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A note on nonresidually solvable hyperlinear one-relator groups

Jon P. Bannon and Nicholas Noble

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# A note on nonresidually solvable hyperlinear one-relator groups

Jon P. Bannon and Nicholas Noblett

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We prove that various nonresidually finite, nonresidually solvable groups of the form  $\langle a, b \mid r^{r^w} = r^2 \rangle$  are sofic.

This paper concerns the sofic property discussed in the survey [Pestov 2008]. Particularly, we address Question 4.10 in that paper: the problem of Nate Brown asking whether or not every one-relator group is sofic. In [Bannon 2010], it is proved that the example in [Baumslag 1969] of a nonresidually finite nonresidually solvable one-relator group is a sofic group. The purpose of this paper is to exhibit more such examples in the following large class of nonresidually solvable one-relator groups introduced in [Baumslag et al. 2007]. Let  $\mathbb{F}_2 = \langle a, b \mid \rangle$  denote the free group on two generators. Let  $r, w \in \mathbb{F}_2$  be two elements that do not commute. In [Baumslag et al. 2007], the authors show that the group

$$\Gamma_{r,w} = \langle a, b \mid r^{r^w} = r^2 \rangle = \langle a, b \mid r = [r, (r^{-1})^w] \rangle$$

has the same finite quotients as the group

$$\langle a, b \mid r \rangle,$$

and is therefore not residually finite. We point out that none of the groups  $\Gamma_{r,w}$  are residually solvable, since  $r = [r, (r^{-1})^w]$  lies in every derived subgroup of  $\Gamma_{r,w}$ . In [Bannon 2010], it is shown that the group  $\Gamma_{ab,a}$  is sofic. The proof in [Bannon 2010] uses [Dykema 2010, Corollary 3.4], that HNN extensions of sofic groups over amenable subgroups remain sofic. The proof in [Bannon 2010] uses the fact that  $\Gamma_{ab,a}$  is an HNN extension of an amenable one-relator group. We shall extend this result to certain other of the groups  $\Gamma_{r,w}$ . If  $r$  and  $w$  generate  $\mathbb{F}_2$ , then  $\Gamma_{r,w}$  embeds naturally as a subgroup of  $\Gamma_{ab,a}$ , and since the sofic property passes to subgroups,  $\Gamma_{r,w}$  is sofic. The first result of this short note is that there exist  $r, w$  that do not generate  $\mathbb{F}_2$ , yet the group  $\Gamma_{r,w}$  is sofic. More precisely, we prove:

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**Theorem 1.** *The group  $\Gamma_{a,b^{-1}ab}$  is sofic.*

*Proof.* Since  $\Gamma_{a,b^{-1}ab} = \langle a, b \mid (bab^{-1})^{-2}a^{-1}(bab^{-1})^{-1}a(bab^{-1})a^{-1}(bab^{-1})a \rangle$ , following [McCool and Schupp 1973], we let  $a_0 = a$  and  $a_{-1} = bab^{-1}$  and realize  $\Gamma_{a,b^{-1}ab}$  as the HNN extension

$$\langle a_0, a_{-1}, t \mid (a_{-1})^{-2}a_0^{-1}(a_{-1})^{-1}a_0a_{-1}a_0^{-1}a_{-1}a_0, t^{-1}a_{-1}t = a_0 \rangle$$

of the one-relator group  $H_1 = \langle a_0, a_{-1} \mid a_0(a_{-1})^{-2}a_0^{-1}(a_{-1})^{-1}a_0a_{-1}a_0^{-1}a_{-1} \rangle$ , where by the Freiheitssatz  $\langle a_{-1} \rangle$  and  $\langle a_0 \rangle$  are copies of  $\mathbb{Z}$  which in the HNN extension we identify by identifying  $a_{-1}$  with  $a_0$ . Letting  $b_1 = a_0a_{-1}a_0^{-1}$  and  $b_0 = a_{-1}$  we may identify  $H_1$  as the HNN extension

$$\langle b_0, b_1, s \mid b_1^{-2}b_0^{-1}b_1b_0, s^{-1}b_1s = b_0 \rangle$$

of the one-relator group  $H_2 = \langle b_0, b_1, s \mid b_1^{-2}b_0^{-1}b_1b_0 \rangle$ , where we identify the two copies  $\langle b_0 \rangle$  and  $\langle b_1 \rangle$  of  $\mathbb{Z}$  as above. By [Ceccherini-Silberstein and Grigorchuk 1997], the group  $H_2$  is amenable, and hence by the argument in [Bannon 2010], the group  $H_1$  is sofic. Since  $\Gamma_{a,b^{-1}ab}$  is an HNN extension of a sofic group with respect to identified copies of the amenable group  $\mathbb{Z}$ , it follows that  $\Gamma_{a,b^{-1}ab}$  is sofic.  $\square$

In this proof we used in an essential way that the identified subgroups are amenable and therefore invoke the full hypotheses of Corollary 3.4 of [Dykema 2010], whereas in [Bannon 2010], the group  $\Gamma_{ab,a}$  is an HNN extension of an amenable group and so any pair of identified subgroups would work. We next illustrate that there are groups of the form  $\Gamma_{r,w}$  that do not in an obvious way fall to the method of [Bannon 2010].

**Theorem 2.** *The group  $\Gamma_{a,b^2} = \langle a, b \mid a = [a, (a^{-1})^{b^2}] \rangle$  is isomorphic to*

$$(G * \mathbb{Z}) *_{\mathbb{F}_2},$$

where  $G$  is a one-relator amenable group.

*Proof.* Since  $\Gamma_{a,b^2} = \langle a, b \mid a^{-2}(b^2ab^{-2})a(b^2ab^{-2})^{-1} \rangle$ , then letting  $a_0 = a$  and  $a_{-2} = b^2ab^{-2}$  we have that  $\Gamma_{a,b^2}$  is isomorphic to the HNN extension

$$\langle a_0, a_{-1}, a_{-2}, t \mid a_0^{-2}a_{-2}a_0(a_{-2})^{-1}, t^{-1}a_{-2}t = a_{-1}, t^{-1}a_{-1}t = a_0 \rangle$$

of the one-relator group  $\langle a_0, a_{-1}, a_{-2} \mid a_0^{-2}a_{-2}a_0(a_{-2})^{-1} \rangle$ , with the isomorphism from the free subgroup  $\langle a_{-2}, a_{-1} \rangle$  with  $\langle a_{-1}, a_0 \rangle$  extending the set map that sends  $a_{-2}$  to  $a_{-1}$  and  $a_{-1}$  to  $a_0$ . But the relator  $a_0^{-2}a_{-2}a_0(a_{-2})^{-1}$  does not involve  $a_{-1}$ , hence  $\langle a_0, a_{-1}, a_{-2} \mid a_0^{-2}a_{-2}a_0(a_{-2})^{-1} \rangle = \langle a_{-1} \rangle * \langle a_0, a_{-2} \mid a_0^{-2}a_{-2}a_0(a_{-2})^{-1} \rangle$ .  $\square$

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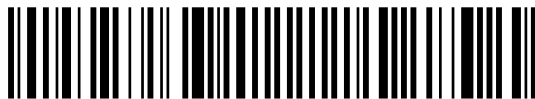
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