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The arithmetic of the natural numbers $\mathbb{N}$ can be extended to arithmetic operations on planar binary trees. This gives rise to a noncommutative arithmetic theory. In this exposition, we describe this *arithmetree*, first defined by Loday, and investigate prime trees.

1. Introduction

J.-L. Loday [2002] published a paper *Arithmetree*, in which he defines arithmetic operations on the set $\mathbb{Y}$ of groves of planar binary trees. These operations extend the usual addition and multiplication on the natural numbers $\mathbb{N}$ in the sense that there is an embedding $\mathbb{N} \hookrightarrow \mathbb{Y}$, and the multiplication and addition he defines become the usual ones when restricted to $\mathbb{N}$. Loday's reasons for introducing these notions have to do with intricate algebraic structures known as dendriform algebras [Loday et al. 2001].

Since the arithmetic extends the usual operations on $\mathbb{N}$, one can ask many of the same questions that arise in the natural numbers. In this exposition, we examine notions of primality, specifically studying *prime trees*. We will see that all trees of prime degree must be prime, but many trees of composite degree are also prime. One should not be misled by the idea that arithmetree is an extension of the usual arithmetic on $\mathbb{N}$. Indeed, away from the image of $\mathbb{N}$ in $\mathbb{Y}$, the arithmetic operations $+$ and $\times$ are noncommutative. Both operations are associative, but multiplication is only distributive on the left with respect to $+$. In the end it is somewhat surprising that there is a very natural copy of $\mathbb{N}$ inside $\mathbb{Y}$.

The paper is organized as follows. Sections 2–6 summarize without proofs the results that we need from [Loday 2002]. Specifically, basic definitions are given in Section 2 to set notation. The embedding $\mathbb{N} \hookrightarrow \mathbb{Y}$ is given in Section 3, and Section 4 discusses the basic operations on groves. Sections 5 and 6 define the

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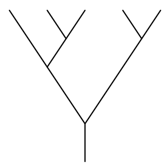
arithmetic on $\mathbb{Y}$. Finally, Section 8 discusses some new results and Section 9 gives a few final remarks.

2. Background

In this section, we give the basic definitions and set notation.

Definition 2.1. A *planar binary tree* is an oriented planar graph drawn in the plane with one root, $n+1$ leaves, and n interior vertices, all of which are trivalent.

Henceforth, by *tree*, we will mean a planar binary tree. We consider trees to be the same if they can be moved in the plane to each other. Thus we can always represent a tree by drawing a root and then having it "grow" upward. The *degree* is the number of internal vertices. Here is an example of a tree of degree four, with five leaves:



Let Y_n be the set of trees of degree n. For example,

$$Y_0 = \{\,|\,\}, \quad Y_1 = \{\vee\}, \quad Y_2 = \{\vee, \vee\}, \quad Y_3 = \{\vee, \vee, \vee, \vee, \vee\}.$$

One can show that the cardinality of Y_n is given by the n-th *Catalan* number,

$$c_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}.$$

The Catalan numbers arise in a variety of combinatorial problems [Stanley 2007].[1]

Definition 2.2. A nonempty subset of Y_n is called a *grove*. The set of all groves of degree n is denoted by $\mathbb{Y}_n$.

For example,

$$\mathbb{Y}_0 = \{\,|\,\}, \quad \mathbb{Y}_1 = \{\vee\}, \quad \mathbb{Y}_2 = \{\vee, \vee, \vee \cup \vee\}.$$

Notice that we are omitting the braces around the sets in $\mathbb{Y}_n$ and use instead $\cup$ to denote the subsets. For example we write $\vee \cup \vee$ as opposed to $\{\vee, \vee\}$ to denote the grove in $\mathbb{Y}_2$ consisting of both trees of degree 2. Let $\mathbb{Y} = \bigcup_{n\in\mathbb{N}} \mathbb{Y}_n$ denote the set of all groves. By definition groves consist of trees of the same degree; hence we get a well-defined notion of degree

$$\deg : \mathbb{Y} \to \mathbb{N}. \tag{1}$$

[1]He currently gives 161 combinatorial interpretations of c_n.

n	$\#Y_n$	$\#\mathbb{Y}_n$
1	1	1
2	2	3
3	5	31
4	14	16383
5	42	4398046511103
6	132	5444517870735015415413993718908291383295
7	429	$\sim 1.386 \times 10^{129}$

Table 1. Number of trees and groves of degree $n \leq 7$.

