $x^2 \equiv a \pmod{p}$ is solvable, then for every $t \geq 2$ the congruence $x^2 \equiv a \pmod{p^t}$ is solvable with precisely 2 distinct solutions.*

(2) *Suppose $p = 2$. If the congruence $x^2 \equiv a \pmod{2^3}$ is solvable, then for every $t \geq 3$ the congruence $x^2 \equiv a \pmod{2^t}$ is solvable with precisely 4 distinct solutions.*

Proposition 4 [Stangl 1996]. *Let p be an odd prime. Then*

$$\#I(x^2, \mathbb{Z}_{p^t}) = \frac{p^{t+1}}{2(p+1)} + (-1)^{t-1}\frac{p-1}{4(p+1)} + \frac{3}{4}. \tag{3}$$

For the special case $p = 2$ we have

$$\#I(x^2, \mathbb{Z}_{2^t}) = \frac{2^{t-1}}{3} + \frac{(-1)^{t-1}}{6} + \frac{3}{2}, \quad t \geq 2. \tag{4}$$

Proposition 5 [Eichhorn et al. 2009].

$$\#I(x+x^{-1}, \mathbb{Z}_{p^t}) = \frac{(p-3)p^{t-1}}{2} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{2(p+1)} + \frac{3}{2}. \tag{5}$$

2. The formulas for $\#I((x+x^{-1})^2, \mathbb{Z}_{p^t})$

The central result of this paper is as follows.

Theorem 6. *For $p=2$ and $t\geq 7$,*

$$\#I((x+x^{-1})^2,\mathbb{Z}_{2^t})=\frac{2^{t-7}}{3}+\frac{(-1)^{t-1}}{6}+\frac{3}{2}. \tag{6}$$

If $p\equiv 1 \pmod 4$ then

$$\#I((x+x^{-1})^2,\mathbb{Z}_{p^t})=\frac{(p-5)p^{t-1}}{4}+\frac{2p^{t-1}+(-1)^{t-1}(p-1)}{2(p+1)}+\frac{3}{2}. \tag{7}$$

If $p\equiv 3 \pmod 4$ then

$$\#I((x+x^{-1})^2,\mathbb{Z}_{p^t})=\frac{(p-3)p^{t-1}}{4}+\frac{2p^{t-1}+(-1)^{t-1}(p-1)}{4(p+1)}+\frac{3}{4}. \tag{8}$$

The proof occupies most of this section.

Proof of Theorem 6, case $p>2$. We will use the squaring map modulo p^t:

$$Q: I(x+x^{-1},\mathbb{Z}_{p^t})\to I((x+x^{-1})^2,\mathbb{Z}_{p^t}),\qquad Q(z)=z^2 \bmod p^t.$$

We note that it preserves coprimeness with p:

$$Q(I'(x+x^{-1},\mathbb{Z}_{p^t}))=I'((x+x^{-1})^2,\mathbb{Z}_{p^t}),$$
$$Q(I''(x+x^{-1},\mathbb{Z}_{p^t}))=I''((x+x^{-1})^2,\mathbb{Z}_{p^t}).$$

Proposition 7. *Let p be an odd prime. For any $a\in I'((x+x^{-1})^2,\mathbb{Z}_{p^t})$, we have $\#Q^{-1}(\{a\})=2$, and consequently*

$$\#I'((x+x^{-1})^2,\mathbb{Z}_{p^t})=\#I'(x+x^{-1},\mathbb{Z}_{p^t})/2. \tag{9}$$

Proof. Let a be an arbitrary element of $I'((x+x^{-1})^2,\mathbb{Z}_{p^t})$. There exists a point $(x_1,y_1)\in\mathcal{H}_{p^t}$ such that

$$(x_1+y_1)^2\equiv a \pmod{p^t}.$$

Since $\gcd(x_1+y_1,p)=1$,

$$x_1+y_1\not\equiv-(x_1+y_1) \pmod{p^t};$$

hence the two distinct elements of $I'((x+x^{-1})^2,\mathbb{Z}_{p^t})$ that Q maps to a are

$$(x_1+y_1) \pmod{p^t}\quad\text{and}\quad -(x_1+y_1) \pmod{p^t}.$$

By Proposition 3, the congruence $x^2\equiv a \pmod{p^t}$ has at most two solutions and we conclude that $\#Q^{-1}(\{a\})=2$. □

