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Identification of localized structure in a nonlinear damped harmonic oscillator using Hamilton's principle

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In the mid-seventeenth century Isaac Newton formalized the language necessary to describe the evolution of physical systems. Newton argued that the evolution of the state of a process can be described entirely in terms of the forces involved with the process. About a century and a half later, William Hamilton was able to establish the whole of Newtonian mechanics without ever using the concept of force. Rather, Hamilton argued that a physical system will evolve in such a way as to extremize the integral of the difference between the kinetic and potential energies. This paradigmatic reformulation allows for a type of reverse engineering of physical systems. This paper will use the Hamiltonian formulation of a nonlinear damped harmonic oscillator with third and fifth order nonlinearities to establish the existence of localized solutions of the governing model. These localized solutions are commonly known in mathematical physics as solitons. The data obtained from the variational method will be used to numerically integrate the equation of motion, and find the exact solution numerically.

1. Introduction

For those that are familiar with mathematical modeling, it is common knowledge that most physical systems can be modeled by considering the sum of the forces acting on the system. These various forces appear explicitly in what is commonly referred to as *the equation of motion* or *the governing system*. These individual terms can be interpreted as forces, accelerations, potentials, external damping, dispersive effects, etc. It is most often the case that this equation is a differential equation. This equation can then be solved using the seemingly endless supply of tricks and techniques that have been developed over recent centuries by mathematicians and others. This approach to model construction was largely formalized by Isaac Newton and his contemporaries in the mid-seventeenth century. Newton offered

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a way to write down physical observations in a concise language which allowed for an accurate prediction of how the process would evolve in time. All one has to do is establish the net forces involved with the process, and solve the associated equation. This development, of course, ran concurrently with the advent of modern calculus. Newton's approach in describing physical processes by examining the net external forces reigned supreme in the collective conscious of scientists, engineers, and natural philosophers for over a century and a half.

William Hamilton, a physicist, later proposed in [1834; 1835] a completely different view of describing physical systems. While the Newtonian school held that a system evolves according to external influences, Hamilton argued that a physical system will evolve in such a way as to extremize certain mathematical quantities. Hamilton's formulation enabled him to establish the whole of Newtonian physics without ever using the concept of force. Instead, Hamilton correctly argued that a physical process will evolve in such a way as to extremize the integral of the difference between the kinetic and potential energies. Hamilton dubbed this quantity *the action*. With this, Hamilton offered humankind a paradigm shift in considering the evolution of physical processes.

Hamilton was able to develop his force-independent theory of physical systems thanks to the mathematics which had been developed over almost two centuries since Newton presented his force-driven theory of the universe. The mathematical roots of Hamilton's theory can be traced back to what is known as the *brachistochrone problem* [Nesbet 2003]. In 1696, Johann Bernoulli proposed this problem, which can be restated thus: *If two points are connected by a wire whose shape is given by an unknown function $y(x)$ in a vertical plane, what shape function minimizes the time of descent of a bead sliding without friction from the higher to the lower point*? This problem was addressed by several of Bernoulli's contemporaries, and what arose from these investigations was a new type of calculus, known today as the calculus of variations and developed into a full mathematical theory by Euler around 1744 (see [Rouse Ball 1901], for instance). The mathematics developed by Euler was extended by Joseph-Louis Lagrange (1736–1813), who discovered that Euler's equation for minimizing a functional integral (later to be named the Euler–Lagrange equation) could be expressed in a compact way by simply using integration by parts (see [O'Connor and Robertson 1999], for instance). It was Lagrange who introduced the integrand of the functional appropriate to mechanics, that is, the difference between potential and kinetic energies.

