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 mathematical sciences publishers

On the maximum number of isosceles right triangles in a finite point set

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(Communicated by Kenneth S. Berenhaut)

Let Q be a finite set of points in the plane. For any set P of points in the plane, $S_Q(P)$ denotes the number of similar copies of Q contained in P . For a fixed n , Erdős and Purdy asked for the maximum possible value of $S_Q(P)$, denoted by $S_Q(n)$, over all sets P of n points in the plane. We consider this problem when $Q = \Delta$ is the set of vertices of an isosceles right triangle. We give exact solutions when $n \leq 9$, and provide new upper and lower bounds for $S_\Delta(n)$.

1. Introduction

Paul Erdős and George Purdy [1971; 1975; 1976] posed the question: *Given a finite set of points Q , what is the maximum number $S_Q(n)$ of similar copies that can be contained in an n -point set in the plane?* This problem remains open in general. However, there has been some progress regarding the order of magnitude of this maximum as a function of n . Elekes and Erdős [1994] noted that $S_Q(n) \leq n(n-1)$ for any pattern Q and they also gave a quadratic lower bound for $S_Q(n)$ when $|Q| = 3$ or when all the coordinates of the points in Q are algebraic numbers. They also proved a slightly subquadratic lower bound for all other patterns Q . Later, Laczkovich and Ruzsa [1997] characterized precisely those patterns Q for which $S_Q(n) = \Theta(n^2)$. In spite of this, the coefficient of the quadratic term is not known for any nontrivial pattern; it is not even known if $\lim_{n \rightarrow \infty} S_Q(n)/n^2$ exists!

Apart from being a natural question in discrete geometry, this problem also arose in connection with the optimization of algorithms designed to look for patterns among data obtained from scanners, digital cameras, telescopes, and so on [Brass 2002; Brass et al. 2005; Brass and Pach 2005].

Our paper considers the case where Q is the set of vertices of an isosceles right triangle. The case where Q is the set of vertices of an equilateral triangle has been considered in [Ábrego and Fernández-Merchant 2000]. To avoid redundancy, we

MSC2000: primary 52C10; secondary 05C35.

Keywords: Erdős problems, similar triangles, isosceles right triangles.

Supported in part by CURM, BYU, and the NSF (DMS-0636648).

refer to an isosceles right triangle as an IRT for the remainder of the paper. We begin with some definitions. Let P denote a finite set of points in the plane. We define $S_{\Delta}(P)$ to be the number of triplets in P that are the vertices of an IRT. Furthermore, let

$$S_{\Delta}(n) = \max_{|P|=n} S_{\Delta}(P).$$

As mentioned before, Elekes and Erdős established that $S_{\Delta}(n) = \Theta(n^2)$ and it is implicit from their work that $1/18 \leq \liminf_{n \rightarrow \infty} S_{\Delta}(n)/n^2 \leq 1$. The main goal of this paper is to derive improved constants that bound the function $S_{\Delta}(n)/n^2$. Specifically, in Sections 2 and 3, we prove:

Theorem 1. $0.433064 < \liminf_{n \rightarrow \infty} \frac{S_{\Delta}(n)}{n^2} \leq \frac{2}{3} < 0.666667.$

We proceed to determine in Section 4 the exact values of $S_{\Delta}(n)$ when $3 \leq n \leq 9$. Several ideas for the proofs of these bounds come from the equivalent bounds for equilateral triangles in [Ábrego and Fernández-Merchant 2000].

2. Lower bound

For $z \in P$, let $R_{\pi/2}(z, P)$ be the $\pi/2$ counterclockwise rotation of P with center z . Let $\deg_{\pi/2}(z)$ be the number of isosceles right triangles in P such that z is the right-angle vertex of the triangle. If $z \in P$, then $\deg_{\pi/2}(z)$ can be computed by simply rotating our point set P by $\pi/2$ about z and counting the number of points in the intersection other than z . Therefore,

$$\deg_{\pi/2}(z) = |P \cap R_{\pi/2}(z, P)| - 1. \quad (1)$$

Since an IRT has only one right angle,

$$S_{\Delta}(P) = \sum_{z \in P} \deg_{\pi/2}(z).$$

That is, the sum computes the number of IRTs in P . From this identity an initial lower bound of $\frac{5}{12}$ can be derived for $\liminf_{n \rightarrow \infty} S_{\Delta}(n)/n^2$ using the set

$$P = \{(x, y) \in \mathbb{Z}^2 : 0 \leq x \leq \sqrt{n}, 0 \leq y \leq \sqrt{n}\}.$$

We now improve this bound.

The following theorem generalizes our method for finding a lower bound. We denote by Λ the lattice generated by the points $(1, 0)$ and $(0, 1)$, and we refer to points in Λ as *lattice points*. The next result provides a formula for the leading term of $S_{\Delta}(P)$ when our points in P are lattice points enclosed by a given shape. This theorem, its proof, and notation, are similar to those of Theorem 2 in [Ábrego and Fernández-Merchant 2000], where we obtained a similar result for equilateral triangles in place of IRTs.