Proposition 8.

$$\#I''(x+x^{-1},\mathbb{Z}_{p^t})=\begin{cases}p^{t-1} & \text{if } p\equiv 1 \pmod 4,\\ 0 & \text{if } p\equiv 3 \pmod 4.\end{cases} \tag{10}$$

Consequently, when $p \equiv 1 \pmod 4$,

$$I''(x+x^{-1}, \mathbb{Z}_{p^t}) = \{kp : k = 0, 1, \ldots, p^{t-1}-1\}.$$

Proof. Define $s_{p^t} : \mathscr{H}_{p^t} \to \mathbb{Z}_{p^t}$ by $s_{p^t}((x, y)) = (x+y) \bmod p^t$ and let

$$\mathscr{H}''_{p^t} = \{(x, y) : (x, y) \in \mathscr{H}_{p^t} \text{ with } s_{p^t}((x, y)) \in I''(x+x^{-1}, \mathbb{Z}_{p^t})\}.$$

If $(x, y) \in \mathscr{H}''_{p^t}$, then $x+y=0 \pmod p$ and consequently $x^2=-1 \pmod p$. Since -1 is a quadratic residue modulo p if and only if $p \equiv 1 \pmod 4$, we obtain the second part of (10).

We now restrict our attention to primes p that are congruent to 1 modulo 4. Since $s_{p^t}(\mathscr{H}''_{p^t}) = I''(x+x^{-1}, \mathbb{Z}_{p^t})$, we prove the first part of (10) by proving the following two assertions:

(i) $\#s_{p^t}^{-1}(\{a\}) = 2$ for any $a \in I''(x+x^{-1}, \mathbb{Z}_{p^t})$.

(ii) $\#\mathscr{H}''_{p^t} = 2p^{t-1}$.

The proof of (i) is as follows. Let $(r, s) \in s_{p^t}^{-1}(\{a\})$. Then $(2r-a)$ and $(2s-a)$ are two distinct roots of the congruence

$$x^2 \equiv (a^2-4) \pmod{p^t}.$$

Since $p \mid a$, we have $\gcd(a^2-4, p) = 1$. Hence by Proposition 3

$$x^2 \equiv (a^2-4) \pmod{p^t}$$

cannot have more than two roots. Consequently $s_{p^t}^{-1}(\{a\}) = \{(r, s), (s, r)\}$.

We now prove (ii). Let (r, s) be an arbitrary element of $\mathscr{H}''_{p^t}$ and let

$$r = d_0 + d_1 p + d_2 p^2 + \cdots + d_{t-1} p^{t-1}$$

be the expansion of r in base p. There are only two possible choices for d_0, specifically, the two roots of $x^2 \equiv -1 \pmod p$, and for each of the other d_i's there are p possible choices: $0, 1, \ldots, p-1$. So there are $2p^{t-1}$ possible r's. Since s is completely determined by the choice of r, we conclude that $\#\mathscr{H}''_{p^t} = 2p^{t-1}$. □

Proposition 9. *If* $p \equiv 1 \pmod 4$ *then*

$$\#I''((x+x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{4(p+1)} + \frac{3}{4}. \tag{11}$$

Proof. By Proposition 8

$$I''(x+x^{-1}, \mathbb{Z}_{p^t}) = \{kp : 0 \le k \le p^{t-1}-1\}.$$

Consequently,

$$\begin{aligned}I''((x+x^{-1})^2, \mathbb{Z}_{p^t}) &= Q(I''(x+x^{-1}, \mathbb{Z}_{p^t})) \\ &= Q(\{kp : 0 \le k \le p^{t-1}-1\}) = \{j^2 \bmod p^t : p \mid j\}.\end{aligned}$$

Therefore,

$$\#I''((x+x^{-1})^2, \mathbb{Z}_{p^t}) = \#\{k^2 \bmod p^t\} - \#\{k^2 \bmod p^t : \gcd(k, p) = 1\}.$$

