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For a group G with generating set $S = \{s_1, s_2, \ldots, s_k\}$, the $\mathbb{G}$-graph of G, denoted by $\Gamma(G, S)$, is the graph whose vertices are distinct cosets of $\langle s_i \rangle$ in G. Two distinct vertices are joined by an edge when the set intersection of the cosets is nonempty. In this paper, we explore the planarity of $\Gamma(G, S)$.

1. Introduction

Let G be a group with a generating set $S = \{s_1, \ldots, s_k\}$. We say that the subset $T_{\langle s_i \rangle} \subset G$ is a *left transversal* for the subgroup $\langle s_i \rangle$ of G if $\{x\langle s_i \rangle \mid x \in T_{\langle s_i \rangle}\}$ is precisely the set of all left cosets of $\langle s_i \rangle$ in G. As in [Bauer et al. 2008], we associate with (G, S) a simple graph $\Gamma(G, S)$ with vertex set $V(\Gamma(G, S)) = \{x_j\langle s_i \rangle \mid x_j \in T_{\langle s_i \rangle}\}$. Two distinct vertices $x_j\langle s_i \rangle$ and $x_l\langle s_k \rangle$ in $V(\Gamma(G, S))$ are joined by an edge if $x_j\langle s_i \rangle \cap x_l\langle s_k \rangle$ is nonempty. The edge set, $E(\Gamma(G, S))$, consists of pairs $(x_j\langle s_i \rangle, x_l\langle s_k \rangle)$. $\Gamma(G, S)$ defined this way has no multiedge or loop.

Let $V_i = \{x_j\langle s_i \rangle \mid x_j \in T_{s_i}\}$. Then $V = \bigcup_{i=1}^{k} V_i$. The number of vertices in V_i is simply the order of G divided by the order of s_i which is the index of $\langle s_i \rangle$ in G, denoted $[G : \langle s_i \rangle]$. The minimum number of elements required to generate a finite group G is called the *rank of* G. A *minimal generating set for* G is a subset $S = \{s_1, \ldots, s_k\}$ such that $G = \langle S \rangle$, where k is the rank of G. This concept is not to be confused with nonredundancy. A *nonredundant* set of generators is a set S such that S generates all of G, that is, $\langle S \rangle = G$, but no proper subset of S generates all of G.

The main object of this paper is to explore the planarity of $\Gamma(G, S)$.

Definition 1.1. A group G is $\mathbb{G}$-planar if there exists a generating set S such that the graph, $\Gamma(G, S)$, is a planar graph.

We recall a fundamental criterion for the $\mathbb{G}$-planarity of a group:

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Theorem 1.2 (Wagner). *A finite graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.*

2. Examples of $\mathbb{G}$-planar groups

The next two theorems give us two classes of $\mathbb{G}$-planar groups.

Theorem 2.1. *All cyclic groups are $\mathbb{G}$-planar.*

Proof. Let G be a cyclic group. Since G is cyclic, there exists an element $b \in G$ such that $\langle b\rangle = G$. Let $S = \{b\}$ be the generating set of G. Then $\Gamma(G, S)$ contains only one vertex and $\Gamma(G, S)$ is a planar graph. Therefore G is a $\mathbb{G}$-planar group. $\square$

For the dihedral group, D_n, let r be a rotation of $360°/n$ and let f be any reflection.

Proposition 2.2. *For $S = \{f, rf\}$, the graph $\Gamma(G, S)$ of the dihedral group D_n is the cycle of length $2n$, C_{2n}.*

Proof. Write

$$V_1 = \{\langle f\rangle, r\langle f\rangle, r^2\langle f\rangle, \dots, r^{n-1}\langle f\rangle\},$$
$$V_2 = \{\langle rf\rangle, r\langle rf\rangle, r^2\langle rf\rangle, \dots, r^{n-1}\langle rf\rangle\}.$$

Since f and rf are both reflections, their composition is a rotation. Denote this rotation by r^m.