Theorem 2. *Let K be a compact set with finite perimeter and area 1. Define*

$$f_K : \mathbb{C} \rightarrow \mathbb{R}^+ \quad \text{as } f_K(z) = \text{Area}(K \cap R_{\pi/2}(z, K)), \quad \text{where } z \in K.$$

If K_n is a similar copy of K intersecting Λ in exactly n points, then

$$S_{\Delta}(K_n \cap \Lambda) = \left(\int_K f_K(z) dz \right) n^2 + O(n^{3/2}).$$

Proof. Given a compact set L with finite area and perimeter, we have

$$|rL \cap \Lambda| = \text{Area}(rL) + O(r) = r^2 \text{Area}(L) + O(r),$$

where rL is the scaling of L by a factor r . Therefore,

$$\begin{aligned} S_{\Delta}(K_n \cap \Lambda) &= \sum_{z \in K_n \cap \Lambda} |(\Lambda \cap K_n) \cap R_{\pi/2}(z, (K_n \cap \Lambda))| - 1 \\ &= \sum_{z \in K_n \cap \Lambda} \text{Area}(K_n \cap R_{\pi/2}(z, K_n)) + O(\sqrt{n}). \end{aligned}$$

We see that each error term in the sum is bounded by the perimeter of K_n , which is finite by hypothesis. Thus,

$$\begin{aligned} S_{\Delta}(K_n \cap \Lambda) &= n^2 \sum_{z \in K_n \cap \Lambda} \frac{1}{n^2} \text{Area}(K_n \cap R_{\pi/2}(z, K_n)) + O(n^{3/2}) \\ &= n^2 \sum_{z \in K_n \cap \Lambda} \frac{1}{n} \text{Area}\left(\frac{1}{\sqrt{n}}(K_n \cap R_{\pi/2}(z, K_n))\right) + O(n^{3/2}) \\ &= n^2 \sum_{z \in K_n \cap \Lambda} \frac{1}{n} \text{Area}\left(\frac{1}{\sqrt{n}}K_n \cap R_{\pi/2}\left(\frac{z}{\sqrt{n}}, \frac{1}{\sqrt{n}}K_n\right)\right) + O(n^{3/2}). \end{aligned}$$

The last sum is a Riemann approximation for the function $f_{(1/\sqrt{n})K_n}$ over the region $(1/\sqrt{n})K_n$; thus

$$S_{\Delta}(K_n \cap \Lambda) = n^2 \left(\int_{\frac{1}{\sqrt{n}}K_n} f_{\frac{1}{\sqrt{n}}K_n}(z) dz + O\left(\frac{1}{\sqrt{n}}\right) \right) + O(n^{3/2}).$$

Since

$$\begin{aligned} \text{Area}\left(\frac{1}{\sqrt{n}}K_n\right) &= \frac{1}{n} \text{Area}(K_n) = \frac{1}{n}(n + O(\sqrt{n})) \\ &= 1 + O\left(\frac{1}{\sqrt{n}}\right) = \text{Area}(K) + O\left(\frac{1}{\sqrt{n}}\right), \end{aligned}$$

it follows that

$$\int_{\frac{1}{\sqrt{n}}K_n} f_{\frac{1}{\sqrt{n}}K_n}(z) dz = \int_K f_K(z) dz + O\left(\frac{1}{\sqrt{n}}\right).$$

As a result,

$$S_{\Delta}(K_n \cap \Lambda) = n^2 \int_{\frac{1}{\sqrt{n}}K_n} f_{\frac{1}{\sqrt{n}}K_n}(z) dz + O(n^{3/2}) = n^2 \int_K f_K(z) dz + O(n^{3/2}). \quad \square$$

The importance of this theorem can be seen immediately. Although our lower bound of $\frac{5}{12}$ for $\liminf_{n \rightarrow \infty} S_{\Delta}(n)/n^2$ was derived by summing the degrees of each point in a square lattice, the same result can be obtained by letting K be the square $\{(x, y) : |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\}$. It follows that $f_K(x, y) = (1 - |x| - |y|)(1 - ||x| - |y||)$ and

$$S_{\Delta}(K_n \cap \Lambda) = \left(\int_K f_K(z) dz \right) n^2 + O(n^{3/2}) = \frac{5}{12} n^2 + O(n^{3/2}).$$

An improved lower bound will follow provided that we find a set K such that the value for the integral in Theorem 2 is larger than $\frac{5}{12}$. We get a larger value for the integral by letting K be the circle $\{z \in \mathbb{C} : |z| \leq 1/\sqrt{\pi}\}$. In this case

$$f_K(z) = \frac{2}{\pi} \arccos\left(\frac{\sqrt{2\pi}}{2}|z|\right) - |z| \sqrt{\frac{2}{\pi} - |z|^2} \quad (2)$$

and

$$S_{\Delta}(K_n \cap \Lambda) = \left(\int_K f_K(z) dz \right) n^2 + O(n^{3/2}) = \left(\frac{3}{4} - \frac{1}{\pi} \right) n^2 + O(n^{3/2}).$$

It was conjectured in [Ábrego and Fernández-Merchant 2000] that not only does $\lim_{n \rightarrow \infty} E(n)/n^2$ exist, but it is attained by the uniform lattice in the shape of a circle. ($E(n)$ denotes the maximum number of equilateral triangles determined by n points in the plane.) The corresponding conjecture in the case of the isosceles right triangle turns out to be false. That is, if $\lim_{n \rightarrow \infty} S_{\Delta}(n)/n^2$ exists, then it must be strictly greater than $\frac{3}{4} - \pi^{-1}$. Define $\bar{\Lambda}$ to be the translation of Λ by the vector $(\frac{1}{2}, \frac{1}{2})$. The following lemma will help us to improve our lower bound.