Combining Stangl's formula (3) with the standard result that the number of quadratic residues modulo p^t is $(p^t - p^{t-1})/2$, we obtain

$$\#I''((x+x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{4(p+1)} + \frac{3}{4},$$

which proves Proposition 9. □

We are now ready to prove formulas (7) and (8). We have

$$\begin{aligned}&\#I((x+x^{-1})^2, \mathbb{Z}_{p^t}) \\ &\quad = \#I'((x+x^{-1})^2, \mathbb{Z}_{p^t}) + \#I''((x+x^{-1})^2, \mathbb{Z}_{p^t}) \\ &\quad = \frac{\#I'(x+x^{-1}, \mathbb{Z}_{p^t})}{2} + \#I''((x+x^{-1})^2, \mathbb{Z}_{p^t}) \\ &\quad = \frac{\#I(x+x^{-1}, \mathbb{Z}_{p^t})}{2} - \frac{\#I''(x+x^{-1}, \mathbb{Z}_{p^t})}{2} + \#I''((x+x^{-1})^2, \mathbb{Z}_{p^t}).\end{aligned}$$

Formula (5) is

$$\#I(x+x^{-1}, \mathbb{Z}_{p^t}) = \frac{(p-3)p^{t-1}}{2} + \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{2(p+1)} + \frac{3}{2}.$$

If $p \equiv 3 \pmod 4$, then $\#I''(x+x^{-1}, \mathbb{Z}_{p^t}) = \#I''((x+x^{-1})^2, \mathbb{Z}_{p^t}) = 0$ by (10). If $p \equiv 1 \pmod 4$, then

$$\#I''(x+x^{-1}, \mathbb{Z}_{p^t}) = p^{t-1}$$

and

$$\#I''((x+x^{-1})^2, \mathbb{Z}_{p^t}) = \frac{2p^{t-1} + (-1)^{t-1}(p-1)}{4(p+1)} + \frac{3}{4},$$

by (10) and (11). We complete the proof with simple algebraic computations. □

Proof of Theorem 6, case $p = 2$. Interestingly this was the most difficult and time consuming part. It was only through experimenting with Maple that we discovered the map f (defined below) that allowed us to prove the formula for powers of 2.

Proposition 10. *Let $t \ge 3$. The image of the map*

$$f : I(x^2, \mathbb{Z}_{2^t}) \to \{0, 1, \dots, 2^{t+6} - 1\}$$

given by

$$f(k^2) = (64k^2+4) \bmod 2^{t+6}$$

is $I((x+x^{-1})^2, \mathbb{Z}_{2^{t+6}})$. *Since* f *is injective we conclude that*

$$\#I((x+x^{-1})^2, \mathbb{Z}_{2^{t+6}}) = \#I(x^2, \mathbb{Z}_{2^t}). \tag{12}$$

Proof. First we show that $I((x+x^{-1})^2, \mathbb{Z}_{2^{t+6}}) \subseteq \text{Image}(f)$. Let $(x, y) \in \mathcal{H}_{2^{t+6}}$. We can write

$$x = 8x_1 + a \quad \text{and} \quad y = 8y_1 + a,$$

with $0 \le x_1, y_1 < 2^{t+3}$ and $a = 1, 3, 5$ or 7. (We are using the fact that each element in $\mathbb{Z}_8^*$ is its own inverse.) The following calculation now shows that $(x+y)^2 \bmod 2^{t+6} \in \text{Image}(f)$.

$$\begin{aligned}(x+y)^2 &= (8x_1+8y_1+2a)^2\\ &= 64x_1^2 - 128x_1y_1 + 64y_1^2 + 256x_1y_1 + 32x_1a + 32y_1a + 4a^2\\ &= 64(x_1-y_1)^2 + 4(64x_1y_1 + 8x_1a + 8y_1a + a^2)\\ &= 64(x_1-y_1)^2 + 4xy\\ &\equiv (64(x_1-y_1)^2+4) \pmod{2^{t+6}}.\end{aligned}$$