The Catalan numbers c_n grow rapidly. Since $\mathbb{Y}_n$ is the set of subsets of Y_n, we see that the cardinality $\#\mathbb{Y}_n = 2^{c_n} - 1$ grows extremely fast (Table 1), necessitating the use of computers even for computations on trees of fairly small degree.

3. The natural numbers

In this section we give an embedding of $\mathbb{N}$ into $\mathbb{Y}$. There is a distinguished grove for each degree given by set of all trees of degree n.

Definition 3.1. The *total grove of degree n* is defined by $\underline{n} = \bigcup_{x \in Y_n} x$.

For example,

$$\underline{0} = |, \quad \underline{1} = \vee, \quad \underline{2} = \curlyvee \cup \curlyvee, \quad \underline{3} = \curlyvee \cup \curlyvee \cup \curlyvee \cup \curlyvee \cup \curlyvee.$$

This gives an embedding $\mathbb{N} \hookrightarrow \mathbb{Y}$. It is clear that the degree map is a one-sided inverse in the sense that $\deg(\underline{n}) = n$ for all $n \in \mathbb{N}$. We will see in Section 7 that under this embedding, *arithmetree* can be viewed as an extension of arithmetic on $\mathbb{N}$.

4. Basic operations

In this section we define a few operations that will be used to define the arithmetic on $\mathbb{Y}$.

4.1. *Grafting.*

Definition 4.1. We say that a tree z is obtain as the *graft* of x and y (notation: $z = x \vee y$) if z is gotten by attaching the root of x to the left leaf and the root of y to the right leaf of $\vee$.

For example, $\curlyvee = \vee \vee |$ and $\curlyvee = \vee \vee \vee$. It is clear that every tree x of degree greater than 1 can be obtained as the graft of trees x^l and x^r of degree less than n.

Specifically, we have that $x = x^l \vee x^r$. We refer to these subtrees as the *left* and *right parts* of x.

Given a tree x of degree n, then one can create a tree of degree $n+1$ that carries much of the structure of x by grafting on $\underline{0} = |$. Indeed, there are two such trees, $x \vee \underline{0}$ and $\underline{0} \vee x$. We will say that such trees are *inherited*.

Definition 4.2. A tree x is said to be *left-inherited* if $x^r = \underline{0}$ and *right-inherited* if $x^l = \underline{0}$. A grove is *left-inherited* (resp. *right-inherited*) if each of its member trees is *left-inherited* (resp. *right-inherited*).

We single out two special sequences of trees L_n and R_n.

Definition 4.3. Let $L_1 = R_1 = \underline{1}$. For $n > 1$, set $L_n = L_{n-1} \vee \underline{0}$ and $R_n = \underline{0} \vee R_{n-1}$. We will call such trees *primitive*.

Notice that L_n is the left-inherited tree such that $L_n^l = L_{n-1}$. Similarly, R_n is the right-inherited tree such that $R_n^r = R_{n-1}$.

4.2. ***Over and under.***

Definition 4.4. For $x \in Y_p$ and $y \in Y_q$ the tree x/y (read x *over* y) in Y_{p+q} is obtained by identifying the root of x with the leftmost leaf of y. Similarly, the tree $x \backslash y$ (read x *under* y) in Y_{p+q} is obtained by identifying the rightmost leaf of x with the root of y.

For example, $\curlyvee / \curlyvee = \curlyvee$ and $\curlyvee \backslash \curlyvee = \curlyvee$.

