

involve

a journal of mathematics

Stability properties of a predictor-corrector implementation of
an implicit linear multistep method

Scott Sarra and Clyde Meador

 mathematical sciences publishers

2011

vol. 4, no. 1

Stability properties of a predictor-corrector implementation of an implicit linear multistep method

Scott Sarra and Clyde Meador

(Communicated by John Baxley)

We examine the stability properties of a predictor-corrector implementation of a class of implicit linear multistep methods. The method has recently been described in the literature as suitable for the efficient integration of stiff systems and as having stability regions similar to well known implicit methods. A more detailed analysis reveals that this is not the case.

1. Introduction

In an undergraduate research project that started as a senior capstone project, Meador [2009] became aware of an explicit ODE method that claimed to have desirable stability properties that are usually only enjoyed by implicit methods. The little known method seemed too good to be true. If it had the claimed stability properties, it deserved to be better known and more widely used in applications. In this work we describe what a more careful study of the method revealed. We calculate the correct stability regions of the methods and verify our claims with numerical experiments.

2. Linear multistep methods

A general s -step linear multistep method (LMM) for the numerical solution of the autonomous ordinary differential equation (ODE) initial value problem (IVP)

$$y' = F(y), \quad y(0) = y_0 \quad (1)$$

is of the form

$$\sum_{m=0}^s \alpha_m y^{n+m} = \Delta t \sum_{m=0}^s \beta_m F(y^{n+m}), \quad n = 0, 1, \dots, \quad (2)$$

MSC2000: 65L04, 65L06, 65L20.

Keywords: linear multistep method, eigenvalue stability, numerical differential equations, stiffness.

where α_m and β_m are given constants. It is conventional to normalize (2) by setting $\alpha_s = 1$. When $\beta_s = 0$ the method is explicit. Otherwise, it is implicit. In order to start multistep methods, the first $s - 1$ time levels have to be calculated by a one-step method such as a Runge–Kutta method. Many of the properties of the method (2) can be described in terms of the characteristic polynomials

$$\rho(\omega) = \sum_{m=0}^s \alpha_m \omega^s \quad \text{and} \quad \sigma(\omega) = \sum_{m=0}^s \beta_m \omega^s. \quad (3)$$

The linear stability region of a numerical ODE method is determined by applying the method to the scalar linear equation

$$y' = \lambda y, \quad y(0) = 1, \quad (4)$$

where λ is a complex number. The exact solution of (4) is $y(t) = e^{\lambda t}$, which approaches zero as $t \rightarrow \infty$ if and only if the real part of λ is negative. The set of all numbers $z = \Delta t \lambda$ such that $\lim_{n \rightarrow \infty} y^n = 0$ is called the linear stability region of the method. For z in the stability domain, the numerical method exhibits the same asymptotic behavior as (4). For stability, all the scaled eigenvalues of the coefficient matrix of a linear system of ODEs must lie in the stability region. For nonlinear systems, the scaled eigenvalues of the Jacobian matrix of the system must lie within the stability region. A numerical ODE method is A-stable if its region of absolute stability contains the entire left half-plane ($\text{Re}(\Delta t \lambda) < 0$).

For LMMs, the boundary of the stability region is found by the boundary locus method which plots the parametric curve of the function

$$r(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}, \quad 0 \leq \theta \leq 2\pi, \quad (5)$$

that is, the ratio of the method's characteristic polynomials (3). Standard references on numerical ODEs can be consulted for more details [Butcher 2003; Hairer et al. 2000; Hairer and Wanner 2000; Iserles 1996; Lambert 1973]

3. Implicit LIL linear multistep methods

In this work we consider a class of LMM that has been referred to as local iterative linearization (LIL) in the literature. The s -stage implicit LIL method also has accuracy of order s . The LIL method has been applied to chaotic dynamical systems in [Danca and Chen 2004; Luo et al. 2007]. The convergence, accuracy, and stability properties of the LIL methods were examined in [Danca 2006].

In [Danca and Chen 2004; Danca 2006; Luo et al. 2007], both the implicit and predictor-corrector versions are referred to as LIL methods. However, the stability properties of the methods are very different and we distinguish between

the methods by calling the implicit method ILIL, and the predictor-corrector implementation PCLIL.