To show the reverse inclusion, let $k^2 \in I(x^2, \mathbb{Z}_{2^t})$. By Proposition 3 the congruence

$$x^2 \equiv 16k^2 + 1 \pmod{2^n}$$

has a solution for all values of n. Let l be any integer such that $l^2 = 16k^2 + 1 \pmod{2^{t+6}}$, and let

$$x = (l-4k) \bmod 2^{t+6}, \qquad y = (l+4k) \bmod 2^{t+6}.$$

The immediate observations that $(x, y) \in \mathcal{H}_{2^{t+6}}$ and

$$(x+y)^2 \equiv 4l^2 \equiv 64k^2 + 4 \pmod{2^{t+6}}$$

complete the proof. □

Now the formula (6) for $\#I((x+x^{-1})^2, \mathbb{Z}_{2^t})$ is obtained by combining (2), (12) and (16). This concludes the proof of Theorem 6. □

We can also derive the formula for $\#I(x^2+x^{-2}, \mathbb{Z}_p)$ as a special case of an old formula for pairs of quadratic residues.

Theorem 11 [Berndt et al. 1998, Theorem 6.3.1, page 197]. *Let* p *be an odd prime and let* c *be an integer relatively prime to* p. *Let* $\epsilon_1 = \pm 1$ *and* $\epsilon_2 = \pm 1$. *Then*

$$\#\left\{n : 0 \le n < p,\ \left(\frac{n}{p}\right) = \epsilon_1,\ \left(\frac{n+c}{p}\right) = \epsilon_2\right\} = \frac{1}{4}\left\{p - 2\epsilon_1\left(\frac{-c}{p}\right) - \epsilon_2\left(\frac{c}{p}\right) - \epsilon_1\epsilon_2\right\}. \tag{13}$$

The special case of this formula with $\epsilon_1 = \epsilon_2 = c = 1$ was first published by Aladov in 1896. The connection between (13) and $\#I(x^2+x^{-2}, \mathbb{Z}_p)$ is as follows.

Theorem 12. *Let $a \in \mathbb{Z}$ with $\gcd(a^2-4, n) = 1$. Then $\mathscr{C}_{a,n} \cap \mathscr{H}_n \neq \varnothing$ if and only if for every prime, p, in the canonical factorization of n we have*

$$\left(\frac{a-2}{p}\right) = \left(\frac{a+2}{p}\right) = 1. \tag{14}$$

Consequently,

$$\#I(x^2+x^{-2}, \mathbb{Z}_p) = \#\left\{a \,:\, 0 \le a < p,\ \left(\frac{a-2}{p}\right) = \left(\frac{a+2}{p}\right) = 1\right\} + 1.$$

Proof. For the "only if" part, let $(r, s) \in \mathscr{C}_{a,n} \cap \mathscr{H}_n$ and let p be an arbitrary prime divisor of n. So, $(r-s)^2 \equiv a-2 \pmod p$ and $(r+s)^2 \equiv a+2 \pmod p$, which leads immediately to (14).

To prove the converse, let $n = \prod_{i=1}^{t} p_i^{e_i}$ be the canonical factorization of n. By Proposition 3, we can lift the square roots (modulo p) of $(a-2)$ and $(a+2)$ to the e_ith power, $p_i^{e_i}$. Let $s_i = \sqrt{a-2} \pmod{p_i^{e_i}}$, and $r_i = \sqrt{a+2} \pmod{p_i^{e_i}}$. Then

$$2^{-1} \cdot (r_i + s_i, r_i - s_i) \in \mathscr{C}_{p_i^{e_i}} \cap \mathscr{H}_{p_i^{e_i}},$$

where 2^{-1} denotes the inverse of 2 modulo $p_i^{e_i}$. Now invoke the Chinese remainder theorem to determine integers r and s such that

$$r \equiv r_i \pmod{p_i^{e_i}} \text{ and } s \equiv s_i \pmod{p_i^{e_i}} \quad \text{for } i = 1, \dots, t.$$