Around the time William Hamilton was developing his new style of physics, a man named John Scott Russell had made the first observation of a phenomena that would one day become relevant in constructing a global telecommunications network. As a civil engineer, Russell was quite active in the advancement of naval-vessel architecture. Russell revolutionized naval architecture by creating a new

system of hull construction, he was the first person to offer steam carriage service between Paisley and Glasgow in 1834, and he is also responsible for some of the first recorded experimental data of the Doppler effect of the sound frequency shift of passing trains [Eilbeck 2007]. One day in 1834, Russell was running an experiment in order to establish a conversion factor between steam power and horse power. As part of his experimental apparatus, he tethered horses to a boat. At some point in the experiment, things went wrong: the ropes binding the horses to the boat snapped. Russell curiously watched as a huge swell of water formed around the hull of the stalled boat. Suddenly this mound of water sprang forward and began propagating down the Union Canal. Russell and his trusty steed followed the traveling water crest until it dissipated in the standing water several kilometers away from the boat. What was so astonishing to Russell was the absolute lack of attenuation or dissipation of the water wave over such a long distance. Water wave dynamics, as understood in the nineteenth century, did not allow for such *waves of permanent form*, as Russell named them. (Today, mathematical solutions of this type of are known as solitons, a name arising arose from the concept of a solitary wave.) Much skepticism surrounded Russell's claim, and he dedicated his remaining days to recreating the phenomena he observed that fateful day along the Union Canal.

The disbelief of Russell's contemporaries lies with the formulation of the model of water wave equations at a relatively shallow depth. It was not until 1895 that two mathematicians, D. J. Korteweg and G. de Vries, successfully constructed a mathematical model which affirmed Russell's observation sixty years earlier. Korteweg and de Vries derived what is known today as the KdV equation:

$$v_t + vv_x + v_{xxx} = 0. \tag{1}$$

Correctly describing the behavior of shallow water waves [Debnath 1997], this is a nonlinear evolution equation (subscript x and t denote differentiation with respect to the space and time coordinates). Russell's contemporaries' disbelief was due to the fact that their models of the behavior of water wave dynamics were linear. Due to the complexity of solving nonlinear evolution equations, research in the area of solitons stalled until the 1960s. Its renaissance was, of course, due to the advent of the modern computer.

In the mid 1960s Martin David Kruskal ran the first numerical simulation of interacting solitons in the KdV equation. This early work contributed much to our current understanding of these rather exotic types of mathematical constructs. What Kruskal was able to demonstrate not only advanced applied mathematics, but forced mathematicians and physicists to reconsider the very notion of what is meant by interacting waves. Kruskal found that when two solitons interact, they will exhibit some level of interference much like any wave phenomena which

is observed in nature, though they do not linearly superimpose on one another. The difference is that upon interacting, the solitons will return to their original state. That is to say, once the interaction had taken place, the soliton reestablishes its original shape, velocity, and other governing physical characteristics with a possible phase shift being the only observable consequence of the interaction. Understanding these types of mathematical constructs has lead to some of the more profound advancements in the last few decades. Most notable of these advances are fiber optic and wireless communication over a global network. This paper will introduce a novel way to establish such localized structure (i.e., solitons) without the difficulties encountered by techniques which require working with the equation from the Newtonian perspective.

2. Hamilton's principle

As mentioned in the previous section, solitons do not exist in linear equations. They only occur in nonlinear differential equations. Throughout the last several decades many techniques have been developed in establishing solutions to nonlinear differential equations [Debnath 1997; Drazin and Johnson 1989]. These techniques are characterized by their limited reach in solving large classes of problems. They are also characterized by being rather complicated. The most famous of these techniques is what is known as the inverse scattering technique. During the late 1960s Kruskal continued his work with solitons and developed what amounts to an analogous form of a Fourier transform for nonlinear differential equations. This work was developed by Kruskal in conjunction with three other mathematicians: Clifford Gardner, John Greene, and Robert Miura [Gardner et al. 1967]. What came out of this work is what is known as the inverse scattering technique (IST). The early development of the IST had a shortcoming: it only applied to integrable equations. In 1972 four young mathematicians, Mark Ablowitz, David Kaup, Alan Newell, and Harvey Segur, established what is known as the AKNS theory [Ablowitz et al. 1974]. This was an extension of the IST that ultimately allowed for analysis of nonintegrable systems. While these techniques have shaped modern applied mathematics, they are complicated and require a great deal of specialized understanding to be used effectively [Drazin and Johnson 1989]. Additionally, IST considers solutions to the nonlinear evolution equation as it is postulated in the Newtonian sense via considering the net forces acting on the system. So let us begin by considering some nonlinear differential operator, Φ, for which there exists some v satisfying

$$\Phi[v] = 0. \tag{2}$$

This represents the Newtonian formulation of the physical system. Depending on the structure of Φ, solutions to (2) most likely cannot be found directly. In fact,