Using the notation $f^n = F(y^n)$, the first four ILIL formulas follow. The $s = 1$ ILIL formula

$$y^{n+1} - y^n = \Delta t f^{n+1} \tag{6}$$

coincides with the implicit Euler method. For $s = 2$ the ILIL algorithm is

$$y^{n+2} - \frac{4}{3}y^{n+1} + \frac{1}{3}y^n = \Delta t \left(\frac{25}{36}f^{n+2} - \frac{1}{18}f^{n+1} + \frac{1}{36}f^n \right); \tag{7}$$

for $s = 3$,

$$y^{n+3} - \frac{5}{3}y^{n+2} + \frac{13}{15}y^{n+1} - \frac{1}{5}y^n = \Delta t \left(\frac{26}{45}f^{n+3} - \frac{1}{9}f^{n+2} + \frac{4}{45}f^{n+1} - \frac{1}{45}f^n \right); \tag{8}$$

and for $s = 4$,

$$y^{n+4} - 2y^{n+3} + \frac{8}{5}y^{n+2} - \frac{26}{35}y^{n+1} + \frac{1}{7}y^n = \Delta t \left(\frac{6463}{12600}f^{n+4} - \frac{523}{3150}f^{n+3} + \frac{383}{2100}f^{n+2} - \frac{283}{3150}f^{n+1} + \frac{223}{12600}f^n \right). \tag{9}$$

The characteristic polynomial coefficients of the ILIL methods are listed in [Table 1](#). The stability regions of the ILIL methods of orders 1 through 4 are shown in [Figure 1](#) (left). The stability regions are exterior to the curves. The innermost curve is associated with the first-order method and the stability region shrinks as the order of the method increases. The first- and second-order methods are A-stable, while the third and fourth-order methods do not include all of the left half-plane. It is well known that the order of an A-stable LMM cannot exceed 2 [[Lambert 1973](#)].

	$s = 1$	$s = 2$	$s = 3$	$s = 4$
α_0/β_0	$-1/0$	$\frac{1}{3}/\frac{1}{36}$	$\frac{-1}{5}/\frac{-1}{45}$	$\frac{1}{7}/\frac{223}{12600}$
α_1/β_1	$1/1$	$\frac{-4}{3}/\frac{-1}{18}$	$\frac{13}{15}/\frac{4}{45}$	$\frac{-26}{35}/\frac{-283}{3150}$
α_2/β_2	-	$1/\frac{25}{36}$	$\frac{-5}{3}/\frac{-1}{9}$	$\frac{8}{5}/\frac{383}{2100}$
α_3/β_3	-	-	$1/\frac{26}{45}$	$-2/\frac{-523}{3150}$
α_4/β_4	-	-	-	$1/\frac{6463}{12600}$

Table 1. Coefficients of the characteristic polynomials (3) for the ILIL algorithms.

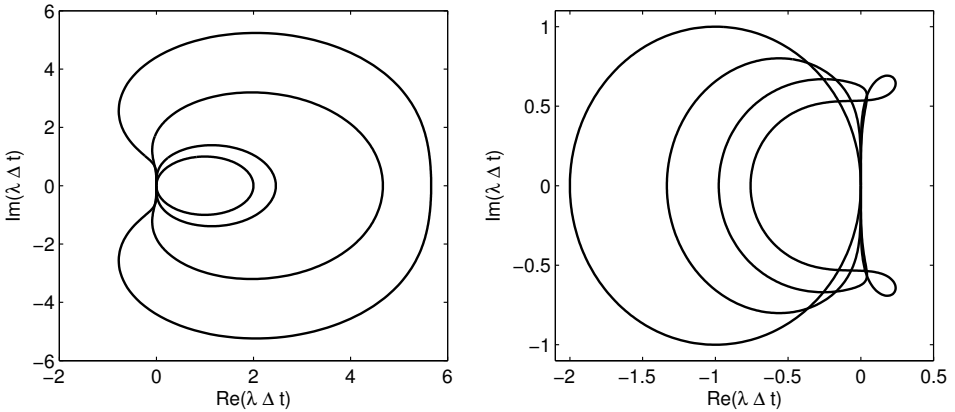


Figure 1. Left: Implicit LIL methods have stability regions consisting of the exterior of the plotted curves. Right: Predictor-corrector implemented LIL methods have bounded stability regions in the interior of the plotted curves.

