involve
a journal of mathematics

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Donald Adams and Vadim Ponomarenko

mathematical sciences publishers

2010 vol. 3, no. 3

INVOLVE 3:3(2010)

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(Communicated by Scott Chapman)

Given $n, k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \mathbb{Z}_n$, we give conditions for the equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 1$ in $\mathbb{Z}_n$ to admit solutions with all the x_i distinct.

A sufficient condition is that $k \leq \phi(n)$ and a_i be invertible in $\mathbb{Z}_n$ for all i.

If $n > 2$ is prime, the following conditions together are necessary and sufficient: $k \leq n$, each a_i is nonzero, and either $k < n$ or not all of the a_i are equal.

1. Linear congruence

Given $n, k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \mathbb{Z}_n$, it is known classically [Uspensky and Heaslet 1939; Vandiver 1924] that the linear congruence

$$a_1x_1 + a_2x_2 + \cdots + a_kx_k = 1 \text{ (in } \mathbb{Z}_n) \tag{1}$$

has a solution if and only if $\gcd(a_1, a_2, \ldots, a_k) \in \mathbb{Z}_n^\times$, the group of units of $\mathbb{Z}_n$. We ask when such a solution exists with *distinct* $x_i \in \mathbb{Z}_n$, a question that appears to have been overlooked in the literature. In general, some additional conditions are necessary; for example, $1x_1 + 1x_2 + 1x_3 = 1$ does not have a solution with distinct $x_i \in \mathbb{Z}_3$.

Our partial solution has a stronger coefficient condition, and another restriction involving $\phi(n)$, the Euler totient. The general case remains open.

Theorem 1. *If $k \leq \phi(n)$ and $a_i \in \mathbb{Z}_n^\times$ $(1 \leq i \leq k)$, then there exist distinct $x_i \in \mathbb{Z}_n$ satisfying* (1).

Proof. We first construct $y_1, y_2, \ldots y_k$ iteratively, as will be explained. For notational convenience, for $i < j$ we set

$$y_{i,j} = y_i(1 - a_{i+1}y_{i+1})(1 - a_{i+2}y_{i+2}) \cdots (1 - a_{j-1}y_{j-1})$$

MSC2000: 11B50, 11D79.

Keywords: linear congruence, minimal zero-sum sequence, property B.

(note that $y_{i,i+1} = y_i$). We set $y_1 = a_1^{-1}$; for $j > 1$ we let y_j be any element chosen from $S_j \setminus T_j$, where

$$S_j = \{y \in \mathbb{Z}_n : 1 - a_j y \in \mathbb{Z}_n^\times\},$$
$$T_j = \{y \in \mathbb{Z}_n : y(1 + a_j y_{i,j}) = y_{i,j} \text{ for some } i \text{ with } 1 \le i < j\}.$$

Note that the defining property of S_j ensures that $1 - a_j y_j$ is invertible, and that T_j ensures that $y_j \neq y_{i,j}(1 - a_j y_j) = y_{i,j+1}$, for all $i < j$.

Now, set $x_i = y_{i,k+1}$ for $1 \le i \le k$. Note that $a_1 x_1 + a_2 x_2 + \cdots + a_k x_k$ conveniently telescopes to 1, because $a_1 y_1 = 1$. Suppose that $x_i = x_j$ (for $i < j$). Then

$$y_{i,k+1} = y_{j,k+1}.$$

We may cancel the common terms, because they were constructed to be invertible, to get $y_{i,j+1} = y_{j,j+1} = y_j$, which contradicts our construction of y_j. Hence the x_i are distinct, and a solution to (1).

It remains to prove that $S_j \setminus T_j$ is nonempty. We first prove that

$$|S_j| = |\mathbb{Z}_n^\times| = \phi(n),$$

by showing that $f(y) = 1 - a_j y$ is a bijection on $\mathbb{Z}_n$, and thus $f(S_j) = \mathbb{Z}_n^\times$. If $f(y) = f(y')$, then $1 - a_j y = 1 - a_j y'$ and $a_j(y - y') = 0$, but a_j is invertible, hence $y = y'$. So f is injective on a finite set and hence bijective. Finally, we prove that $|T_j| \le j - 1 \le k - 1 < k \le \phi(n)$, by showing that $y(1 + a_j y_{i,j}) = y_{i,j}$ has at most one solution y. If $(1 + a_j y_{i,j})$ is invertible, then $y = (1 + a_j y_{i,j})^{-1} y_{i,j}$ is unique. If not, then there is some $m > 1$ with $m|n$ and $m|(1 + a_j y_{i,j})$. If there is a solution y then also $m|y_{i,j}$, so $m|(1 + a_j y_{i,j}) - a_j y_{i,j} = 1$, a contradiction. □

If n is prime, we can do better, solving the problem completely. Clearly it is necessary that $k \le n$, and that not all a_i are zero, that is, $\gcd(a_1, a_2, \ldots, a_k) \in \mathbb{Z}_n^\times$.

