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Combinatorial proofs of Zeckendorf representations of Fibonacci and Lucas products

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In 1998, Filipponi and Hart introduced many Zeckendorf representations of Fibonacci, Lucas and mixed products involving two variables. In 2008, Artz and Rowell proved the simplest of these identities, the Fibonacci product, using tilings. This paper extends the work done by Artz and Rowell to many of the remaining identities from Filipponi and Hart's work. We also answer an open problem raised by Artz and Rowell and present many Zeckendorf representations of mixed products involving three variables.

1. Preliminaries

Definition 1.1. The *n-th Fibonacci number* is the term f_n of the *Fibonacci sequence* defined recursively by

$$f_0 = 1, \quad f_1 = 1, \quad f_n = f_{n-1} + f_{n-2}.$$

This definition is shifted relative to the standard Fibonacci sequence, which begins at 0. This is done to ensure that the combinatorial interpretation matches our sequence without having to shift indices.

Benjamin and Quinn [2003] presented a combinatorial interpretation for the Fibonacci sequence: f_n is the number of possible tilings of an $1 \times n$ board with 1×2 dominoes and 1×1 squares.[1] They also gave a combinatorial interpretation for a related sequence introduced by Edouard Lucas:

Definition 1.2. The *n-th Lucas number* is the term L_n of the *Lucas sequence*, defined recursively by

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2}.$$

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[1] The $1 \times n$ board, or *n-board*, is divided into 1×1 squares, called *cells*. In a *tiling*, the board is entirely covered by tiles without overlap. (A *tile* is either a domino or a square.) Two tilings are equivalent if, given any pair of cells, they belong to the same tile in one tiling if and only if they belong to the same tile in the other.

L_n is the number of possible square-and-domino tilings of an *n-bracelet*, that is, an n-board with ends identified. (One can think of such a board as a ring of curved cells.) We do *not* consider as equivalent tilings superimposable by a rotation or reflection; the equivalence relation is the same as for a linear board (see note 1). An n-bracelet has a designated starting cell and ending cell. If these two cells are covered by the same domino, we say that the board is *out of phase*. Otherwise, the board is *in phase*.

The combinatorial interpretation of f_n and L_n given by Benjamin and Quinn is easy to prove by induction. (For instance, in the linear case, consider the first cell of the n-board: either it's covered by a domino, in which case there are, by the induction assumption, f_{n-2} possible tilings of the $n-2$ leftover cells, or it's covered by a square, in which case there are f_{n-1} possibilities.) Since the introduction of these interpretations, many Fibonacci and Lucas identities have been proved combinatorially. Some identities are presented below and will be used repeatedly throughout the paper.

Lemma 1.1. *For any positive integer $n \geq 0$,*

$$f_n = \begin{cases} f_0 + f_2 + \cdots + f_{n-1} & \text{for } n \text{ odd}, \\ f_1 + f_3 + \cdots + f_{n-1} + 1 & \text{for } n \text{ even}. \end{cases}$$

A combinatorial proof of the odd case of Lemma 1.1 appears as Identity 2 in [Benjamin and Quinn 2003]. The even case can be proved similarly.

In the next proof and later one, we say that a tiling *has a fault at m* if the m-th and $(m+1)$-st cells belong to different tiles.

Lemma 1.2. *For any positive integers $m, n \geq 1$,*

$$f_{m+n} - f_m f_n = f_{m-1} f_{n-1}.$$

Proof. Consider the tilings of an $(m+n)$-board; we know there are f_{m+n} of them. Divide the board into an m-board and an n-board. For tilings that have a fault at m, there are f_m possibilities for the m-board and f_n for the n-board, for a total of $f_m f_n$ possibilities. The complementary case is where there is a domino straddling tiles m and $m+1$. Then we're left with subboards of lengths $m-1$ and $n-1$, and there are $f_{m-1} f_{n-1}$ such possibilities. □

Lemma 1.3. *For any positive integer $n \geq 2$,*

$$L_n = f_n + f_{n-2}.$$

A combinatorial proof of this appears under Identity 32 in [Benjamin and Quinn 2003]. We will repeatedly apply this lemma in our identities that involve Lucas products so that we can work with n-boards rather than bracelets. For example,

$$L_m L_n = f_m f_n + f_{m-2} f_n + f_m f_{n-2} + f_{m-2} f_{n-2}.$$

Each of the four terms on the right-hand side are each of the combinations of two bracelets either being in or out of phase.

Edouard Zeckendorf, an amateur mathematician and a doctor in the Belgian army, proved [1972] an interesting property of Fibonacci numbers (here $\mathbb{N}$ stands for the natural numbers, not including 0):

Theorem 1.4. *Every $N \in \mathbb{N}$ can be expressed uniquely as a sum*

$$\sum_{i=1}^{M} f_{a_i} = N,$$

where $M \in \mathbb{N}$, $a_i \in \mathbb{N}$ for $1 \leq i \leq M$, and $a_{i+1} > a_i + 1$ for $1 \leq i < M$.