Lemma 3. *If $(j, k) \in \mathbb{R}^2$ and $\Lambda' = \Lambda$ or $\Lambda' = \bar{\Lambda}$, then*

$$R_{\pi/2}((j, k), \Lambda') \cap \Lambda' = \begin{cases} \Lambda' & \text{if } (j, k) \in \Lambda \cup \bar{\Lambda}, \\ \emptyset & \text{else.} \end{cases}$$

Proof. Observe that

$$R_{\pi/2}((j, k), (s, t)) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s-j \\ t-k \end{pmatrix} + \begin{pmatrix} j \\ k \end{pmatrix} = \begin{pmatrix} k-t+j \\ s-j+k \end{pmatrix}.$$

First suppose $(s, t) \in \Lambda$. Since $s, t \in \mathbb{Z}$, then $(k-t+j, s-j+k) \in \Lambda$ if and only if $k-j \in \mathbb{Z}$ and $k+j \in \mathbb{Z}$. This can only happen when either both j and k are half-integers (i.e., $(j, k) \in \bar{\Lambda}$), or both j and k are integers (i.e., $(j, k) \in \Lambda$). Now suppose $(s, t) \in \bar{\Lambda}$. In this case, because both s and t are half-integers, we conclude

that $(k - t + j, s - j + k) \in \bar{\Lambda}$ if and only if both $k - j \in \mathbb{Z}$ and $k + j \in \mathbb{Z}$. Once again this occurs if and only if $(j, k) \in \Lambda \cup \bar{\Lambda}$. \square

Recall that if K denotes the circle of area 1, then $(\frac{3}{4} - \pi^{-1})n^2$ is the leading term of $S_{\Delta}(K_n \cap \Lambda)$. The previous lemma implies that, if we were to adjoin a point $z \in \mathbb{R}^2$ to $K_n \cap \Lambda$ such that z has half-integer coordinates and is located near the center of the circle formed by the points of $K_n \cap \Lambda$, then $\deg_{\pi/2}(z)$ will approximately equal $|K_n \cap \Lambda|$. We obtain the next theorem by further exploiting this idea.

Theorem 4. $0.43169 \approx \frac{3}{4} - \frac{1}{\pi} < 0.433064 < \liminf_{n \rightarrow \infty} \frac{S_{\Delta}(n)}{n^2}$.

Proof. Let K be the circle of area 1 and set $A = K_{m_1} \cap \Lambda$, $B = K_{m_2} \cap \bar{\Lambda}$. Position B so that its points are centered on the circle formed by the points in A (see Figure 1). We let $n = m_1 + m_2 = |A \cup B|$ and $m_2 = x \cdot m_1$, where $0 < x < 1$ is a constant to be determined.

We proceed to maximize the leading coefficient of $S_{\Delta}(A \cup B)$ as x varies from 0 to 1. By Lemma 3, there cannot exist an IRT whose right-angle vertex lies in A while one $\pi/4$ vertex lies in A and the other lies in B . Similarly, there cannot exist an IRT whose right angle-vertex lies in B while one $\pi/4$ vertex lies in A and the other lies in B . Therefore, each IRT with vertices in $A \cup B$ must fall under one of four cases:

Case 1: All three vertices in A. By Theorem 2, there are $(\frac{3}{4} - \pi^{-1})m_1^2 + O(m_1^{3/2})$ IRTs in this case. Since $m_1 = n/(1+x)$, the number of IRTs in terms of n equals

$$\left(\frac{3}{4} - \frac{1}{\pi}\right) \frac{n^2}{(1+x)^2} + O(n^{3/2}). \quad (3)$$

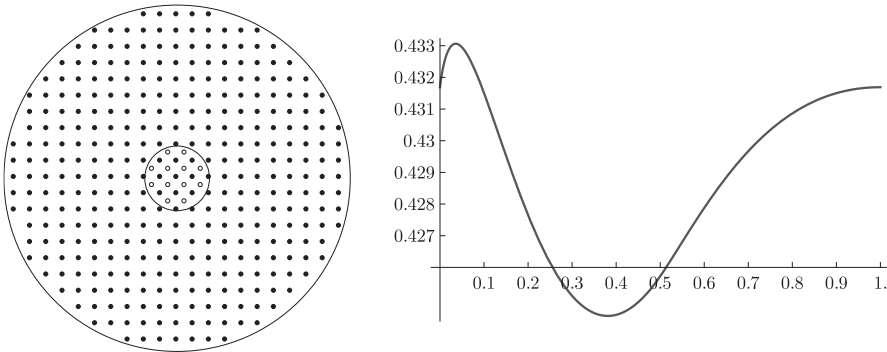


Figure 1. Left: set B (open dots) centered on set A (black dots). Right: plot of the n^2 coefficient of $S_{\Delta}(A \cup B)$ for x from 0 to 1.

Case 2: All three vertices in B. By Theorem 2, there are $(\frac{3}{4} - \pi^{-1})m_2^2 + O(m_2^{3/2})$ IRTs in this case. This time $m_2 = nx/(1+x)$ and the number of IRTs in terms of n equals

$$\left(\frac{3}{4} - \frac{1}{\pi}\right) \frac{n^2 x^2}{(1+x)^2} + O(n^{3/2}). \quad (4)$$

Case 3: Right-angle vertex in B, $\pi/4$ vertices in A. The relationship given by Lemma 3 allows us to slightly adapt the proof of Theorem 2 in order to compute the number of IRTs in this case. The integral approximation to the number of IRTs in this case is given by

$$\sum_{z \in K_{m_2} \cap \bar{\Lambda}} |(K_{m_1} \cap \Lambda) \cap R_{\pi/2}(z, (K_{m_1} \cap \Lambda))| = m_1^2 \left(\int_{\frac{1}{\sqrt{m_1}} K_{m_2}} f_{\frac{1}{\sqrt{m_1}} K_{m_1}}(z) dz \right) + O(m_1^{3/2}).$$

But

$$\text{Area}\left(\frac{1}{\sqrt{m_1}} K_{m_2}\right) = \text{Area}\left(\sqrt{\frac{m_2}{m_1}} K\right) + O(\sqrt{m_1}),$$

so

$$m_1^2 \left(\int_{\frac{1}{\sqrt{m_1}} K_{m_2}} f_{\frac{1}{\sqrt{m_1}} K_{m_1}}(z) dz \right) + O(m_1^{3/2}) = m_1^2 \left(\int_{\sqrt{\frac{m_2}{m_1}} K} f_K(z) dz \right) + O(m_1^{3/2}).$$