Clearly $(r, s) \in \mathscr{C}_n \cap \mathscr{H}_n$. □

3. The formulas for $\#I(x^2 + y^2, \mathbb{Z}^2_{p^t})$

We now determine the formulas for $\#I(x^2 + y^2, \mathbb{Z}^2_{p^t})$ to contrast them to

$$\#I(x^2 + x^{-2}, \mathbb{Z}_{p^t}).$$

Theorem 13. *Let p be an odd prime. Then*

$$\#I(x^2+y^2, \mathbb{Z}^2_{p^t}) = \begin{cases} p^t & \text{if } p \equiv 1 \pmod 4, \\ p & \text{if } p \equiv 3 \pmod 4 \text{ and } t = 1, \\ p^t - \sum\limits_{j=0}^{[t/2]-1} \varphi(p^{t-1-2j}) & \text{if } p \equiv 3 \pmod 4 \text{ and } t > 1, \end{cases} \tag{15}$$

When $p = 2$ we have

$$\#I(x^2 + y^2, \mathbb{Z}^2_{2^t}) = \varphi(2^t) + 1. \tag{16}$$

As is typically the case, the formula for powers of two, 2^t, will require a separate argument. We first prove (15).

Proof of formula (15). We treat each case separately.

• $p \equiv 1 \pmod 4$. Let $a \in \{0, 1, \dots, p^t - 1\}$. The simultaneous congruences

$$x - y \equiv 1 \pmod{p^t} \quad \text{and} \quad x + y \equiv a \pmod{p^t}$$

have the solutions

$$x = ((a+1) \cdot (2^{-1} \bmod p^t)) \bmod p^t,$$
$$y = ((a-1) \cdot (2^{-1} \bmod p^t)) \bmod p^t.$$

It immediately follows that $x^2 + (i_{p^t} y)^2 \equiv a \pmod{p^t}$, where

$$i_{p^t}^2 \equiv -1 \pmod{p^t}.$$

• $p \equiv 3 \pmod 4$, $t = 1$. Let $a \in \{0, 1, \dots, p-1\}$. By (3), $\#I(x^2, \mathbb{Z}_p) = (p+1)/2$ and therefore $\#(a - I(x^2, \mathbb{Z}_p)) = (p+1)/2$. Since

$$\#I(x^2, \mathbb{Z}_p) + \#(a - I(x^2, \mathbb{Z}_p)) = p + 1,$$

it follows that there is an element $(a - x_1^2) \in (a - I(x^2, \mathbb{Z}_p))$ and an element $x_2^2 \in I(x^2, \mathbb{Z}_p)$ such that $(a - x_1^2) \equiv x_2^2 \pmod p$.

• $p \equiv 3 \pmod 4$, $t \geq 2$. The key is to prove that an element $a \in \{0, 1, 2, \dots, p^t - 1\}$ satisfies $a \equiv x^2 + y^2 \pmod{p^t}$ if and only if $a = p^k b$, with $\gcd(p, b) = 1$ and k *even*.

($\Leftarrow$) Since p^k is a square in $\mathbb{Z}$, it is sufficient to prove this for integers a that are relatively prime to p. We argue by induction. The previous case shows that the result holds for $t = 1$. Let us assume it is true for t. So

$$a \equiv (x^2 + y^2) \pmod{p^t}.$$

If $p^{t+1} \mid (a - x^2 - y^2)$, there is nothing to prove. So let us assume that $(a - x^2 - y^2) = p^t l$, with $\gcd(l, p) = 1$. Since $\gcd(a, p) = 1$ either $\gcd(x, p) = 1$ or $\gcd(y, p) = 1$. Without loss of generality we assume the former. We now define $s \in \mathbb{Z}$, with $1 \leq s < p$, to be the solution of the congruence

$$2xs \equiv l \pmod p.$$

An immediate calculation shows that

$$a \equiv (x + sp^t)^2 + y^2 \pmod{p^{t+1}}.$$

($\Rightarrow$) We argue by contradiction. Suppose $a = p^k b$, with $a < p^t$, $\gcd(b, p) = 1$,and k odd, be the sum of two squares modulo p^t. So there are integers $x = p^{e_1} x_1$, $y = p^{e_2} y_1$, with $\gcd(x_1 y_1, p) = 1$, such that

$$p^k b \equiv (x^2 + y^2) \pmod{p^t},$$

that is,

$$p^k b \equiv (p^{2e_1}x_1^2 + p^{2e_2}y_1^2) \pmod{p^t}.$$

Since $b \not\equiv 0 \bmod p$ and k is odd we have $\min\{2e_1, 2e_2\} < k$. Without loss of generality we may assume that $e_1 \leq e_2$. We can reduce the congruence