it is difficult to make any generalization about (2) without assuming some sort of additional structure on Φ. Instead of restricting Φ to a certain class of nonlinear differential operators, consider a paradigmatic reformulation of (2). Suppose this nonlinear operator is the derivative (in some sense) of some associated *energy functional* L, a fact we write as

$$\Phi[v] = \nabla(L[v]). \tag{3}$$

Equation (2) may now be written in terms of the energy functional:

$$\nabla(L[v]) = 0. \tag{4}$$

This establishes a duality in which solutions to (2) are seen as the critical points of the functional $L[\,\cdot\,]$. This is the heart of Hamilton's principle. This approach enabled Hamilton to describe the entirety of Newtonian mechanics without having to consider the evolution of a system in terms of external forces. In modern mathematics this energy functional is termed the *Lagrangian*, and its formulation depends on the physical system of interest.

Suppose the expression of the Lagrangian is known, and that it is a functional of the variable $v(t)$, which itself may be a scalar, vector or tensor quantity. In the present work, we shall only consider a one-dimensional scalar case, where the integration variable is t. If we let $\mathcal{D}$ be the domain of support of the function v, the action, $S[v]$, is defined by

$$S = \int_{\mathcal{D}} L[v]\,dt. \tag{5}$$

Hamilton's principle states that the evolution of a dynamical system between two specific states is an extremum of the action functional given by (5). More formally, Hamilton's principle states that the solution to a given dynamical system $v(t)$ is determined by (6) for any bounded variation $\delta v(t)$, provided that this variation vanishes at any and all end points of the domain $\mathcal{D}$ [Kaup and Vogel 2007]. Note that this also defines the quantity $(\delta L/\delta v)(t)$, which is called the (first) variational derivative of L:

$$\lim_{\epsilon\to 0} \frac{S[v(t)+\epsilon\delta v(t)] - S[v(t)]}{\epsilon} = \int_{\mathcal{D}} \frac{\delta L}{\delta v}[v(t)]\,\delta v(t)\,dt = 0. \tag{6}$$

In terms of the nonlinear differential operator Φ, this establishes a connection between the governing equation of motion and the first variational derivative of the Lagrangian:

$$\Phi[\,\cdot\,] = \frac{\delta L[\,\cdot\,]}{\delta v}. \tag{7}$$

This paradigm shift offered by Hamilton allows for a rather novel approach to approximating solutions of evolution equations for which a Lagrangian can be

established. Suppose the physical characteristics (geometric or otherwise) of a particular type of solution to the equation of motion given by (2) are known. For instance, an ordinary soliton could be described in terms of a traveling "lump" having some associated amplitude and width. Of course, depending on the governing system, the solution could have other identifying characteristics such as position, velocity, chirp, phase, etc. An *ansatz*, or tentative functional form, can then be constructed in terms of parameters representing those physical characteristics. Let $v_0(t; q_i)$ be the ansatz, where the q_i from a finite collection of parameters representing the physical characteristics in question, and on which v_0 is dependent; these parameters could also depend on other independent variables, such as t. With the functional form of v_0 fixed, we can vary the q_i, and this variation gives a set of equations expressing the extremum principle:

$$\frac{\partial S}{\partial q_i} = \frac{\partial}{\partial q_i} \int_{\mathcal{D}} L[v_0]\, dt = 0. \tag{8}$$

(This notation presumes the structure of the q_i is constant. If the parameters are assumed to be dependent on time, the partial derivative would become a functional derivative.) Once this is done, we have the q_i determined in the sense that we have the (algebraic or differential) equations whose solutions represent a best fit for the parameter values according to Hamilton's principle. The nonlinear differential equation considered for analysis in this paper is

$$v'' + \kappa v' + \varphi v + v^3 + \omega v^5 = 0, \tag{9}$$

where $\kappa, \varphi, \omega \in \mathbb{R}$ and $v(\xi) : \mathbb{R} \to \mathbb{R}$. The prime indicates the differentiation with respect to the independent variable ξ.