4. LIL predictor-corrector

Two types of methods that are commonly used to solve the nonlinear difference equations of implicit methods are functional iteration and Newton's method. A third approach, which does not involve solving nonlinear equations, that can be used to implement an implicit ODE method is a predictor-corrector approach. An explicit formula, the predictor, is used to get a preliminary approximation \hat{y}^{n+s} of y^{n+s} . Then the corrector step uses formulas like the implicit LIL methods (6)–(9), with \hat{y}^{n+s} in place of y^{n+s} when calculating f^{n+s} , to get a more accurate approximation of y^{n+s} . The predictor-corrector approach turns the implicit method into one that is implemented in the manner of an explicit method. However, the stability properties of the predictor-corrector method will be inferior to those of the original implicit method. The predictors for the PCLIL methods are listed in Table 2.

s	order s LIL predictor
1	$\hat{y}^{n+1} = y^n$
2	$\hat{y}^{n+2} = 2y^{n+1} - y^n$
3	$\hat{y}^{n+3} = 3y^{n+2} - 3y^{n+1} + y^n$
4	$\hat{y}^{n+4} = 4y^{n+3} - 6y^{n+2} + 4y^{n+1} - y^n$

Table 2. The predictor stages for the predictor-corrector LIL algorithms.

Applying the PCLIL methods to the stability test problem (4) reveals that the α coefficients of the characteristic polynomial (3) remain the same as the implicit LIL methods. However, the β coefficients are modified to be $\hat{\beta}$ which lead to different stability regions. The $\hat{\beta}$ coefficients for the PCLIL methods are listed in Table 3. The details of finding the $\hat{\beta}$ coefficients are illustrated with the second-order PCLIL method:

$$\begin{aligned} \alpha_2 y^{n+2} + \alpha_1 y^{n+1} + \alpha_0 y^n &= \Delta t (\beta_2 f^{n+2} + \beta_1 f^{n+1} + \beta_0 f^n) \\ &= \Delta t (\beta_2 \lambda (2y^{n+1} - y^n) + \beta_1 \lambda y^{n+1} + \beta_0 \lambda y^n) \\ &= \Delta t ((\beta_1 + 2\beta_2) \lambda y^{n+1} + (\beta_0 - \beta_2) \lambda y^n) \\ &= \Delta t (\hat{\beta}_1 f^{n+1} + \hat{\beta}_0 f^n). \end{aligned}$$

The stability regions for the PCLIL methods of orders 1 through 4 are shown in the right image of Figure 1. Since the stability regions consist of the regions that are interior to the curves, PCLIL methods are not A-stable. It is well known that A-stable explicit LMMs do not exist [Nevanlinna and Sipilä 1974].

5. Numerical examples

Many problems arising from various fields result in systems of ODEs that have a property called stiffness. A formal definition can be formulated (see [Lambert 1973], for example), but the essence of a stiff problem can be explained by the fact the coefficient matrix of a linear ODE system (or Jacobian matrix of a nonlinear ODE system) has some eigenvalues with large negative real parts. Thus, explicit methods with their bounded stability regions may be required to take much smaller time steps for stability than are necessary for accuracy. Implicit methods, particularly A-stable methods, with their unbounded stability regions are well suited for stiff problems.

	$s = 1$	$s = 2$	$s = 3$	$s = 4$
$\hat{\beta}_0$	1	$-\frac{2}{3}$	$\frac{5}{9}$	$\beta_0 - \beta_4$
$\hat{\beta}_1$	0	$\frac{4}{3}$	$-\frac{74}{45}$	$\beta_1 + 4\beta_4$
$\hat{\beta}_2$	-	0	$\frac{73}{45}$	$\beta_2 - 6\beta_4$
$\hat{\beta}_3$	-	-	0	$\beta_3 + 4\beta_4$
$\hat{\beta}_4$	-	-	-	0

Table 3. Modified β coefficients of the characteristic polynomials (3) for the LIL algorithms implemented as predictor-correctors.

Linear example. We consider the linear ODE system

$$\begin{aligned} y_1' &= -21y_1 + 19y_2 - 20y_3, & y_1(0) &= 1, \\ y_2' &= 19y_1 - 21y_2 + 20y_3, & y_2(0) &= 0, \\ y_3' &= 40y_1 - 40y_2 - 40y_3, & y_3(0) &= -1, \end{aligned} \quad (10)$$

which may be considered stiff. The coefficient matrix

$$A = \begin{bmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{bmatrix} \quad (11)$$

has eigenvalues $\lambda_1 = -2$, $\lambda_2 = -40 + 40i$, and $\lambda_3 = -40 - 40i$.