4.3. ***Involution.*** The symmetry around the axis passing through the root defines an involution σ on Y. For example, $\sigma(\curlyvee) = \curlyvee$ and $\sigma(\curlyvee) = \curlyvee$. The involution can be extended to an involution on $\mathbb{Y}$, by letting σ act on each tree in the grove. One can easily check that for trees x, y:

(i) $\sigma(x \vee y) = \sigma(y) \vee \sigma(x)$,

(ii) $\sigma(x/y) = \sigma(y) \backslash \sigma(x)$,

(iii) $\sigma(x \backslash y) = \sigma(y) / \sigma(x)$.

We will see that this involution also respects the arithmetic of groves.

5. Addition

Before we define addition, we first put a partial ordering on Y_n.

5.1. ***Partial ordering.*** We say that the inequality $x < y$ holds if y is obtained from x by moving edges of x from left to right over a vertex. This induces a partial ordering on Y_n by imposing:

(i) $(x \vee y) \vee z \leq x \vee (y \vee z)$.

(ii) If $x < y$ then $x \vee z < y \vee z$ and $z \vee x < z \vee y$ for all $z \in Y_n$.

For example,  $<$  $<$  $<$ . Note that the primitive trees are extremal elements with respect to this ordering.

5.2. *Sum.*

Definition 5.1. The *sum* of two trees x and y is the following disjoint union of trees

$$x + y := \bigcup_{x/y \leq z \leq x \backslash y} z\ .$$

All the elements in the sum have the same degree, namely $\deg(x) + \deg(y)$. The definition of addition extends to groves by distributing. Namely, for groves $x = \bigcup_i x_i$ and $y = \bigcup_j y_i$,

$$x + y := \bigcup_{ij} (x_i + y_j). \tag{2}$$

We remark that it is not immediate that the result of the sum is a grove since it is not obvious that the trees arising in the union are all distinct. Loday shows that this is indeed the case for total groves

$$\underline{n} + \underline{m} = \underline{n + m},$$

and deduces the general case from this as every grove is a subset of some total grove.

Proposition 5.2 (Recursive property of addition). *Let $x = x^l \vee x^r$ and $y = y^l \vee y^r$ be nonzero trees. Then*

$$x + y = x^l \vee (x^r + y) \cup (x + y^l) \vee y^r.$$

The recursive property of addition says that the sum of two trees x and y is naturally a union of two sets, which we call the *left* and *right sum* of x and y:

$$x \dashv y = x^l \vee (x^r + y) \quad \text{and} \quad x \vdash y = (x + y^l) \vee y^r.^2 \tag{3}$$

Note that $x + y = x \dashv y \cup x \vdash y$. You can think about this as splitting the plus sign $+$ into two signs $\dashv$ and $\vdash$. From (2) and the definition, we see that the definition for left sum and right sum can also be extended to groves by distributing.

With the definition of inherited trees/groves and (3), one can easily check that left (respectively right) inheritance is passed along via right (respectively left) sums. More precisely,

Lemma 5.3. *Let y be a left-inherited tree. Then $x \vdash y$ is left-inherited. Similarly, if x is right-inherited, then $x \dashv y$ is right-inherited.*

[2] We set $x \vdash \underline{0} = \underline{0} \dashv y = \underline{0}$.

5.3. *Universal expression.* It turns out that every tree can expressed as a combination of left and right sums of $\vee$. This expression is unique modulo the failure of left and right sum to be associative. More precisely,

Proposition 5.4. *Every tree x of degree n can be written in as an iterated Left and Right sum of n copies of $\vee$. This is called the* universal expression *of x, and we denote it by $w_x(\vee)$. This expression is unique modulo*:

(i) $(x \dashv y) \dashv z = x \dashv (y + z)$,
(ii) $(x \vdash y) \dashv z = x \vdash (y \dashv z)$,
(iii) $(x + y) \vdash z = x \vdash (y \vdash z)$.

For example,

$$ {}^{\vee}\!\!\vee = \vee \vdash \vee \quad \text{and} \quad {}^{\vee\vee}\!\!\vee = \vee \vdash \vee \dashv \vee. $$

Loday gives a algorithm for computing the universal expression of a tree x.