We call this decomposition the *Zeckendorf representation of N*. Note that, since $a_{i+1} > a_i + 1$, repeated or consecutive Fibonacci numbers cannot appear in the representation.

An open exercise in [Benjamin and Quinn 2003] lists a number of identities involving Zeckendorf representations of multiples of Fibonacci numbers and asks for combinatorial proofs:

$$\begin{aligned}
2f_n &= f_{n-2} + f_{n+1},\\
3f_n &= f_{n-2} + f_{n+2},\\
4f_n &= f_{n-2} + f_n + f_{n+2},\\
&\vdots
\end{aligned}$$

Wood [2007] provided combinatorial proofs for several of these identities, but without a unified method. Gerdemann [2009] gave a combinatorial algorithm for finding the Zeckendorf representation of any particular mf_n, but it does not give a general closed-form representation.

Artz and Rowell [2009] found combinatorial proofs of certain Zeckendorf representations of $f_m f_n$ originally proved in [Filipponi and Hart 1998] by other means:

Theorem 1.5. *For $n > 2k + 1$,*

$$f_{2k+1} f_n = \sum_{i=1}^{k+1} f_{n-2k-4+4i}.$$

Theorem 1.6. *For $n > 2k$,*

$$f_{2k} f_n = f_{n-2k} + \sum_{i=1}^{k} f_{n-2k-1+4i}.$$

To sketch the proof for the case of $f_{2k+1} f_n$, one must break the set of all tilings of an $(n + 2k + 1)$-board with a fault at n into many disjoint sets where the closest square is i dominoes away from the fault at n. Further our closest square can be no further than k dominoes away from the fault; therefore, $0 \leq i \leq k$.

In Sections 2 and 3 we provide combinatorial proofs of additional Zeckendorf representations of Fibonacci and Lucas products given in [Filipponi and Hart 1998], namely those for $2f_m f_n$ and $L_m L_n$. In Section 4 we answer an open problem from [Artz and Rowell 2009] and present many new Fibonacci and Lucas product Zeckendorf representations.

2. The Zeckendorf representation of $2f_m f_n$

A Zeckendorf representation for $2f_m f_n$ was given in [Filipponi and Hart 1998]. We provide a combinatorial proof for this identity, extending the combinatorial methods from [Artz and Rowell 2009].

Theorem 2.1. *For integers k and n such that $n > 2k+1 > 0$,*

$$2f_{2k+1}f_n = f_{n+2k+1} + \sum_{i=1}^{k} f_{n+2k+3-4i} + f_{n-2k-2}.$$

Proof. The tilings of an $(n+2k+1)$-board having a fault at n make up a $f_n f_{2k+1}$-element set. We will partition this set into a union of four sequences of subsets R_i, S_i, T_i, and U_i, for $0 \le i \le k$, according to Figure 1. Specifically, given a $(n+2k+1)$-board tiling having a fault at n, let i be the number of dominos between the fault and a square closest to the fault: then $i \le k$ (there is at least one square in the $(2k+1)$-board to the right of the fault). Next assign this tiling to the set

- R_i if there are i dominos adjacent to the fault on each side, followed by a square on each side;
- S_i if there are i dominos adjacent to the fault on each side, followed by yet another domino on the left and a square on the right;

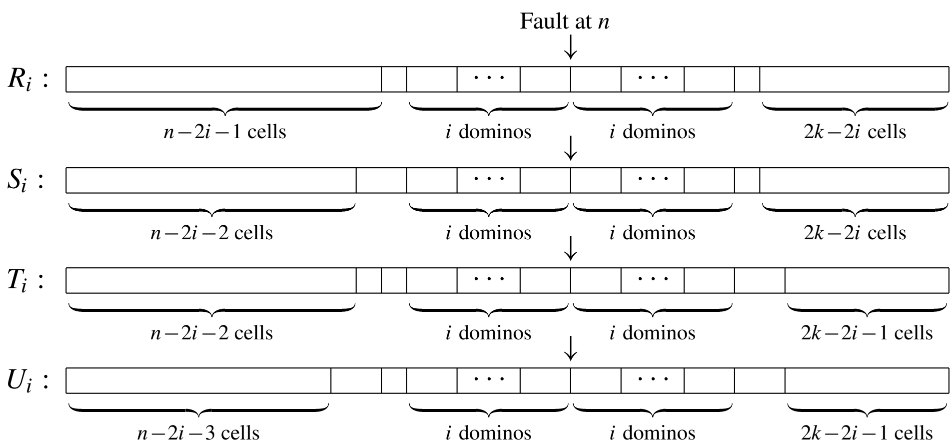

Figure 1. Configurations charaterizing membership in the sets R_i, S_i, T_i and U_i.

T_i if there are i dominos adjacent to the fault on each side, followed by two squares on the left and a domino on the right;

U_i if there are i dominos adjacent to the fault on each side, followed by a square and a domino on the left and a domino on the right.