In [Figure 2](#) the stability region of the third-order ILIL is the outside of the dashed curve and the stability region of the third-order PCLIL is the interior of solid curve. The eigenvalues of the linear ODE system (10) scaled by $\Delta t = 0.017$ are in the left image and scaled by $\Delta t = 0.012$ in the right image.

The unstable PCLIL solution of the $y_1(t)$ component of the system using $\Delta t = 0.017$ is shown in the left image in [Figure 3](#) and the stable solution using $\Delta t = 0.012$ is shown on the right. The system can be integrated with the implicit LIL methods with any size time step and the method will remain stable.

Note that for linear problems it is possible to derive an explicit expression from the implicit LIL formulas and that an iterative method is not required. For example,

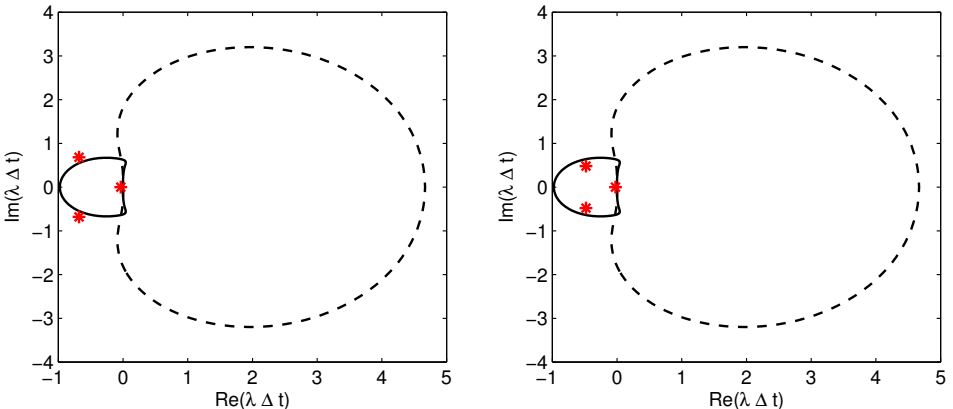


Figure 2. Color dots indicate the eigenvalues of the linear ODE system (10) scaled by $\Delta t = 0.017$ (left) and $\Delta t = 0.012$ (right). The third-order ILIL is stable for eigenvalues outside the dashed curve, and the third-order PCLIL for those inside the solid curve.

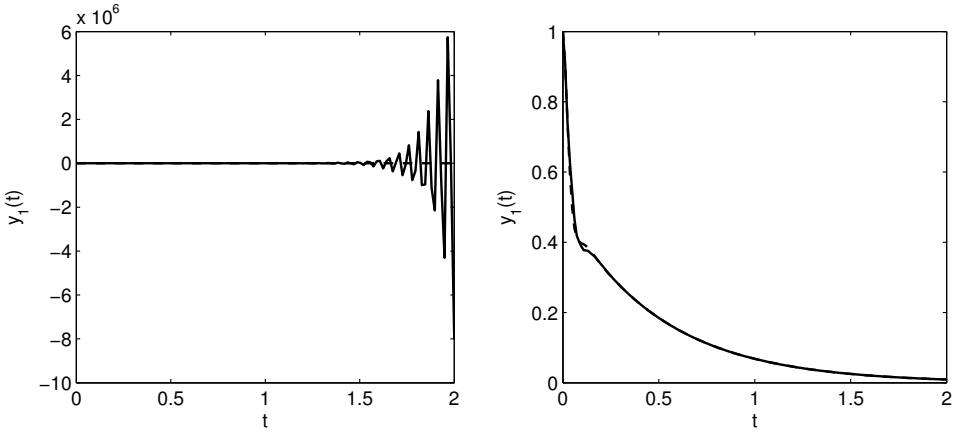


Figure 3. Left: unstable PCLIL solution of the $y_1(t)$ component of the system (10) using $\Delta t = 0.017$. Right: stable solution using $\Delta t = 0.012$.

the second-order implicit LIL method applied to the linear ODE system (10) can be evaluated as

$$y^{n+1} = \left(I - \frac{25\Delta t}{36}A\right)^{-1} \left(\frac{4}{3}I - \frac{\Delta t}{18}A\right)y^n + \left(I - \frac{25\Delta t}{36}A\right)^{-1} \left(\frac{-1}{3}I - \frac{\Delta t}{36}A\right)y^{n-1},$$

where I is the 3×3 identity matrix.