Proposition 5.5 (Recursive property for universal expression). *Let x be a tree of degree greater than* 1*. The algorithm for determining $w_x(\vee)$ is given through the recursive relation*

$$ w_x(\vee) = w_{x^l}(\vee) \vdash \vee \dashv w_{x^r}(\vee). $$

6. Multiplication

Essentially, we define the multiplication to distribute on the left over the universal expression.

Definition 6.1. The product $x \times y$ is defined by

$$ x \times y = w_x(y). $$

This means to compute the product $x \times y$, first compute the universal expression for x, then replace each occurrence of $\vee$ by the tree y, then compute the resulting Left and Right sums. For example, one can easily check that ${}^{\vee}\!\!\vee = \vee \vdash \vee$. This means for any tree y, ${}^{\vee}\!\!\vee \times y = y \vdash y$. In particular,

$$ {}^{\vee}\!\!\vee \times \vee^{\vee} = \vee^{\vee} \vdash \vee^{\vee} $$

is the tree shown in the figure on page 2.

Note that the definition of $x \times y$ as stated still makes sense if y is a grove. We can further extend the definition of multiplication to the case when x is a grove by declaring multiplication to be distributive on the left over disjoint unions:

$$ (x \cup x') \times y = x \times y \cup x' \times y = w_x(y) \cup w_{x'}(y). $$

7. Properties

We list a few properties of *arithmetree*.

- The addition $+ : \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ is associative, but not commutative.
- The multiplication $\times : \mathbb{Y} \times \mathbb{Y} \to \mathbb{Y}$ is associative, but not commutative. It is distributive on the left with respect to $+$, but it is not right distributive.
- There is an injective map $\mathbb{N} \hookrightarrow \mathbb{Y}$, $n \mapsto \underline{n}$ (defined in Section 3) that respects the arithmetic. Namely,

$$\underline{m+n} = \underline{m} + \underline{n} \quad \text{and} \quad \underline{mn} = \underline{m} \times \underline{n} \quad \text{for all } m, n \in \mathbb{N}.$$

- Degree gives a surjective map $\deg : \mathbb{Y} \to \mathbb{N}$ that respects the arithmetic and is a one-sided inverse to the injection above . For every $x, y \in \mathbb{Y}$,

$$\deg(x + y) = \deg(x) + \deg(y) \quad \text{and} \quad \deg(x \times y) = \deg(x)\deg(y).$$

- $\deg(\underline{n}) = n$ for all $n \in \mathbb{N}$.
- The neutral element for $+$ is $\underline{0} = |$.
- The neutral element for $\times$ is $\underline{1} = \vee$.
- The involution σ satisfies

$$\sigma(x + y) = \sigma(y) + \sigma(x) \quad \text{and} \quad \sigma(x \times y) = \sigma(x) \times \sigma(y).$$

8. Results

The recursive properties of addition and multiplication allowed us to implement arithmetree on a computer using PARI/GP [2005]. The computational experimentation was done using Loday's [2002] naming convention for trees.

8.1. *Counting trees.* Since each grove $x \in \mathbb{Y}$ is just a subset of trees, there is another measure of the "size" of x other than degree.

Definition 8.1. Let $x \in \mathbb{Y}$ be a grove. The *count* of x, denoted $C(x)$ is defined as the cardinality of x.

It turns out that count function gives a coarse measure of how complicated a grove x is in terms of arithmetree. Namely, if x is the sum (resp. product) of other groves, then the count of x is at least as large as the count of any of the summands (resp. factors).

Lemma 8.2. *Let $x, y \in \mathbb{Y}$ be two nonzero groves. Then*

(i) $C(x \dashv y) \geq C(x)C(y)$, *with equality if and only if x is a left-inherited grove.*

(ii) $C(x \vdash y) \geq C(x)C(y)$, *with equality if and only if y is a right-inherited grove.*