3. Ordinary solitons

This paper will establish the existence of two different types of solitons for (9). The first type is known as *ordinary solitons*: localized solutions that occur in a region of the extrinsic parameter space, in this case $(\kappa, \varphi, \omega)$-space, for which the linear eigenmodes are exponential. The most frequent type of localized structure identified in nonlinear evolution equations are those solutions for which $v(\xi) \to 0$ as $\xi \to \pm\infty$. Ordinary solitons for which $v(\xi) \to 0$ as $\xi \to \pm\infty$ are referred to as *bright solitons*, while those with the asymptotic behavior $v(\xi) \to c$ where $c \in \mathbb{R}$ as $\xi \to \pm\infty$ are called *dark*. We will only consider bright solitons in the current work. Requiring a vanishing amplitude for very large values of ξ means that the eigenvalues of the linearized problem must remain real (otherwise, there would oscillatory behavior in the eigenmodes). The eigenvalues of the linearization of

(9) are given by

$$\lambda_{\pm} = \frac{-\kappa \pm \sqrt{\kappa^2 - 4\varphi}}{2}. \tag{10}$$

In order to keep them real-valued, it will be necessary to impose on the extrinsic parameter space the condition that $\kappa^2 - 4\varphi > 0$.

To begin the process of using Hamilton's principle to identify ordinary solitons, it is necessary to have the Lagrangian associated with (9) and an ansatz representing the geometry of the desired solution. A combination of inspection and trial and error shows that the Lagrangian from which (9) arises is given by

$$L(\xi, v, v') = \frac{e^{\kappa\xi}}{2}\Big(\varphi v^2 - (v')^2 + \frac{v^4}{2} + \frac{\omega v^6}{3}\Big); \tag{11}$$

this is the L that which recovers the equation of motion (9) under the associated Euler–Lagrange equation $L_v - (d/d\xi)L_{v'} = 0$. The ansatz for the soliton will be taken as

$$v_0(\xi; a, \rho) = a \exp\Big(-\frac{\xi^2}{\rho^2}\Big), \tag{12}$$

that is, a Gaussian function of amplitude a and core width ρ. There are two good reasons for choosing a trial function such as this: (i) it offers a relatively good geometric description of an ordinary soliton; and (ii) the Lagrangian evaluated at the ansatz is easy to integrate over $\mathbb{R}$. While other functional forms such $\operatorname{sech}^2(\xi)$ have similar geometric properties, it may become quite difficult to calculate the action. It could become necessary, for instance, to lift the integration into the complex plane to calculate the associated action.

Calculating the action as defined in (5), where the function $v(\xi)$ is evaluated at the ansatz ($v = v_0$) gives rise to the action

$$\begin{aligned} &S(a, \rho) \\ &= -\frac{a^2 e^{\frac{\rho^2\kappa^2}{24}}\sqrt{\pi}}{144\rho}\Big(-18a^2 e^{\frac{\rho^2\kappa^2}{48}}\rho^2 + 9\sqrt{2}e^{\frac{\rho^2\kappa^2}{12}}(4+\rho^2(\kappa^2-4\varphi)) - 4\sqrt{6}a^4\rho^2\omega\Big). \end{aligned} \tag{13}$$

As discussed in Section 2, Hamilton's principle states that solutions of (9) will evolve in such a way as to extremize the action. While Hamilton's principle cannot be satisfied in general by using the trial function, it is possible to establish what values of variational parameters bring the ansatz *closest* to the exact solution. In general these parameters could vary with respect to some independent variable (such as time). If this were the case, the action would be varied with respect to a and ρ by way of the functional derivative (i.e., $S_{q_i} - (d/d\xi)S_{q_i'}$, where the q_i represents the variational parameters). Since the current analysis presumes the structure of the variational parameters to be constant, varying the action amounts to taking the partial derivative with respect to the variational parameters.