Nonlinear example. We consider the Rabinovich–Fabrikant (RF) equations, a set of differential equations in three variables with two constant parameters a and b :

$$\begin{aligned} x' &= y(z - 1 + y^2) + ax, \\ y' &= x(3z + 1 - x^2) + ay, \\ z' &= -2z(b + xy). \end{aligned}$$

PCLIL methods have been used extensively in the study of this system [Danca and Chen 2004; Luo et al. 2007; Danca 2006].

In our numerical work, we encountered severe stability issues while using the PCLIL methods with certain settings of the parameters. For instance, with $a = 0.33$ and $b = 0.5$, a very small step size of $\Delta t = 0.0001$ was needed to stably integrate the system to $t = 200$ with the fourth-order PCLIL method. The resulting attractor is shown in Figure 4. The fourth-order ILIL method was implemented and was an improvement in many cases. However, due to the method not being A-stable, we still had stability problems for some parameter settings.

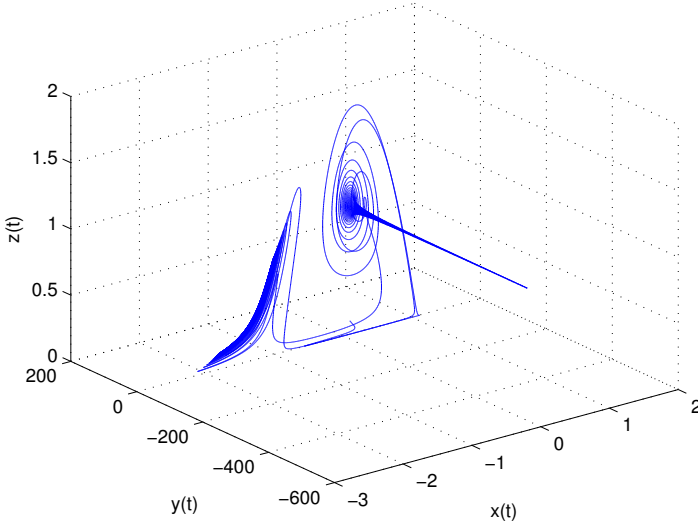


Figure 4. Phase plots of the Rabinovich–Fabrikant equations for parameter settings $a = 0.33$ and $b = 0.5$.

We note that the most efficient method that we found for our numerical exploration of the RF system was an implicit Runge–Kutta method. Using the 4-stage, eighth-order accurate, A-stable Gauss method [Butcher 1964; Ehle 1968; Hairer and Wanner 2000; Sanz-Serna and Calvo 1994], we were able to accurately approximate the attractor in Figure 4 with a step size as large as $\Delta t = 0.2$.

6. Conclusions

Previously, the predictor-corrector implementation of the LIL method has been analyzed in [Danca 2006] where of the PCLIL method it was said that “The time stability of LIL method is more efficient than that of other known algorithms and is comparable with time stability of the Gear’s algorithm” and that the LIL method is suitable for stiff problems. Additionally, in [Danca and Chen 2004; Luo et al. 2007] the PCLIL was applied to chaotic dynamical systems that had stiff characteristics and was presented as a method well suited to this type of problem. As we have shown here, this is not the case. The PCLIL methods are explicit and have bounded stability regions that decrease in area as the order of the method increases. The PCLIL methods are not well suited for stiff problems as they will require very small time steps in order to remain stable. It is possible that in the previous application to nonlinear chaotic systems that very small time steps were always used for accuracy purposes and thus stability issues were not encountered.