Proof. We first consider Lemma 8.2(i). Since $\dashv$ is distributive over unions, it suffices to prove the case when x and y are trees. Namely, we must show that for all nonzero trees x and y, $C(x \dashv y) \geq 1$, with equality if and only if x is a left-inherited tree. It is immediate that $C(x \dashv y) \geq 1$; it remains to show that equality is only attained when x is left-inherited. From the definition of left sum, $x \dashv y = x^l \vee (x^r + y)$. If x is not left-inherited, then $x^r \neq \underline{0}$ and

$$\begin{aligned} C(x \dashv y) &= C(x^l \vee (x^r + y)) = C(x^r + y) \\ &= C(x^r \dashv y \cup x^r \vdash y) = C(x^r \dashv y) + C(x^r \vdash y) \\ &> 1. \end{aligned}$$

On the other hand, if x is left-inherited, then $x^r = \underline{0}$ and

$$C(x \dashv y) = C(x^l \vee (x^r + y)) = C(x^l \vee y) = 1.$$

Item (ii) follows similarly. □

Proposition 8.3. *Let $x, y \in \mathbb{Y}$ be two nonzero groves. Then*

(i) $C(x + y) \geq 2C(x)C(y)$, *with equality if and only if x is a left-inherited and y is right-inherited.*

(ii) $C(x \times y) \geq C(x)C(y)^{\deg(x)}$.

Proof. Since $x + y = x \dashv y \cup x \vdash y$, Proposition 8.3(i) follows immediately from Lemma 8.2. For Proposition 8.3(ii), we note that multiplication is left distributive over unions, and so it suffices to prove the case when x is a tree. Namely we must show that for a tree x and a grove y, $C(x \times y) \geq C(y)^{\deg(x)}$.

Let w_x be the universal expression of the tree x. Then $x \times y = w_x(y)$ is some combination of left and right sums of y. By distributivity of left and right sum over unions and repeated usage of Lemma 8.2, the result follows. □

8.2. *Primes.*

Definition 8.4. A grove x is said to be *prime* if x is not the product of two groves different from $\underline{1}$.

Since $\deg(x \times y) = \deg(x)\deg(y)$ for all groves x, y, it is immediate that any grove of prime degree is prime. However, there are also prime groves of composite degree. For example, by taking all possible products of elements of $\mathbb{Y}_2$, one can check by hand that the primitive tree L_4 is a prime grove of degree 4.

We turn our focus to prime trees, which are prime groves with count 1. It turns out that composite trees have a nice description in terms of inherited trees. Namely, a composite tree must have an inherited tree as a right factor and a primitive tree as a left factor.

Theorem 8.5. *Let z be a composite tree of degree n. Then there exists a proper divisor $d \neq 1$ of n and a tree $T \in Y_{d-1}$ such that*

$$z = L_{n/d} \times (\underline{0} \vee T) \quad \textit{or} \quad z = R_{n/d} \times (T \vee \underline{0}).$$

Proof. Let $z = x \times y$ be a composite tree of degree n. By Proposition 8.3, x and y must also be trees. Since $n = \deg(z) = \deg(x)\deg(y)$, it follows that there exists a proper divisor $d \neq 1$ of n such that $\deg(y) = d$ and $\deg(x) = n/d$.

We proceed by induction on the degree of x. Suppose x is a tree of degree 2. Then $x = \vee \dashv \vee$ or $x = \vee \vdash \vee$. If $x = \vee \vdash \vee$, then $x = L_2$ is primitive and

$$1 = C(x \times y) = C(y \vdash y).$$

From Proposition 8.3, it follows that y is right-inherited. Similarly, if $x = | \dashv |$, then $x = R_2$ and y is left-inherited.

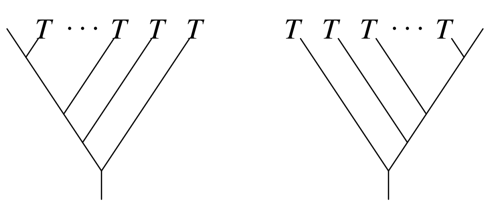
Now suppose x is a tree of degree k such that $x \times y$ is a tree of degree n. From Proposition 5.5 and the definition of multiplication, it follows that

$$\begin{aligned} x \times y &= w_x(y) \\ &= w_{x^l}(y) \vdash y \dashv w_{x^r}(y) \\ &= (x^l \times y) \vdash y \dashv (x^r \times y). \end{aligned}$$