Expressing this value in terms of n gives

$$\left(\int_{\sqrt{x} K} f_K(z) dz \right) \frac{n^2}{(1+x)^2} + O(n^{3/2}). \quad (5)$$

Case 4: Right-angle vertex in A, $\pi/4$ vertices in B. As in Case 3, the number of IRTs is given by

$$\begin{aligned} \sum_{z \in K_{m_1} \cap \Lambda} |(K_{m_2} \cap \bar{\Lambda}) \cap R_{\pi/2}(z, (K_{m_2} \cap \bar{\Lambda}))| \\ = m_2^2 \left(\int_{\frac{1}{\sqrt{m_2}} K_{m_1}} f_{\frac{1}{\sqrt{m_2}} K_{m_2}}(z) dz \right) + O(m_2^{3/2}). \end{aligned} \quad (6)$$

Now recall that $f_{(1/\sqrt{m_2})K_{m_2}}(z) = \text{Area}\left((1/\sqrt{m_2})K_{m_2} \cap R_{\pi/2}(z, (1/\sqrt{m_2})K_{m_2})\right)$. It follows that $f_{(1/\sqrt{m_2})K_{m_2}}(z_0) = 0$ if and only if z_0 is farther than $\sqrt{2/\pi}$ from the center of $(1/\sqrt{m_2})K_{m_2}$. Thus for small enough values of m_2 , the region of integration in (6) is actually $(\sqrt{2/m_2})K_{m_2}$, so it does not depend on m_1 . We consider two subcases.

First, if $x \leq \frac{1}{2}$ (i.e., $m_2 \leq m_1/2$), then

$$\sqrt{\frac{2}{\pi}} = \frac{1}{\sqrt{m_2}} \frac{\sqrt{2m_2}}{\sqrt{\pi}} \leq \frac{1}{\sqrt{m_2}} \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sqrt{m_1}}{\sqrt{2}} = \frac{1}{\sqrt{m_2}} \sqrt{\frac{m_1}{\pi}}.$$

The left side of this inequality is the radius of $(\sqrt{2/m_2})K_{m_2}$, while the right side is the radius of $(1/\sqrt{m_2})K_{m_1}$; thus the region of integration where $f_{(1/\sqrt{m_2})K_{m_2}}$ is nonzero equals $(\sqrt{2/m_2})K_{m_2}$. Hence, the number of IRTs equals

$$\begin{aligned} m_2^2 \left(\int_{\sqrt{\frac{2}{m_2}}K_{m_2}} f_{\frac{1}{\sqrt{m_2}}K_{m_2}}(z) dz \right) + O(m_2^{3/2}) &= m_2^2 \left(\int_{\sqrt{2}K} f_K(z) dz \right) + O(m_2^{3/2}) \quad (7) \\ &= \left(\int_{\sqrt{2}K} f_K(z) dz \right) \frac{n^2 x^2}{(1+x)^2} + O(n^{3/2}). \end{aligned}$$

Now we consider the case $x > \frac{1}{2}$ (i.e., $m_2 > m_1/2$). In this case, $f_{(1/\sqrt{m_2})K_{m_2}}$ is nonzero for all points in $(1/\sqrt{m_2})K_{m_1}$. Thus the number of IRTs is then

$$\begin{aligned} m_2^2 \left(\int_{\frac{1}{\sqrt{m_2}}K_{m_1}} f_{\frac{1}{\sqrt{m_2}}K_{m_2}}(z) dz \right) + O(m_2^{3/2}) &= m_2^2 \left(\int_{\sqrt{\frac{m_1}{m_2}}K} f_K(z) dz \right) + O(m_2^{3/2}) \quad (8) \\ &= \left(\int_{\sqrt{\frac{1}{x}}K} f_K(z) dz \right) \frac{n^2 x^2}{(1+x)^2} + O(n^{3/2}). \end{aligned}$$

By (2), we have, for $t > 0$,

$$\begin{aligned} \int_{tK} f_K(z) dz &= 2\pi \int_0^{t/\sqrt{\pi}} \left(\frac{2}{\pi} \arccos \left(\frac{\sqrt{2\pi}}{2} r \right) - r \sqrt{\frac{2}{\pi} - r^2} \right) r dr \\ &= \frac{1}{2\pi} \left(4t^2 \arccos \frac{t}{\sqrt{2}} + 2 \arcsin \frac{t}{\sqrt{2}} - t(t^2 + 1)\sqrt{2-t^2} \right). \end{aligned}$$

Therefore, putting all four cases together — i.e., expressions (3)–(5), and either (7) or (8) — we obtain that the n^2 coefficient of $S_\Delta(A \cup B)$ equals

$$\frac{1}{4\pi(x+1)^2} \left(8x \arccos \sqrt{\frac{x}{2}} + 4 \arcsin \sqrt{\frac{x}{2}} + (5\pi - 4)x^2 + (3\pi - 4) - 2(x+1)\sqrt{2x-x^2} \right)$$

if $0 < x \leq \frac{1}{2}$, or

$$\begin{aligned} \frac{1}{4\pi(x+1)^2} \left(8x \left(\arccos \sqrt{\frac{x}{2}} + \arccos \sqrt{\frac{1}{2x}} \right) + 4 \arcsin \sqrt{\frac{x}{2}} + 4x^2 \arcsin \sqrt{\frac{1}{2x}} \right. \\ \left. + (3\pi - 4)(x^2 + 1) - 2(x+1)(\sqrt{2x-x^2} + \sqrt{2x-1}) \right), \end{aligned}$$

if $\frac{1}{2} < x < 1$. Letting x vary from 0 to 1, this coefficient is maximized (see Figure 1) when $x \approx 0.0356067$, corresponding to a radius of B approximately 18.87% of the radius of A . Letting x equal this value gives 0.433064 as a decimal approximation to the maximum value attained by the n^2 coefficient. \square