Varying the action with respect to a and ρ gives, respectively,

$$S_a = -\frac{a e^{\frac{\rho^2\kappa^2}{24}}\sqrt{\pi}}{24\rho}\left(-12a^2 e^{\frac{\rho^2\kappa^2}{48}}\rho^2 + 3\sqrt{2}e^{\frac{\rho^2\kappa^2}{12}}(4+\rho^2(\kappa^2-4\varphi)) - 4\sqrt{6}a^4\rho^2\omega\right), \quad (14)$$

$$S_\rho = -\frac{a^2 e^{\frac{\rho^2\kappa^2}{24}}\sqrt{\pi}\left(+27\sqrt{2}e^{\frac{\rho^2\kappa^2}{12}}(-16+8\rho^2(\kappa^2-2\varphi)+\rho^4(\kappa^4-4\kappa^2\varphi))\right)}{1728\rho^2}$$
$$-\frac{a^2 e^{\frac{\rho^2\kappa^2}{24}}\sqrt{\pi}\left(-27a^2 e^{\frac{\rho^2\kappa^2}{48}}\rho^2(8+\rho^2\kappa^2) - 4\sqrt{6}a^4\rho^2(12+\rho^2\kappa^2)\omega\right)}{1728\rho^2}. \quad (15)$$

The variational solution space is five-dimensional — there are three extrinsic parameters κ, ω, and φ, plus two variational parameters ρ and a. We are interested in the points $(\kappa, \varphi, \omega; a, \rho)$ that represent best parameter fits for the ansatz (that is, satisfy $S_\rho = 0$ and $S_a = 0$) and also satisfy the condition $\kappa^2 - 4\varphi > 0$, which we established by considerations from the linear spectrum. It is clear from the expressions (14) and (15) for S_ρ and S_a that solutions must be obtained numerically.

Since there are too many degrees of freedom, we fix two of the parameters on any given run — we chose the variational amplitude a and the linear damping κ — and take a third parameter to be an independent variable. We chose for this role the core width, ρ, which is always positive. Thus we step through values of ρ and search numerically for values of φ and ω satisfying the Euler–Lagrange equations, i.e., making (14) and (15) vanish. We discard solutions that do not satisfy the condition $\kappa^2 - 4\varphi > 0$. Some results of this process can be seen in Figure 1. As expected, the variational method indicates the persistence of many solution curves satisfying the Euler–Lagrange equations. It is often the case that ordinary solitons in a nonlinear evolution equation occur in infinite families. The geometric characteristics of ordinary solitons (such as the amplitude) very often depend continuously on

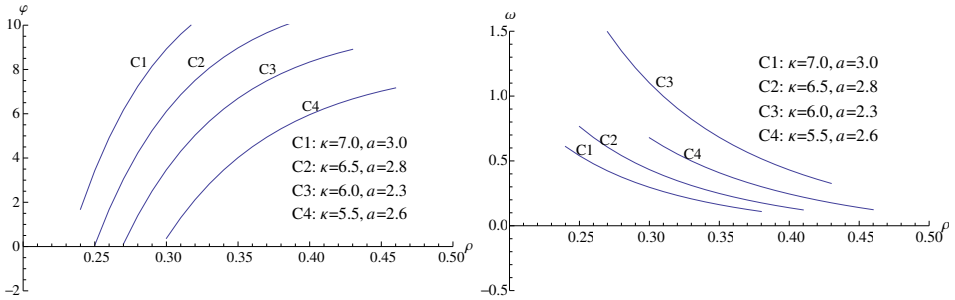

Figure 1. Projections in the (φ, ρ)- and (ω, ρ)-planes of some curves in parameter space satisfying $S_a = 0$ and $S_\rho = 0$. A point in parameter space is given by $(\kappa, \varphi, \omega; a, \rho)$; given a starting point, the numerical integration of (9) yields an "exact" solution.

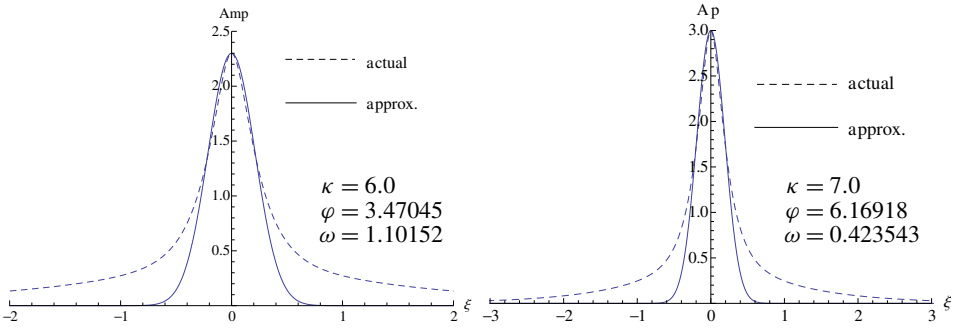

Figure 2. Ordinary solitons with the given extrinsic parameter values.

some extrinsic parameter in the governing model (for instance, the wave speed). Thus it is not surprising to find continuous curves in parameter space for which the variational method picks up localized structure.