$$p^k b \equiv (x^2 + y^2) \pmod{p^t}$$

to $p^{k-2e_1}b \equiv x_1^2 + p^{2(e_2-e_1)}y_1^2 \pmod{p^{k-2e_1}}$, which in turns reduces to

$$x_1^2 + p^{2(e_2-e_2)}y_1^2 \equiv 0 \pmod{p}.$$

Since $x_1 \not\equiv 0 \pmod{p}$ we must have $p^{2(e_2-e_2)}y_1^2 \not\equiv 0 \pmod{p}$, that is $e_2 = e_1$, and consequently $(x_1^2 + y_1^2) \equiv 0 \pmod{p}$, with $\gcd(x_1y_1, p) = 1$. But this gives us the contradiction that $x^2 \equiv -1 \pmod{p}$ is solvable for a prime p with $p \equiv 3 \pmod{4}$. This concludes the proof of (15). □

Proposition 14. *Let $t \geq 3$ and $0 < m < 2^t$. Then $m \in I(x^2 + y^2, \mathbb{Z}_{2^t}^2)$ if and only if $m = 2^j \cdot a$, with $j < t$ and $a \equiv 1 \pmod{4}$.*

Proof. ($\Leftarrow$) Let $a \equiv 1 \pmod{4}$. Since 2^j is a sum of squares (in $\mathbb{Z}$) we only need to show that a is a sum of two squares modulo 2^t. If $a \equiv 1 \pmod{8}$ then a is a square modulo 2^t by Proposition 3. If $a \equiv 5 \pmod{8}$, then $a - 4 \equiv 1 \pmod{8}$ and is therefore a square modulo 2^t. Consequently a is a sum of two squares modulo 2^t.

($\Rightarrow$) We now assume that $a \equiv 3 \pmod{4}$ and argue by contradiction. Let

$$x^2 + y^2 \equiv m \pmod{2^t}.$$

We look at four possible cases.

(1) $j = 0$: We obtain the contradiction that

$$x^2 + y^2 \equiv 3 \pmod{4}.$$

(2) $j = 1$: We obtain the contradiction that

$$x^2 + y^2 \equiv 6 \pmod{8}.$$

(3) $j \geq 2$, $j \leq (t-2)$: We have $x = 2^{e_1} \cdot x_1$ and $y = 2^{e_2} \cdot y_1$, with x_1, y_1 odd and $j = \min\{2e_1, 2e_2\}$. Without loss of generality we may assume that $e_1 \leq e_2$. We now obtain the contradiction

$$x_1^2 + 4^{e_2-e_1}y_1^2 \equiv a \equiv 3 \pmod{4}.$$

(4) $j = t - 1$: Then

$$m = 2^{t-1} \cdot a \geq 2^{t-1} \cdot 3 > 2^t,$$

contradicting the fact that the elements of $I(x^2+y^2, \mathbb{Z}_{2^t}^2)$ are less than 2^t. □

Proof of formula (16). Let M_t denote the set

$$M_t = \{m : 0 < m < 2^t,\ m = 2^j \cdot a,\ j < t,\ a \equiv 1 \pmod 4\}.$$

In our previous proposition we proved that

$$I(x^2 + y^2, \mathbb{Z}_{2^t}^2) \setminus \{0\} = M_t.$$

We now make the following two observations about elements in M_t:

(i) If $m \in M_t$, then $(m + 2^t) \in M_{t+1}$ provided $m \neq 2^{t-1}$.

(ii) If $m \in M_{t+1}$ with $m > 2^t$, then $(m - 2^t) \in M_t$.

From these two observations we conclude that

$$M_{t+1} \setminus \{2^t\} = M_t \cup \{m + 2^t : m \in M_t \setminus \{2^{t-1}\}\},$$

and consequently $\#M_{t+1} = 2 \cdot \#M_t$. An inductive argument now proves that $\#M_t = \varphi(2^t)$ and therefore $\#I(x^2 + y^2, \mathbb{Z}_{2^t}^2) = \varphi(2^t) + 1$. □

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