At this point, one might be tempted to further increase the quadratic coefficient by placing a third set of lattice points arranged in a circle and centered on the circle formed by B . It turns out that forming such a configuration does not improve the results in the previous theorem. This is due to Lemma 3. More specifically, given our construction from the previous theorem, there is no place to adjoin a point z to the center of $A \cup B$ such that $z \in \Lambda$ or $z \in \bar{\Lambda}$. Hence, if we were to add the point z to the center of $A \cup B$, then any new IRTs would have their right-angle vertex located at z with one $\pi/4$ vertex in A and the other $\pi/4$ vertex in B . Doing so can produce at most $2m_2 = 2xm_1 \approx 0.0712m_1$ new IRTs (recall that $x \approx 0.0356066$ in our construction). On the other hand, adding z to the perimeter of A , gives us $m_1 f_K(1/\sqrt{\pi}) \approx 0.1817m_1$ new IRTs.

3. Upper bound

We now turn our attention to finding an upper bound for $S_\Delta(n)/n^2$. It is easy to see that $S_\Delta(n) \leq n^2 - n$, since any pair of points can be the vertices of at most six IRTs. Our next theorem improves this bound. The idea is to prove that there exists a point in P that does not belong to many IRTs. First, we need the following definition.

For every $z \in P$, let $R_{\pi/4}^+(z, P)$ and $R_{\pi/4}^-(z, P)$ be the dilations of P , centered at z , with factor $\sqrt{2}$ and $1/\sqrt{2}$, respectively, followed by a $\pi/4$ counterclockwise rotation with center z . Let $\deg_{\pi/4}^+(z)$ and $\deg_{\pi/4}^-(z)$ be the number of isosceles right triangles zxy with $x, y \in P$ such that zxy occur in counterclockwise order, and zy , respectively zx , is the hypotenuse of the triangle zxy .

Much like the case of $\deg_{\pi/2}$, $\deg_{\pi/4}^+$ and $\deg_{\pi/4}^-$ can be computed with the identities

$$\deg_{\pi/4}^+(z) = |P \cap R_{\pi/4}^+(z, P)| - 1, \quad \deg_{\pi/4}^-(z) = |P \cap R_{\pi/4}^-(z, P)| - 1.$$

Theorem 5. $S_\Delta(n) \leq \lfloor \frac{2}{3}(n-1)^2 - \frac{5}{3} \rfloor$ for $n \geq 3$.

Proof. By induction on n . If $n = 3$, then $S_\Delta(3) \leq 1 = \lfloor \frac{1}{3}(2 \cdot 4 - 5) \rfloor$. Now suppose the theorem holds for $n = k$. We must show this implies the theorem holds for $n = k + 1$. Suppose that there is a point $z \in P$ such that $\deg_{\pi/2}(z) + \deg_{\pi/4}^+(z) + \deg_{\pi/4}^-(z) \leq \lfloor \frac{1}{3}(4n - 5) \rfloor$. Then, by induction,

$$\begin{aligned} S_\Delta(k+1) &\leq \deg_{\pi/2}(z) + \deg_{\pi/4}^+(z) + \deg_{\pi/4}^-(z) + S_\Delta(k) \\ &\leq \lfloor \frac{1}{3}(4k-1) \rfloor + \lfloor \frac{2}{3}(k-1)^2 - \frac{5}{3} \rfloor = \lfloor \frac{2}{3}k^2 - \frac{5}{3} \rfloor. \end{aligned}$$

The last equality can be verified by considering the three possible residues of k when divided by 3. Hence, our theorem is proved if we can find a point $z \in P$ with the desired property.

Let $x, y \in P$ be points such that x and y form the diameter of P . In other words, if $w \in P$, then the distance from w to any other point in P is less than or equal to the distance from x to y . We now prove that either x or y is a point with the desired property mentioned above. We begin by analyzing $\deg_{\mathcal{G}_{\pi/4}}^-$. We use the notation from [Ábrego and Fernández-Merchant 2000, Theorem 1].

Define $N_x = P \cap R_{\pi/4}^-(x, P) \setminus \{x\}$ and $N_y = P \cap R_{\pi/4}^-(y, P) \setminus \{y\}$. It follows from our identities that, $\deg_{\mathcal{G}_{\pi/4}}^-(x) = |N_x|$ and $\deg_{\mathcal{G}_{\pi/4}}^-(y) = |N_y|$. Furthermore, by the inclusion-exclusion principle for finite sets, we have

$$|N_x| + |N_y| = |N_x \cup N_y| + |N_x \cap N_y|.$$

We shall prove by contradiction that $|N_x \cap N_y| \leq 1$. Suppose that there are two points $u, v \in N_x \cap N_y$. This means that there are points $u_x, v_x, u_y, v_y \in P$ such that the triangles $xu_xu, xv_xv, yu_yu, yv_yv$ are IRTs oriented counterclockwise with right angle at either u or v .