Theorem 2. *Let n be an odd prime, $k \le n$, and $\gcd(a_1, a_2, \ldots, a_k) \in \mathbb{Z}_n^\times$. Then there exist distinct $x_i \in \mathbb{Z}_n$ satisfying* (1), *if and only if either* (a) *$k < n$, or* (b) *not all of the a_i are equal.*

Proof. The nonzero a_i are in $\mathbb{Z}_n^\times$, and $\phi(n) = n - 1$, so unless there are n nonzero a_i, we can apply Theorem 1, and arbitrarily assign leftover distinct elements from $\mathbb{Z}_n$ to those x_i where $a_i = 0$. If $k = n$ and $a_1 = \cdots = a_k = t$, then there is only one possible solution, and it fails because $t(0 + 1 + \cdots + n) = tn(n + 1)/2 = 0$ in $\mathbb{Z}_n$.

Remaining is the case where $k = n$, the a_i are all nonzero and not all equal. Set $a_i' = a_i - a_1$. More than zero, but less than n, of the a_i' are nonzero, so we can find

distinct $x_i \in \mathbb{Z}_n$ with $a_1' x_1 + \cdots a_n' x_n = 1$. But now we have

$$\begin{aligned} a_1 x_1 + \cdots + a_n x_n &= (a_1' + a_1)x_1 + \cdots + (a_n' + a_1)x_n \\ &= (a_1' x_1 + \cdots + a_n' x_n) + a_1(x_1 + \cdots + x_n) \\ &= 1 + a_1(0 + 1 + \cdots + n) \\ &= 1 + a_1 n(n + 1/2) = 1 \quad \text{in } \mathbb{Z}_n. \end{aligned}$$ □

In fact, we believe that a similar result holds for composite n; this is supported by preliminary computer calculations. For example, consider $n = 6$, $k = 5$, $(a_1, a_2, a_3, a_4, a_5) = (2, 2, 2, 3, 3)$. Neither of the strong conditions of Theorem 1 are met; however $(x_1, x_2, x_3, x_4, x_5) = (2, 4, 5, 0, 1)$ satisfies (1).

Conjecture 3. Let $k < n$ and $\gcd(a_1, a_2, \ldots a_k) \in \mathbb{Z}_n^\times$. Then there exist distinct $x_i \in \mathbb{Z}_n$ satisfying (1).

2. Application

Fix the finite abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$. We consider multisets[1] of elements such that their sum is zero; we call these zero-sum multisets. They have a rich literature and history [Geroldinger and Halter-Koch 2006], arising from fundamental number theoretic questions about nonunique factorization.

It is well known that the largest minimal (i.e. containing no other nontrivial zero-sum multiset) zero-sum multiset is of size $2n - 1$. Recently it has been shown [Gao et al. 2010] that any zero-sum multiset of this size contains some element of multiplicity $n - 1$. In [Gao and Geroldinger 2003] it was shown that the remaining multiplicities $a_1, a_2, \ldots a_k$ (where $a_1 + a_2 + \cdots + a_k = n$) must admit a solution to (1) in distinct elements of $\mathbb{Z}_n$, leaving open the question of when this occurs.

Corollary 4. *Let*

$$n > 0, \quad k \le \phi(n) \quad \text{and} \quad a_i \in \mathbb{N}, \quad \text{with} \begin{cases} a_1 + \cdots + a_k = n, \\ \gcd(a_i, n) = 1. \end{cases}$$

Then there is an irreducible zero-sum multiset in $\mathbb{Z}_n \times \mathbb{Z}_n$ whose elements have multiplicities $n - 1, a_1, a_2, \ldots, a_k$.

Corollary 5. *Let $n > 0$ be prime, $k \le n$, and $a_i \in \mathbb{N}$ with*

$$a_1 + \cdots + a_k = n \quad \text{and} \quad \gcd(a_1, a_2, \ldots, a_k, n) = 1.$$

Then there is an irreducible zero-sum multiset in $\mathbb{Z}_n \times \mathbb{Z}_n$ whose elements have multiplicities $n - 1, a_1, a_2, \ldots, a_k$ if and only if $1 < k < n$.

[1] For historical reasons these are called *sequences* in the literature, although the elements are not ordered.

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Received: 2010-07-07 Revised: 2010-09-29 Accepted: 2010-09-29

DJUNIOR82@gmail.com *Arizona State University, Tempe, AZ 85287-1804, United States*

vadim@sciences.sdsu.edu *San Diego State University, Department of Mathematics and Statistics, 5500 Campanile Dr., San Diego, CA 92182-7720, United States*
http://www-rohan.sdsu.edu/~vadim/

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

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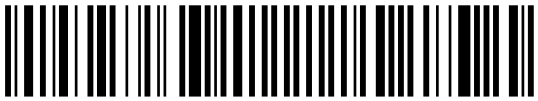
Typeset in LaTeX

involve
2010 vol. 3 no. 3



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