With an abundance of data indicating the existence of localized structure, it is now time to numerically integrate (9) using this data, and find the exact (numerical) solutions. Integrating (9) requires two initial conditions: $v(0) = a$, where a is obtained from the variational data, and $v_\xi(0) = 0$ (from symmetry considerations). Figure 2 illustrates the result of this process for some selected solutions. Overlaid with the results of numerically integrating the model (9) is the ansatz evaluated at the variational solution data.

4. Embedded solitons

Embedded solitons get their name from the peculiar place in which they reside in the linear spectrum. Recall that for the case of ordinary solitons, asymptotic considerations lead us to require (in a very natural way) that the linear eigenmodes remain exponential. As it turns out, localized structure can exist in the spectrum of radiation modes. This breed of soliton is one of the relatively new types discovered in the last decade and a half [Yang et al. 1999; Champneys et al. 2001; Kaup and Malomed 2003]. It is possible that there are discrete values in the parameter space in which the nonlinearity is capable of "switching off" the radiation present in the linear eigenmodes. For this reason, these types of solitons became known as embedded solitons since they exist for parameter values embedded in the spectrum of radiation modes. A comprehensive explanation for the existence of such anomalous solutions is offered in [Champneys et al. 2001]. Thus we will restrict our attention to the region of parameter space where the linear eigenvalues determined by (10) are complex-valued. Once again, this solution will be established by way of considering a variational approximation. The ansatz is modified to allow for an additive radiation term:

$$v_0(\xi; a, \rho, \alpha) = a \exp\left(-\frac{\xi^2}{\rho^2}\right) + \alpha \cos(\psi\xi). \tag{16}$$

This variational trial function has been used in identifying embedded solitons in other systems [Kaup and Malomed 2003]. The $\cos(\psi\xi)$ structure is intentionally chosen to adhere to the symmetry of the core of the variational ansatz. The parameter ψ appearing in the phase of the radiation is *not* an additional parameter; it is a constant that will be determined in terms of variational and extrinsic parameters.

As indicated in Section 2, the variational trial function is then used to establish the action. This becomes quite tricky with an ansatz of the form (16). In general, this action integral will not converge. Upon inserting (16) into (11), some of terms can be integrated over all space, while others cannot. Here is the trick. To begin, the $\exp(\kappa\xi)$ factor arising in the Lagrangian from the damping is combined with the Gaussian structures by completing the square (the particulars of the substitution will vary from term to term). Then an effort is made, using various trigonometric identities, to isolate terms which are pure radiation. Such terms don't converge in the strict sense, but if the action is considered to be an *averaged* integral, the radiation can be considered to have a net zero contribution over all space. This approach was established in [Kaup and Malomed 2003]. Throughout the process of applying trig identities, terms which are not pure radiation (and are divergent) are generated. This is where the extra degree of freedom, ψ, in the phase of the radiation comes into play: ψ is established in such a way as to cause the remaining divergent terms to vanish. For this particular situation, ψ was calculated to be

$$\psi = \sqrt{\varphi + \tfrac{3}{8}\alpha^2 + \tfrac{1}{3}\alpha^4\omega}. \tag{17}$$

Upon taking the variational ansatz determined by (16) with a phase adjustment given by (17) and averaging out radiation terms, the effective action is found to be