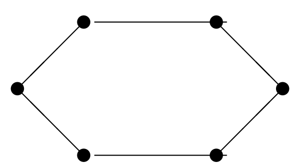
Choose a vertex from V_1, $r^s\langle f\rangle$. Since

$$r^s \in r^s\langle f\rangle \cap r^s\langle rf\rangle,$$

the edge $(r^s\langle f\rangle, r^s\langle rf\rangle)$ is in E. Now we need to show that there is another edge between $r^s\langle f\rangle$ and V_2. By simple calculation, we have $r^s f = r^{(s+m) \bmod n} rf$; moreover $(r^s\langle f\rangle, r^{(s+m) \bmod n}\langle rf\rangle)$ is in E.

Therefore the degree of each vertex in V_1 is 2. By similar arguments, the degree of each vertex in V_2 is 2 and $\Gamma(G, S)$ is a cycle. $\square$

Example 2.3. Let $G = D_3$ and $S = \{f, rf\}$. Then the $\mathbb{G}$-graph is the cycle C_6:



Theorem 2.4. *All dihedral groups are $\mathbb{G}$-planar.*

Proof. Let $G = D_n$ and $S = \{f, rf\}$. Since $\Gamma(G, S)$ is a cycle, $\Gamma(G, S)$ is a planar graph and G is a $\mathbb{G}$-planar group. $\square$

From [DeWitt et al. $\geq$ 2010], we have a few other examples of $\mathbb{G}$-planar groups.

Example 2.5. The modular group M has presentation

$$\langle s, t \mid s^8 = t^2 = e, st = ts^5 \rangle.$$

Let $S = \{s, ts\}$. From [DeWitt et al. $\geq$ 2010], $\Gamma(M, S)$ is $K_{2,2}$. Therefore $\Gamma(M, S)$ is a planar graph and M is a $\mathbb{G}$-planar group.

Example 2.6. The quasihedral group QS has presentation

$$\langle s, t \mid s^8 = t^2 = e, st = ts^3 \rangle.$$

Let $S = \{s, ts\}$. From [DeWitt et al. $\geq$ 2010], $\Gamma(QS, S)$ is $K_{2,4}$. Therefore $\Gamma(QS, S)$ is a planar graph and QS is a $\mathbb{G}$-planar group.

Recall that the generalized quaternion group Q_{2^n} has presentation

$$\langle s, t \mid s^{2^{n-1}} = e, s^{2^{n-2}} = t^2, tst^{-1} = s^{-1} \rangle.$$

Theorem 2.7. *The generalized quaternion group Q_{2^n} is $\mathbb{G}$-planar.*

Proof. Let $G = Q_{2^n}$ and $S = \{ts^k, ts^m\}$, where k is odd and m is even. $\Gamma(G, S)$ is a bipartite connected graph with every vertex of degree 2 [DeWitt et al. $\geq$ 2010]. Therefore, $\Gamma(G, S)$ is a cycle and Q_{2^n} is $\mathbb{G}$-planar. □

3. Finite abelian groups

The fundamental theorem of finite abelian groups tells us that every finite abelian group of rank k is isomorphic to a direct product of cyclic groups of prime-power order, that is, $G \cong \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_k}$. A *standard generating set for G* is a subset $S = \{s_1, \ldots, s_k\}$ such that $G = \langle s_1 \rangle \times \cdots \times \langle s_k \rangle$. Let G be an abelian group with standard generating set $S = \{s_1, \ldots, s_k\}$, then G is isomorphic to

$$\mathbb{Z}_{|s_1|} \times \mathbb{Z}_{|s_2|} \times \cdots \times \mathbb{Z}_{|s_k|}.$$

From Theorem 2.1, we know that all finite abelian groups with 1 generator are $\mathbb{G}$-planar. We now consider three cases: finite abelian groups with 4 or more generators, 3 generators or 2 generators.

Let G be a group with generating set S. There exists a subset of S, S', that is nonredundant and generates G. From [Bretto and Gillibert 2004], $\Gamma(G, S')$ is necessarily a subgraph of $\Gamma(G, S)$. If $\Gamma(G, S')$ is not a planar graph, then $\Gamma(G, S)$ is not planar. Therefore, it is only necessary to consider generating sets that are nonredundant.