But notice that the line segments u_xu_y and v_xv_y are simply the $(\pi/2)$ -counterclockwise rotations of xy about centers u and v , respectively. Hence, $u_xu_yv_xv_y$ is a parallelogram with two sides having length xy as shown in Figure 2, left. This is a contradiction since one of the diagonals of the parallelogram is longer than any of its sides. Thus, $|N_x \cap N_y| \leq 1$. Furthermore, $x \notin N_y$ and $y \notin N_x$, so $|N_x \cup N_y| \leq n - 2$ and thus $\deg_{\mathcal{G}_{\pi/4}}^-(x) + \deg_{\mathcal{G}_{\pi/4}}^-(y) = |N_x \cup N_y| + |N_x \cap N_y| \leq n - 2 + 1 = n - 1$. This also implies that $\deg_{\mathcal{G}_{\pi/4}}^+(x) + \deg_{\mathcal{G}_{\pi/4}}^+(y) \leq n - 1$, since we can follow the same argument applied to the reflection of P about the line xy .

We now look at $\deg_{\mathcal{G}_{\pi/2}}(x)$ and $\deg_{\mathcal{G}_{\pi/2}}(y)$. We claim that, for every $p \in P$, at most one of $R_{\pi/2}(x, p)$ or $R_{\pi/2}(y, p)$ belongs to P . Indeed, let $p_x = R_{\pi/2}(x, p)$ and $p_y = R_{\pi/2}(y, p)$ (see Figure 2, right). The distance p_xp_y is exactly the distance xy but scaled by $\sqrt{2}$. This contradicts the fact that xy is the diameter of P .

Define a graph G with vertex set $V(G) = P \setminus \{x, y\}$ and edge set given by saying that $uv \in E(G)$ if and only if $v = R_{\pi/2}(x, u)$ or $v = R_{\pi/2}(y, u)$. We show that

$$0 \leq \deg_{\mathcal{G}_{\pi/2}}(x) + \deg_{\mathcal{G}_{\pi/2}}(y) - |E(G)| \leq 1. \tag{9}$$

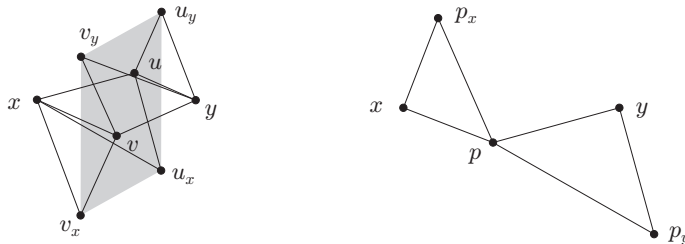


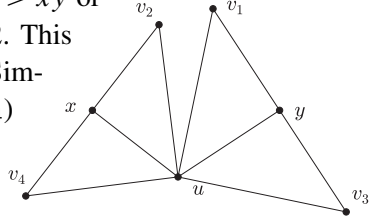
Figure 2. Proof of Theorem 5.

The left inequality follows from the fact each edge counts an IRT in either $\deg_{\mathbb{S}_{\pi/2}}(x)$ or $\deg_{\mathbb{S}_{\pi/2}}(y)$ and possibly in both. However, if uv is an edge of G so that $v = R_{\pi/2}(x, u)$ and $u = R_{\pi/2}(y, v)$, then $xuyv$ is a square, so this can only happen for at most one edge.

Now, let $\deg_G(u)$ be the number of edges in $E(G)$ incident to u . We claim that

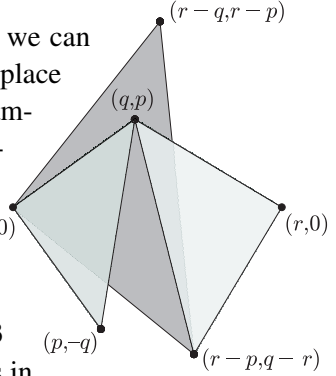
$$\deg_G(u) \leq 2 \quad \text{for every } u \in V(G). \quad (10)$$

Indeed, take $uv_1 \in E(G)$; we can assume, without loss, that $u = R_{\pi/2}(y, v_1)$. If $v_3 = R_{\pi/2}(y, u) \in P$, then we conclude that $xv_3 > xy$ or $xv_1 > xy$, because $\angle xyv_3 \geq \pi/2$ or $\angle xyv_1 \geq \pi/2$. This contradicts the fact that xy is the diameter of P . Similarly, if v_2 and v_4 are defined as $u = R_{\pi/2}(x, v_4)$ and $v_2 = R_{\pi/2}(x, u)$, then at most one of v_2 or v_4 can be in P .



Claim. All paths in G have length at most 2.

Proof. We prove this claim by contradiction. Suppose we can have a path of length 3 or more. To assist us, let us place our points on a cartesian coordinate system with our diameter xy relabeled as the points $(0, 0)$ and $(r, 0)$, furthermore, assume $p, q \geq 0$ and that the four vertices of the path of length 3 are $(p, -q)$, (q, p) , $(r-p, q-r)$, and $(r-q, r-p)$. Our aim is to show that the distance between $(r-q, r-p)$ and $(r-p, q-r)$ contradicts that r is the diameter of P . Now, if paths of length 3 were possible, the distance between every pair of points in the figure on the right must be less than or equal to r . Since $d((p, -q), (q, p)) \leq r$, we have $p^2 + q^2 \leq r^2/2$.