Suppose $x^r \neq \underline{0}$. Then $x^r \times y \neq \underline{0}$ and $C(y \dashv (x^r \times y)) = 1$. Then by Proposition 8.3, y is left-inherited. Let $T = y \dashv (x^r \times y)$. By Lemma 5.3, T is also left-inherited. Since $C((x^l \times y) \vdash T) = 1$ and $T \neq \underline{0}$, we must have that either T is also right-inherited, or $(x^l \times y) = \underline{0}$. The only tree that is both left and right-inherited is the tree $\underline{1} = |$. It follows that $(x^l \times y) = \underline{0}$, and hence $x^l = \underline{0}$. By the inductive hypothesis, x^r is a right-primitive tree, and hence $x = R_k$.

Now suppose $x^r = \underline{0}$. Then $x^l \neq \underline{0}$, and an analogous argument shows that y is left-inherited and $x = L_k$. □

From this theorem, we get a nice picture of the possible shapes of composite trees:



Indeed, one computes that the product $L_k \times (\underline{0} \vee T)$ has the form shown on the left, and and $R_k \times (T \vee \underline{0})$ the form on the right.

It follows that the primitive trees (L_k and R_k) and the inherited trees ($\underline{0} \vee T$ and $T \vee \underline{0}$) are prime. More precisely:

Proposition 8.6. *A nonzero tree is either* $\vee$, *prime*, *or the product of exactly two prime trees. Furthermore*, *the factors are exactly the ones given in Theorem 8.5*, *and can be read off from the shape of the tree.*

The following combinatorial formula is a consequence of Proposition 8.6:

Corollary 8.7. *Let* a_n *denote the number of composite trees of degree* n. *Then*

$$\frac{a_n}{2} = -c_1 - c_n + \sum_{d|n} c_d, \quad \text{where } c_d \text{ is the } d\text{-th Catalan number.}$$

9. Final remarks

9.1. ***Unique factorization.*** Loday [2002] conjectures that arithmetree possesses unique factorization. Namely, when a grove x is written as a product of prime groves, the ordered sequence of factors is unique. Very narrowly interpreted, this statement is false. For example since multiplication in $\mathbb{N}$ is commutative and multiplication in $\mathbb{Y}$ extends arithmetic on $\mathbb{N}$, we see that for $n \in \mathbb{N}$, if $n = p_1 p_2 \cdots p_k$, then

$$\underline{n} = \underline{p_{\sigma(1)}} \times \underline{p_{\sigma(2)}} \times \cdots \times \underline{p_{\sigma(k)}},$$

for any permutation σ. However, away from the image of $\mathbb{N}$ in $\mathbb{Y}$, it appears that this narrow interpretation is true. Specifically, computer experimentation on groves of degree up to 12 yielded a unique ordered sequence of prime factors for each grove outside of the image of $\mathbb{N}$ in $\mathbb{Y}$.

If we interpret the image of $\mathbb{N}$ in $\mathbb{Y}$ in terms of the count function, we see that it is precisely the set of groves with maximal count:

$$\mathbb{Y}^{\max} = \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{Y}_n \mid C(x) = c_n\}.$$

This subset $\mathbb{Y}^{\max}$ possesses unique factorization up to permutation of the factors. On the other extreme, the trees are precisely the set of groves with minimal count;

$$\mathbb{Y}^{\min} = \bigcup_{n \in \mathbb{N}} \{x \in \mathbb{Y}_n \mid C(x) = 1\}.$$

It follows from Proposition 8.6 that $\mathbb{Y}^{\min}$ possesses unique factorization in the narrow sense. The question of unique factorization for all of $\mathbb{Y}$ is open.

9.2. ***Additively irreducible.*** From Proposition 8.3 we see that not every grove can be written as a sum of groves. In fact it is easy to see that every tree is *additively irreducible* in the sense that it cannot be written as the sum of two groves. It would be interesting to study additively irreducible groves. In an analogue to the question of unique factorization, one could ask if arithmetree possesses *unique partitioning*. Namely, when a grove is written as a sum of additively irreducible elements, is the ordered sequence of summands unique?

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