Example 3.1. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_6$ and $S = \{(1, 0), (0, 0), (0, 2), (0, 3), (0, 4)\}$. The subset $S' = \{(1, 0), (0, 2), (0, 3)\}$ of S is a nonredundant generating set of G. The set $S'' = \{(1, 0), (0, 1)\}$ is a minimal generating set of G that is also nonredundant.

Lemma 3.2. *Let G be a finite abelian group and let $S = \{s_1, s_2, s_3, \dots, s_k\}$ be a nonredundant generating set, then $|s_i| \geq 2$ for all i.*

Proof. Assume $|s_i| < 2$. Then $|s_i| = 1$ and $\langle s_i \rangle = \{e\}$. Therefore s_i is not needed to generate G and $S \setminus \{s_i\}$ generates G. This is a contradiction. Therefore, $|s_i| \geq 2$. □

Finite abelian groups G with 4 or more generators.

Lemma 3.3. *Let G be a finite abelian group and let $S = \{s_1, s_2, s_3, s_4, \dots, s_k\}$ be a nonredundant generating set of G with $k \geq 4$. Consider the subgroup H of G that is generated by $S' = \{s_1, s_2, s_3, s_4\}$. The vertices $\langle s_1 \rangle$, $\langle s_2 \rangle$, $\langle s_3 \rangle$, $\langle s_4 \rangle$, $s_1\langle s_2 \rangle$, $s_2\langle s_1 \rangle$, $s_2\langle s_3 \rangle$, $s_3\langle s_2 \rangle$, $s_3\langle s_4 \rangle$, $s_4\langle s_3 \rangle$ of $\Gamma(H, S')$ are all unique.*

Proof. To see that each of these vertices is unique, assume $\langle s_1 \rangle, s_2\langle s_1 \rangle \in V_1$ are not distinct, that is, $\langle s_1 \rangle = s_2\langle s_1 \rangle$. So there exists $k \in \mathbb{Z}^+$ such that $s_2 = s_1^k$ which contradicts the fact that S is a nonredundant generating set of G. The proofs of the other cases are similar. □

Theorem 3.4. *Let G be a finite abelian group and let $S = \{s_1, s_2, s_3, s_4, \dots, s_k\}$ be a nonredundant generating set of G with $k \geq 4$. Then $\Gamma(G, S)$ is not a planar graph.*

Proof. Consider the subgroup H of G generated by $S' = \{s_1, s_2, s_3, s_4\}$. Define a contraction Γ of $\Gamma(H, S')$ in this way: Let $\overline{V}_1, \overline{V}_2, \overline{V}_3, \overline{V}_4, \overline{V}_5 \in V(\Gamma)$ with

$$\{\langle s_1 \rangle\} = \overline{V}_1, \quad \{\langle s_2 \rangle\} = \overline{V}_2, \quad \{\langle s_3 \rangle\} = \overline{V}_3, \quad \{\langle s_4 \rangle\} = \overline{V}_4,$$
$$\{s_1\langle s_2 \rangle, s_2\langle s_1 \rangle, s_2\langle s_3 \rangle, s_3\langle s_2 \rangle, s_3\langle s_4 \rangle, s_4\langle s_3 \rangle\} = \overline{V}_5.$$

Then, $e \in (\overline{V}_1 \cap \overline{V}_2)$, $e \in (\overline{V}_1 \cap \overline{V}_3)$, $e \in (\overline{V}_1 \cap \overline{V}_4)$, $s_1 \in (\overline{V}_1 \cap \overline{V}_5)$, $e \in (\overline{V}_2 \cap \overline{V}_3)$, $e \in (\overline{V}_2 \cap \overline{V}_4)$, $s_2 \in (\overline{V}_2 \cap \overline{V}_5)$, $e \in (\overline{V}_3 \cap \overline{V}_4)$, $s_3 \in (\overline{V}_3 \cap \overline{V}_5)$, and $s_4 \in (\overline{V}_4 \cap \overline{V}_5)$. Then $(\overline{V}_i, \overline{V}_j) \in E(\Gamma)$ for all $i \neq j$ and $\Gamma = K_5$. So, $\Gamma(H, S')$ has K_5 as a minor and $\Gamma(H, S')$ is not planar. From [Bretto et al. 2005], $\Gamma(H, S')$ is a subgraph of $\Gamma(G, S)$. Therefore, $\Gamma(G, S)$ is not a planar graph. □

Corollary 3.5. *Let G be a finite abelian group of rank 4 or more. Then G is not $\mathbb{G}$-planar.*

Finite abelian groups G with 3 generators.