$$\begin{aligned}
S(a,\rho,\alpha) &= \frac{a\sqrt{\pi}}{1440\rho} \\
&\times\Big(180a^3e^{\rho^2\kappa^2/16}\rho^2 - 90\sqrt{2}ae^{\rho^2\kappa^2/8}(4+\rho^2(\kappa^2-4\varphi)) + 40\sqrt{6}a^5e^{\rho^2\kappa^2/24}\rho^2\omega \\
&\quad + 288\sqrt{5}a^4e^{\rho^2(\kappa^2-\psi^2)/20}\rho^2\alpha\omega\cos{}^{P}\!/_{10} + 480\sqrt{3}a^2e^{\rho^2(\kappa^2-\psi^2)/12}\rho^2\alpha\cos{}^{P}\!/_{6} \\
&\quad + 900a^3e^{\rho^2(\kappa^2-4\psi^2)/16}\rho^2\alpha^2\omega\big(e^{\rho^2\psi^2/4} + \cos{}^{P}\!/_{4}\big) \\
&\quad + 1440e^{\rho^2(\kappa^2-\psi^2)/4}\rho^2\alpha\varphi\cos{}^{P}\!/_{2} + \sqrt{2}ae^{\rho^2(\kappa^2-4\psi^2)/8}\rho^2\alpha^2\big(e^{\rho^2\psi^2/2} + \cos{}^{P}\!/_{2}\big) \\
&\quad + 400\sqrt{3}a^2e^{\rho^2(\kappa^2-9\psi^2)/12}\rho^2\alpha^3\omega\big(3e^{2\rho^2\psi^2/3}\cos{}^{P}\!/_{6} + \cos{}^{P}\!/_{2}\big) \\
&\quad + 225\sqrt{2}ae^{\rho^2(\kappa^2-16\psi^2)/8}\rho^2\alpha^4\omega\big(3e^{2\rho^2\psi^2} + 4e^{3\rho^2\psi^2/2}\cos{}^{P}\!/_{2} + \cos P\big) \\
&\quad + 360e^{\rho^2(\kappa^2-9\psi^2)/4}\rho^2\alpha^3\big(3e^{2\rho^2\psi^2}\cos{}^{P}\!/_{2} + \cos{}^{3P}\!/_{2}\big)\big) \\
&\quad + 90e^{\rho^2(\kappa^2-25\psi^2)/4}\rho^2\alpha^5\omega\big(10e^{6\rho^2\psi^2}\cos{}^{P}\!/_{2} + 5e^{4\rho^2\psi^2}\cos{}^{3P}\!/_{2} + \cos{}^{5P}\!/_{2}\big) \\
&\quad - 1440e^{\rho^2(\kappa^2-\psi^2)/}4\rho^2\alpha\psi\big(\psi\cos{}^{P}\!/_{2} + \kappa\sin{}^{P}\!/_{2}\big)\Big),
\end{aligned}$$

where we have introduced the shorthand $P = \rho^2\kappa\psi$.

The action is then varied with respect to the three variational parameters a, ρ, and α. Since the embedded soliton itself has no radiation present (it is a purely localized solution) α will be taken to be zero *after* varying the action with respect to each parameter. This approach is discussed in [Kaup and Malomed 2003]. This results in the following three associated Euler–Lagrange equations:

$$(S_a)_{\alpha=0} = -24\sqrt{2}ae^{\rho^2\kappa^2/8} + 24a^3e^{\rho^2\kappa^2/16}\rho^2 - 6\sqrt{2}ae^{\rho^2\kappa^2/8}\rho^2\kappa^2 + 24\sqrt{2}ae^{\rho^2\kappa^2/8}\rho^2\varphi + 8\sqrt{6}a^5e^{\rho^2\kappa^2/24}\rho^2\omega, \quad (18)$$

$$(S_\rho)_{\alpha=0} = -27a^2e^{\rho^2\kappa^2/48}\rho^2(8+\rho^2\kappa^2) + 27\sqrt{2}e^{\rho^2\kappa^2/12}\big(-16+8\rho^2(\kappa^2-2\varphi)+\rho^4(\kappa^4-4\kappa^2\varphi)\big) - 4\sqrt{6}a^4\rho^2(12+\rho^2\kappa^2)\omega, \quad (19)$$

$$(S_\alpha)_{\alpha=0} = 3\sqrt{5}a^4\omega\cos\big(\tfrac{1}{10}\rho^2\kappa\sqrt{\varphi}\big) + 5\sqrt{3}a^2e^{\rho^2(\kappa^2-\varphi)/30}\cos\big(\tfrac{1}{6}\rho^2\kappa\sqrt{\varphi}\big) - 15e^{\rho^2(\kappa^2-\varphi)/5}\kappa\sqrt{\varphi}\sin\big(\tfrac{1}{2}\rho^2\kappa\sqrt{\varphi}\big). \quad (20)$$

These equations are then solved numerically in a similar fashion to the Euler–Lagrange equations in Section 3. Though this time there are three algebraic constraints. Possible solutions to these equations are also vetted according to the condition $\kappa^2 - 4\varphi < 0$ to ensure there is radiation present in the linear eigenmodes. Unlike the variational solution data obtained in Section 3, embedded solitons do not normally occur as continuous curves in parameter space. Rather, embedded solitons usually exist for a discrete value in the parameter space. We found one such solution, shown in Figures 3 and 4.