References

- [Butcher 1964] J. C. Butcher, “Implicit Runge–Kutta processes”, *Math. Comp.* **18** (1964), 50–64. [MR 28 #2641](#) [Zbl 0123.11701](#)
- [Butcher 2003] J. C. Butcher, *Numerical methods for ordinary differential equations*, John Wiley & Sons Ltd., Chichester, 2003. [MR 2004e:65069](#) [Zbl 1040.65057](#)
- [Danca 2006] M.-F. Danca, “A multistep algorithm for ODEs”, *Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms* **13**:6 (2006), 803–821. [MR 2007k:65097](#) [Zbl 1111.65065](#)
- [Danca and Chen 2004] M.-F. Danca and G. Chen, “Bifurcation and chaos in a complex model of dissipative medium”, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **14**:10 (2004), 3409–3447. [MR 2107556](#) [Zbl 1129.37314](#)
- [Ehle 1968] B. L. Ehle, “High order A-stable methods for the numerical solution of systems of D.E.’s”, *Nordisk Tidskr. Informationsbehandling (BIT)* **8** (1968), 276–278. [MR 39 #1119](#)
- [Hairer and Wanner 2000] E. Hairer and G. Wanner, *Solving ordinary differential equations, II: Stiff and differential-algebraic problems*, Springer Series in Computational Math. **14**, Springer, 2000.
- [Hairer et al. 2000] E. Hairer, S. Norsett, and G. Wanner, *Solving ordinary differential equations, I: Nonstiff problems*, Springer Series in Computational Math. **8**, Springer, 2000.
- [Iserles 1996] A. Iserles, *A first course in the numerical analysis of differential equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1996. [MR 1384977 \(97m:65003\)](#)
- [Lambert 1973] J. D. Lambert, *Computational methods in ordinary differential equations*, Wiley, New York, 1973. [MR 54 #11789](#) [Zbl 0258.65069](#)
- [Luo et al. 2007] X. Luo, M. Small, M.-F. Danca, and G. Chen, “On a dynamical system with multiple chaotic attractors”, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **17**:9 (2007), 3235–3251. [MR 2008k:37081](#) [Zbl 1185.37081](#)
- [Meador 2009] C. Meador, “A comparison of two 4th-order numerical ordinary differential equation methods applied to the Rabinovich–Fabrikant equations”, 2009, http://www.scottsarra.org/math/papers/ClydeMeador_SeniorCapstone_2009.pdf.
- [Nevanlinna and Sipilä 1974] O. Nevanlinna and A. H. Sipilä, “A nonexistence theorem for explicit A-stable methods”, *Math. Comp.* **28** (1974), 1053–1056. [MR 50 #1515](#) [Zbl 0293.65055](#)
- [Sanz-Serna and Calvo 1994] J. M. Sanz-Serna and M. P. Calvo, *Numerical Hamiltonian problems*, Applied Mathematics and Mathematical Computation **7**, Chapman & Hall, London, 1994. [MR 95f:65006](#) [Zbl 0816.65042](#)

Received: 2010-03-01

Revised: 2011-03-23

Accepted: 2011-05-07

sarra@marshall.edu

Department of Mathematics, Marshall University, One John Marshall Drive, Huntington, WV 25755-2560, United States
<http://www.scottsarra.org/>

meador16@marshall.edu

Department of Mathematics, Marshall University, One John Marshall Drive, Huntington, WV 25755-2560, United States

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Ken Ono	University of Wisconsin, USA ono@math.wisc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	Robert J. Plemmons	Wake Forest University, USA plmmons@wfu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Sat Gupta	U of North Carolina, Greensboro, USA sgupta@uncg.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Natalia Hritonenko	Prairie View A&M University, USA nahritonenko@pvamu.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Karen Kafadar	University of Colorado, USA karen.kafadar@cudenver.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
David Larson	Texas A&M University, USA larson@math.tamu.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu

PRODUCTION

Silvio Levy, Scientific Editor

Sheila Newbery, Senior Production Editor

Cover design: ©2008 Alex Scorpan

See inside back cover or <http://pjm.math.berkeley.edu/involve> for submission instructions.

The subscription price for 2011 is US \$100/year for the electronic version, and \$130/year (+\$35 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY
 **mathematical sciences publishers**
<http://msp.org/>

A NON-PROFIT CORPORATION

Typeset in L^AT_EX

Copyright ©2011 by Mathematical Sciences Publishers

involve

2011

vol. 4

no. 1

The arithmetic of trees	1
ADRIANO BRUNO AND DAN YASAKI	
Vertical transmission in epidemic models of sexually transmitted diseases with isolation from reproduction	13
DANIEL MAXIN, TIMOTHY OLSON AND ADAM SHULL	
On the maximum number of isosceles right triangles in a finite point set	27
BERNARDO M. ÁBREGO, SILVIA FERNÁNDEZ-MERCHANT AND DAVID B. ROBERTS	
Stability properties of a predictor-corrector implementation of an implicit linear multistep method	43
SCOTT SARRA AND CLYDE MEADOR	
Five-point zero-divisor graphs determined by equivalence classes	53
FLORIDA LEVIDIOTIS AND SANDRA SPIROFF	
A note on moments in finite von Neumann algebras	65
JON BANNON, DONALD HADWIN AND MAUREEN JEFFERY	
Combinatorial proofs of Zeckendorf representations of Fibonacci and Lucas products	75
DUNCAN MCGREGOR AND MICHAEL JASON ROWELL	
A generalization of even and odd functions	91
MICKI BALAIKH AND MATTHEW ONDRUS	