Example 3.6. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with standard generating set

$$S = \{s_1, s_2, s_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.$$

The graph $\Gamma(G, S)$, illustrated in Figure 1, is a planar graph; hence G is a $\mathbb{G}$-planar group.

Next we show that this example is the only abelian group of rank three that is $\mathbb{G}$-planar.

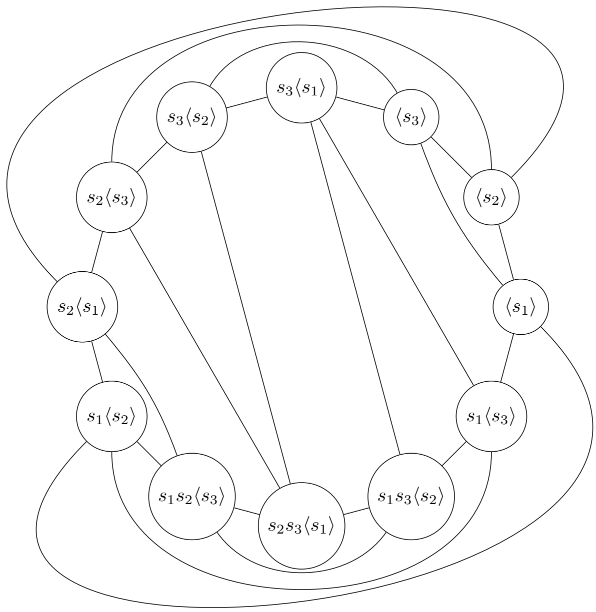


Figure 1. The graph $\Gamma(G, S)$, with $G = \mathbb{Z}_2{}^3$ and $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.

Lemma 3.7. *Let G be a finite abelian group with nonredundant generating set $S = \{s_1, s_2, s_3\}$ such that $|s_i| \geq 3$ for at least one i. Then the graph $\Gamma(G, S)$ contains at least* 16 *vertices.*

Proof. Without loss of generality, assume that $|s_3| \geq 3$. There are at least 6 vertices in V_1. They are $\langle s_1\rangle$, $s_2\langle s_1\rangle$, $s_3\langle s_1\rangle$, $s_2s_3\langle s_1\rangle$, $s_3^2\langle s_1\rangle$, $s_2s_3^2\langle s_1\rangle$. To see that each of these vertices is unique, assume $\langle s_1\rangle, s_2s_3\langle s_1\rangle \in V_1$ are not distinct, that is, $\langle s_1\rangle = s_2s_3\langle s_1\rangle$. So there exists $k \in \mathbb{Z}^+$ such that $s_2s_3 = s_1^k$ which contradicts the fact that S is a nonredundant generating set of G. The proofs of the other cases are similar.

Likewise, there are at least 6 unique vertices in V_2 and 4 unique vertices on V_3. They are $\langle s_2\rangle$, $s_1\langle s_2\rangle$, $s_3\langle s_2\rangle$, $s_1s_3\langle s_2\rangle$, $s_3^2\langle s_2\rangle$, $s_1s_3^2\langle s_2\rangle$ and $\langle s_3\rangle$, $s_1\langle s_3\rangle$, $s_2\langle s_3\rangle$, $s_1s_2\langle s_3\rangle$. □

Theorem 3.8. *Let G be a finite abelian group with nonredundant generating set $S = \{s_1, s_2, s_3\}$ such that $|s_i| \geq 3$ for at least one i. Then $\Gamma(G, S)$ is not a planar graph.*