Now let us analyze the square of the distance from $(r-q, r-p)$ to $(r-p, q-r)$. Because $2(p^2 + q^2) \geq (p+q)^2$, it follows that

$$\begin{aligned} d^2((r-q, r-p), (r-p, q-r)) &= (-q+p)^2 + (2r-p-q)^2 \\ &= 4r^2 - 4r(p+q) + 2(p^2+q^2) \\ &\geq 4r^2 - 4\sqrt{2}r\sqrt{p^2+q^2} + 2(p^2+q^2) \\ &= (2r - \sqrt{2(p^2+q^2)})^2. \end{aligned}$$

But $\sqrt{2(p^2+q^2)} \leq r$, so $(2r - \sqrt{2(p^2+q^2)}) \geq r$ and thus

$$d^2((r-q, r-p), (r-p, q-r)) \geq r^2.$$

Equality occurs if and only if $p = r/2$ and $q = r/2$; otherwise, the distance between

$(r-q, r-p)$ and $(r-p, q-r)$ is strictly greater than r , contradicting the fact that the diameter of P is r . Therefore if $p \neq r/2$ or $q \neq r/2$ then there is no path of length 3. In the case that $p = r/2$ and $q = r/2$ the points (q, p) and $(r-q, r-p)$ become the same and so do the points $(p, -q)$ and $(r-p, q-r)$. Thus we are left with a path of length 1. \square

It follows from (10) and the Claim that all paths of length 2 are disjoint. Thus, G is the union of disjoint paths of length at most 2. If a denotes the number of paths of length 2 and b the number of paths of length 1, then

$$|E(G)| = 2a + b \quad \text{and} \quad 3a + 2b \leq n - 2.$$

Recall from (9) that either

$$\deg_{\mathfrak{g}_{\pi/2}}(x) + \deg_{\mathfrak{g}_{\pi/2}}(y) = |E(G)| \quad \text{or} \quad \deg_{\mathfrak{g}_{\pi/2}}(x) + \deg_{\mathfrak{g}_{\pi/2}}(y) = |E(G)| + 1.$$

If $\deg_{\mathfrak{g}_{\pi/2}}(x) + \deg_{\mathfrak{g}_{\pi/2}}(y) = |E(G)|$, then

$$2|E(G)| = 4a + 2b \leq n - 2 + a \leq n - 2 + \frac{n-2}{3},$$

so $\deg_{\mathfrak{g}_{\pi/2}}(x) + \deg_{\mathfrak{g}_{\pi/2}}(y) = |E(G)| \leq \frac{2}{3}(n-2)$. Moreover, if

$$\deg_{\mathfrak{g}_{\pi/2}}(x) + \deg_{\mathfrak{g}_{\pi/2}}(y) = |E(G)| + 1,$$

then $b \geq 1$ and we get a minor improvement,

$$2|E(G)| = 4a + 2b \leq n - 2 + a \leq n - 4 + \frac{n-2}{3},$$

so $\deg_{\mathfrak{g}_{\pi/2}}(x) + \deg_{\mathfrak{g}_{\pi/2}}(y) = |E(G)| + 1 \leq (2n-7)/3 < \frac{2}{3}(n-2)$.

We are now ready to put everything together. Between the two points x and y , we derived the bounds:

$$\begin{aligned} \deg_{\mathfrak{g}_{\pi/2}}(x) + \deg_{\mathfrak{g}_{\pi/2}}(y) &\leq \frac{2}{3}(n-2), \\ \deg_{\mathfrak{g}_{\pi/4}}^+(x) + \deg_{\mathfrak{g}_{\pi/4}}^+(y) &\leq (n-1), \\ \deg_{\mathfrak{g}_{\pi/4}}^-(x) + \deg_{\mathfrak{g}_{\pi/4}}^-(y) &\leq (n-1). \end{aligned}$$

Because the degree of a point must take on an integer value, it must be the case that either x or y satisfies $\deg_{\mathfrak{g}_{\pi/2}} + \deg_{\mathfrak{g}_{\pi/4}}^+ + \deg_{\mathfrak{g}_{\pi/4}}^- \leq \lfloor (4n-5)/3 \rfloor$. \square

4. Small cases

In this section we determine the exact values of $S_{\Delta}(n)$ when $3 \leq n \leq 9$.

Theorem 6. For $3 \leq n \leq 9$, $S_{\Delta}(3) = 1$, $S_{\Delta}(4) = 4$, $S_{\Delta}(5) = 8$, $S_{\Delta}(6) = 11$, $S_{\Delta}(7) = 15$, $S_{\Delta}(8) = 20$, and $S_{\Delta}(9) = 28$.

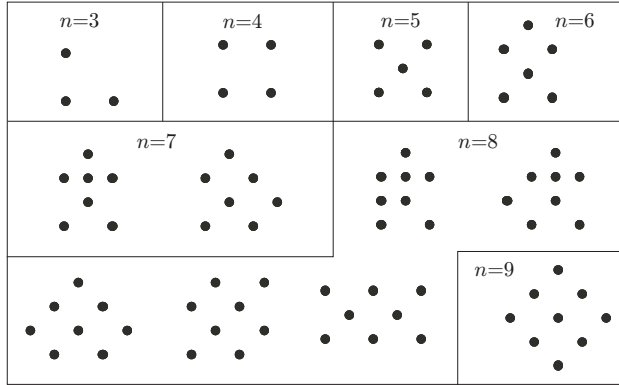


Figure 3. Optimal sets achieving equality for $S_{\Delta}(n)$.

The corresponding optimal sets are shown in Figure 3.

Proof. We begin with $n = 3$. Since three points uniquely determine a triangle, and there is an IRT with three points, shown in Figure 4(a), this situation becomes trivial and we conclude that $S_{\Delta}(3) = 1$.

Now let $n = 4$. In Figure 4(b) we show a point set P such that $S_{\Delta}(P) = 4$. This implies that $S_{\Delta}(4) \geq 4$. However, $S_{\Delta}(4)$ is also bounded above by $\binom{4}{3} = 4$. Hence, $S_{\Delta}(4) = 4$.

To continue with the proof for the remaining values of n , we need the following two lemmas.