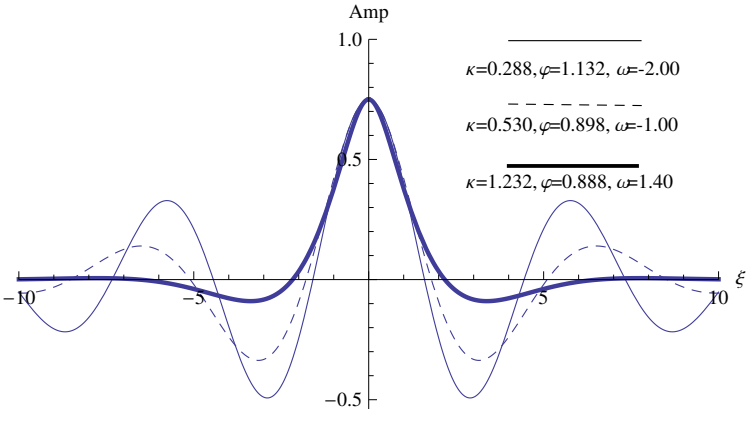

Figure 3. Delocalized solitons: results of numerically integrating (9) with (κ, ϕ, ω) values near those of the embedded soliton. The radiation dissipates as the parameter values approach the limit.

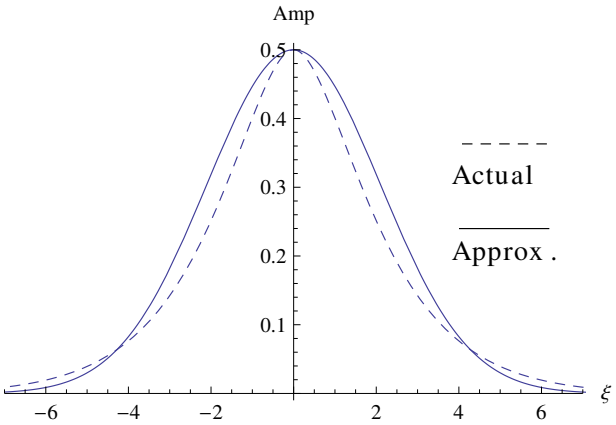

Figure 4. Exact solution (obtained via numerical integration) versus the variational trial function for extrinsic parameter values $\kappa = 1.1595$, $\varphi = 0.366215$, and $\omega = -0.101605$.

This figure also offers an overlay of the variational ansatz evaluated at the parameter values obtained by solving (18)–(20). Also included in Figure 3 is the result of integrating the equation of motion (9) *near* values for which the embedded soliton exists. These are commonly referred to as delocalized solitons. The closer the parameter values get to that of the embedded soliton the closer the amplitude of the radiation gets to zero.

5. Conclusions

The results obtained in this paper demonstrate the effectiveness and relative accuracy of using Hamilton's principle to establish localized structure in nonlinear evolution equations. These techniques are not limited to nonlinear equations. They can be implemented on just about any type of differential equation whether it is nonlinear or linear, partial or ordinary. As long as the Lagrangian can be established and there is some general understanding of the geometric characteristics of the desired solution, a variational method can be implemented. It has been shown [Kaup and Vogel 2007] that the variational method can fail to give reasonably accurate results in situations such as tracking soliton versus soliton interactions in a governing system. The approach outlined in this paper has the advantage of being able to establish solutions with relative ease when compared to some of the more complicated approaches available (e.g., inverse scattering techniques, calculating homoclinic orbits in phase space, etc.). In fact this methodology is accessible enough that advanced undergraduate students (such as the coauthor of this paper) with a good deal of mathematical maturity can use it in their own research projects.

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