Proof. Define a contraction Γ of $\Gamma(G, S)$ by setting

$$\overline{V}_1 = \{\langle s_1\rangle, \langle s_2\rangle\}, \qquad \overline{V}_2 = \{s_1\langle s_2\rangle, s_1s_2\langle s_3\rangle, s_1s_3\langle s_2\rangle\},$$
$$\overline{V}_3 = \{s_1\langle s_3\rangle, s_3^2\langle s_1\rangle, s_3^2\langle s_2\rangle\}, \qquad \overline{V}_4 = \{\langle s_3\rangle, s_3\langle s_2\rangle, s_3\langle s_1\rangle, s_2s_3\langle s_1\rangle\},$$
$$\overline{V}_5 = \{s_2\langle s_1\rangle, s_2\langle s_3\rangle, s_2s_3^2\langle s_1\rangle, s_1s_3^2\langle s_2\rangle\}.$$

Then

$$\begin{aligned} s_1 &\in (\overline{V}_1 \cap \overline{V}_2), & s_1 &\in (\overline{V}_1 \cap \overline{V}_3), \\ e &\in (\overline{V}_1 \cap \overline{V}_4), & s_2 &\in (\overline{V}_1 \cap \overline{V}_5), \\ s_1 &\in (\overline{V}_2 \cap \overline{V}_3), & s_1 s_2 s_3 &\in (\overline{V}_2 \cap \overline{V}_4), \\ s_1 s_2 &\in (\overline{V}_2 \cap \overline{V}_5), & s_3^2 &\in (\overline{V}_3 \cap \overline{V}_4), \\ s_3^2 s_2 &\in (\overline{V}_3 \cap \overline{V}_5), & s_2 s_3 &\in (\overline{V}_4 \cap \overline{V}_5). \end{aligned}$$

It follows that $(\overline{V}_i, \overline{V}_j) \in E(\Gamma)$ for all $i \neq j$ and $\Gamma = K_5$. So, $\Gamma(G, S)$ has K_5 as a minor and is not a planar graph. □

Corollary 3.9. *Let G be a finite abelian group of rank 3 such that $G \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then G is not a $\mathbb{G}$-planar group.*

Finite abelian groups G with 2 generators. Since we have results for groups of rank 1 and for groups of rank 3 or more, the only case left to consider is that of groups of rank 2. Notice that any finite abelian group of rank 2 is isomorphic to the direct product $\mathbb{Z}_m \times \mathbb{Z}_n$ with $gcd(m, n) \neq 1$.

Lemma 3.10. *Let G be a finite abelian group of rank 2 and let S be a nonredundant generating set of G. If $|S| \geq 3$, then $\Gamma(G, S)$ is not a planar graph.*

Proof. If $|S| > 3$, then $\Gamma(G, S)$ is not planar by Theorem 3.4. Assume that $|S| = 3$, that is, $S = \{s_1, s_2, s_3\}$ and that $|s_i| < 3$ for $i = 1, 2, 3$. Since S is nonredundant $|s_i| > 1$ and therefore $|s_i| = 2$ for $i = 1, 2, 3$. Consider the subset

$$H = \langle s_1 \rangle \langle s_2 \rangle = \{hk \mid h \in \langle s_1 \rangle, k \in \langle s_2 \rangle\} = \{e, s_1, s_2, s_1 s_2\}$$

of G. Since G is abelian, this subset is a subgroup. Now consider the subset

$$K = H\langle s_3 \rangle = \{hk \mid h \in H, k \in \langle s_3 \rangle\} = \{e, s_1, s_2, s_1 s_2, s_3, s_1 s_3, s_2 s_3, s_1 s_2 s_3\}$$

of G. Again K is necessarily a subgroup of G.

Now assume that $g \in G$. Since S generates G, there exists n, m, l such that $g = s_1^n s_2^m s_3^l$. Since the order of each generator is 2, n, m, l are congruent to 0 or 1 modulo 2 and $g \in K$. Therefore $G = K$. Since the order of each element in G is two, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. This is a contradiction since G is a group of rank 2. Therefore, $|s_i| \geq 3$ for at least one i and by Theorem 3.8 the graph, $\Gamma(G, S)$, is not planar. □

Theorem 3.11. *Let G be a finite abelian group of rank 2. G is $\mathbb{G}$-planar if and only if $G \cong \mathbb{Z}_2 \times \mathbb{Z}_k$, for some $k \in \mathbb{N}$.*