Lemma 7. *Suppose $|P| = 4$ and $S_{\Delta}(P) \geq 2$. The sets in parts (b)–(e) of Figure 4 are the only possibilities for such a set P , not counting symmetric repetitions.*

Proof. Having $S_{\Delta}(P) \geq 2$ implies that we must always have more than one IRT in P . Hence, we can begin with a single IRT and examine the possible ways of adding a point and producing more IRTs. We accomplish this task in Figure 4(a). The 10 numbers in the figure indicate the location of a point, and the total number of IRTs after its addition to the set of black dots. All other locations not labeled with a number do not increase the number of IRTs. Therefore, except for symmetries, all the possibilities for P are shown in Figure 4(b)–(e). \square

Lemma 8. *Let P be a finite set with $|P| = n$. Suppose that $S_{\Delta}(A) \leq b$ for all $A \subseteq P$ with $|A| = k$. Then*

$$S_{\Delta}(P) \leq \left\lfloor \frac{n(n-1)(n-2)b}{k(k-1)(k-2)} \right\rfloor.$$

Proof. Suppose that within P , every k -point configuration contains at most b IRTs. The number of IRTs in P can then be counted by adding all the IRTs in every k -point subset of P . However, in doing so, we end up counting a fixed IRT exactly

$\binom{n-3}{k-3}$ times. Because $S_\Delta(A) \leq b$ we get,

$$\binom{n-3}{k-3} S_\Delta(P) = \sum_{\substack{A \subseteq P \\ |A|=k}} S_\Delta(A) \leq \binom{n}{k} b.$$

Notice that $S_\Delta(P)$ can only take on integer values so,

$$S_\Delta(P) \leq \left\lfloor \frac{\binom{n}{k} b}{\binom{n-3}{k-3}} \right\rfloor = \left\lfloor \frac{n(n-1)(n-2)b}{k(k-1)(k-2)} \right\rfloor. \quad \square$$

Now suppose $|P| = 5$. If $S_\Delta(A) \leq 1$ for all $A \subseteq P$ with $|A| = 4$, then by Lemma 8, $S_\Delta(P) \leq 2$. Otherwise, by Lemma 7, P must contain one of the four sets shown in Figure 4(b)–(e). The result now follows by examining the possibilities for producing more IRTs by placing a fifth point in the four distinct sets. In Figure 4(b)–(e), we accomplish this task. Just as in Lemma 7, every number in a figure indicates the location of a point, and the total number of IRTs after its addition to the set of black dots. It follows that the maximum value achieved by placing a fifth point is 8 and so $S_\Delta(5) = 8$. The point set that uniquely achieves equality is shown in Figure 4(f). Moreover, there is exactly one set P with $S_\Delta(P) = 6$ (shown in Figure 4(g)), and two sets P with $S_\Delta(P) = 5$ (Figures 4(h) and 4(i)).

Now suppose $|P| = 6$. If $S_\Delta(A) \leq 4$ for all $A \subseteq P$ with $|A| = 5$, then by Lemma 8, $S_\Delta(P) \leq 8$. Otherwise, P must contain one of the sets in Figure 4(f)–(i). We now check all possibilities for adding more IRTs by joining a sixth point to our four distinct sets. This is shown in Figure 4(f)–(i). It follows that the maximum value achieved is 11 and so $S_\Delta(6) = 11$. The point set that uniquely achieves equality is shown in Figure 4(j). Also, except for symmetries, there are exactly three sets P with $S_\Delta(P) = 10$ (Figure 4(k)–(m)) and only one set P with $S_\Delta(P) = 9$ (Figure 4(n)).

Now suppose $|P| = 7$. If $S_\Delta(A) \leq 8$ for all $A \subseteq P$ with $|A| = 6$, we have $S_\Delta(P) \leq 14$, by Lemma 8. Otherwise, P must contain one of the sets in parts (j)–(n) of Figure 4. We now check all possibilities for adding more IRTs by joining a seventh point to our 5 distinct configurations. We complete this task in parts (j)–(n). Because the maximum value achieved is 15, we deduce that $S_\Delta(7) = 15$. In this case, there are exactly two point sets that achieve 15 IRTs.

The proof for the values $n = 8$ and $n = 9$ follows along the same lines, but there are many more intermediate sets to be considered. We omit the details. \square

Inspired by our method used for proving exact values of $S_\Delta(n)$, a computer algorithm was devised to construct the best one-point extension of a given base set. This algorithm, together with appropriate heuristic choices for some initial sets, led to the construction of point sets with many IRTs giving us our best lower

n	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
bound	35	43	52	64	74	85	97	112	124	139	156	176	192	210	229	252

Table 1. Best lower bounds for $S_{\Delta}(n)$.

bounds for $S_{\Delta}(n)$ when $10 \leq n \leq 25$. These lower bounds are shown in Table 1 and the point sets achieving them in Figure 5.

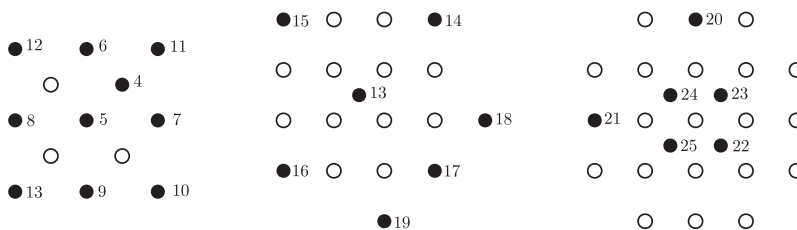


Figure 5. Best constructions A_n for $n \leq 25$. Each set A_n is obtained as the union of the starting set (in white) and the points with label $\leq n$. The value $S_{\Delta}(A_n)$ is given by Table 1.

Acknowledgements

We thank Virgilio Cerna who, as part of the CURM mini-grant that supported this project, helped to implement the program that found the best lower bounds for smaller values of n . We also thank an anonymous referee for some useful suggestions and improvements to the presentation.

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Received: 2010-01-14 Revised: 2011-02-27 Accepted: 2011-02-27

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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