Proof. ($\Leftarrow$) Let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_k$ and let $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_k, S)$ be the associated $\mathbb{G}$-graph of $\mathbb{Z}_2 \times \mathbb{Z}_k$ with $S = \{(1, 0), (0, 1)\}$. There exist an isomorphism $\phi : \mathbb{Z}_2 \times \mathbb{Z}_k \to G$. Let $(x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_k$. There exists a, b such that $(x, y) = a(1, 0) + b(0, 1)$. Then $\phi(x, y) = \phi(a(1, 0) + b(0, 1)) = a\phi(1, 0) \oplus b\phi(0, 1)$. So $\phi(S) = \{\phi(1, 0), \phi(0, 1)\}$

Rank	Group	Planarity
1	all G	planar
2	$G \cong \mathbb{Z}_2 \times \mathbb{Z}_k$ $G \ncong \mathbb{Z}_2 \times \mathbb{Z}_k$	planar not planar
3	$G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ $G \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	planar not planar
4 or more	all G	not planar

Table 1. $\mathbb{G}$-planarity of finite abelian groups.

generates G. $\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_k, \mathrm{S})$ is $K_{k,2}$, so $K_{k,2} \cong \Gamma(G, \phi(S))$. Since $K_{k,2}$ is planar, $\Gamma(G, \phi(S))$ is planar. Therefore G is $\mathbb{G}$-planar.

($\Rightarrow$) Let G be a finite abelian $\mathbb{G}$-planar group of rank 2 and let S be a generating set such that $\Gamma(G, S)$ is a planar graph. From Lemma 3.10, $|S| = 2$, that is, $S = \{s_1, s_2\}$.

Case 1. Assume that $|s_1| = 2$. Let $|G| = n$, $|V_1| = [G : \langle s_1\rangle] = n/2$. So

$$V_1 = \{\langle s_1\rangle, s_2\langle s_1\rangle, s_2^2\langle s_1\rangle, \cdots, s_2^{n/2-1}\langle s_1\rangle\},$$

and the elements of G are of the form

$$s_2, s_2^2, \ldots, s_2^{n/2-1}, e \quad \text{and} \quad s_1s_2, s_1s_2^2, \ldots, s_1s_2^{n/2-1}, s_1.$$

Therefore $|s_2| = n/2$ and G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{n/2}$.

Case 2. Assume that $|s_1|, |s_2| > 2$. Consider the vertex induced subgraph generated by the six vertices $\langle s_1\rangle, s_2\langle s_1\rangle, s_2^2\langle s_1\rangle, \langle s_2\rangle, s_1\langle s_2\rangle, s_1^2\langle s_2\rangle$. This graph is $K_{3,3}$. Since this subgraph is not planar, $\Gamma(G, S)$ is not planar. This contradicts the supposition that S is a generating set such that $\Gamma(G, S)$ is a planar graph. Therefore, if G is $\mathbb{G}$-planar, then $G \cong \mathbb{Z}_2 \times \mathbb{Z}_k$. □

Table 1 summarizes the results for all finite abelian groups.

References

[Bauer et al. 2008] C. M. Bauer, C. K. Johnson, A. M. Rodriguez, B. D. Temple, and J. R. Daniel, "Paths and circuits in $\mathbb{G}$-graphs", *Involve* **1**:2 (2008), 135–144. MR 2009j:05108 Zbl 1146.05306

[Bretto and Gillibert 2004] A. Bretto and L. Gillibert, "Graphical and computational representation of groups", pp. 343–350 in *Computational science* (ICCS 2004) (Kraków, 2004), edited by M. Bubak et al., Lecture Notes in Comput. Sci. **3039**, Springer, Berlin, 2004. MR 2233213 Zbl 1102.68735

[Bretto et al. 2005] A. Bretto, L. Gillibert, and B. Laget, "Symmetric and semisymmetric graphs construction using G-graphs", pp. 61–67 in *Algorithms and Computation: 16th International Symposium, ISAAC* (Sanya, Hainan, China), edited by X. Deng and D.-Z. Du, ACM, New York, 2005. MR 2280530

[DeWitt et al. $\geq$ 2010] A. DeWitt, A. Rodriguez, and J. Daniel, "Paths and circuits in $\mathbb{G}$-graphs of certain non-abelian groups", *Furman Univ. Electr. J. Undergrad. Math.*. To appear.

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