(Note that T_k and U_k are empty.) Thus, the sets R_i, S_i, T_i, U_i for $0 \le i \le k$ account exactly once for each tiling having a fault at n.

Further, we take a second copy of each of these sets, denoting them by R_i^*, S_i^*, T_i^*, and U_i^*, and we define

$$A_i = R_i \cup R_i^* \cup S_i \cup T_i \cup T_i^* \cup U_i, \qquad B_i = S_i^* \cup U_i^*.$$

It follows that the sets A_i and B_i, for $0 \le i \le k$, account exactly twice for each tiling having a fault at n. Therefore

$$\sum_{i=0}^{k} |A_i \cup B_i| = 2 f_n f_{2k+1},$$

by the first sentence of the proof. To complete the proof, we will show the following equalities:

$$\begin{aligned} |A_0| &= f_{n+2k+1}; \\ |A_i \cup B_{i-1}| &= f_{n+2k+3-4i} \quad \text{for } 1 \le i \le k; \\ |B_k| &= f_{n-2k-2}. \end{aligned}$$

We prove each equality by exhibiting a bijection from the set of tilings of a board of the appropriate size to the set in the left-hand side of the equality. For instance, to show that $|A_0| = f_{n+2k+1}$, we start from the set of all tilings of the $(n+2k+1)$-board; this set, as we know, has f_{n+2k+1} elements. So consider any tiling of the $(n+2k+1)$-board.

- If the tiling has a fault at n and a square next to the fault, on either or both sides, do nothing. This gives an element of $R_0 \cup S_0 \cup T_0 \cup U_0$.
- If the tiling has a fault at n and a domino on both sides of the fault, replace the domino to the left of the fault with two squares, obtaining an element of T_0^*.
- If the tiling does not have a fault at n, split the domino covering cells n and $n+1$ into two squares, obtaining an element of R_0^*.

Since $A_0 = R_0 \cup R_0^* \cup S_0 \cup T_0 \cup T_0^* \cup U_0$ and all elements of the component sets are accounted for, we have shown that $|A_0| = f_{n+2k+1}$.

Next we show that $|A_i \cup B_{i-1}| = f_{n+2k+3-4i}$ for $1 \le i \le k$. Consider any tiling of an $(n+2k+3-4i)$-board, and remove the last tile. Suppose first that the removed tile was a domino, which leaves an $(n+2k+1-4i)$-board.

- If the tiling has a fault at $n-2i$ and a square next to the fault, on either or both sides, insert $2i$ dominos at the fault. This gives an element of $R_i \cup S_i \cup T_i \cup U_i$.
- If the tiling has a fault at $n-2i$ and a domino on both sides of the fault, replace the domino to the left of the fault with two squares and insert $2i$ dominos at the fault, obtaining an element of T_i^*.
- If the tiling does not have a fault at $n-2i$, replace the domino covering the fault with two squares and insert $2i$ dominos between the two squares, obtaining an element of R_i^*.

This accounts for each element of A_i once. Now suppose instead that the tile we removed was a square, which leaves an $(n+2k+2-4i)$-board.

- If the tiling has a fault at $n-2i$, insert $2i-1$ dominos followed by a square at the fault, obtaining an element of S_{i-1}^*.
- If the tiling does not have a fault at $n-2i$, insert a square followed by $2i-1$ dominos just before the domino that covers cell $n-2i$. This gives an element of U_{i-1}^*.

This accounts for each element of B_{i-1} once. Thus $A_i \cup B_{i-1}$ is in bijection with the set of tilings of the $(n+2k+3-4i)$-board.

Lastly, we must show that $|B_k| = f_{n-2k-2}$. Given any tiling of an $(n-2k-2)$-board, append $2k+1$ dominos followed by a square at the right edge, to obtain an element of $B_k = S_k^*$ (recall that U_k^* is empty). This concludes the proof. □

We only present, but do not prove, the case $2f_{2k}f_n$. Its proof is similar to the case presented above and is left to the interested reader.

Theorem 2.2. *For integers k and n such that $n > 2k+1 > 0$,*

$$2f_{2k}f_n = f_{n+2k} + \sum_{i=1}^{k} f_{n+2k+2-4i} + f_{n-2k}.$$

3. Zeckendorf representations of $L_m L_n$

Also given in [Filipponi and Hart 1998] is a Zeckendorf representation of $L_m L_n$. We again extend the notion of squares closest to a given fault to prove our theorem combinatorially.

Lemma 3.1. *Let m and n be positive integers such that $n > m > 1$. Then*

$$f_n f_{m-2} - f_{n-1} f_{m-1} = (-1)^m f_{n-m}.$$

Proof. Let $A^{\{n+m-2,n\}}$ be the set of all tilings of an $(n+m-2)$-board with a fault at n.

For $0 \le i \le \lfloor (m-2)/2 \rfloor$, let $A_{2i}^{\{n+m-2,n\}}$ be the set of all tilings of an $(n+m-2)$-board with a fault at n, i dominos on both sides of the fault and a square at cell

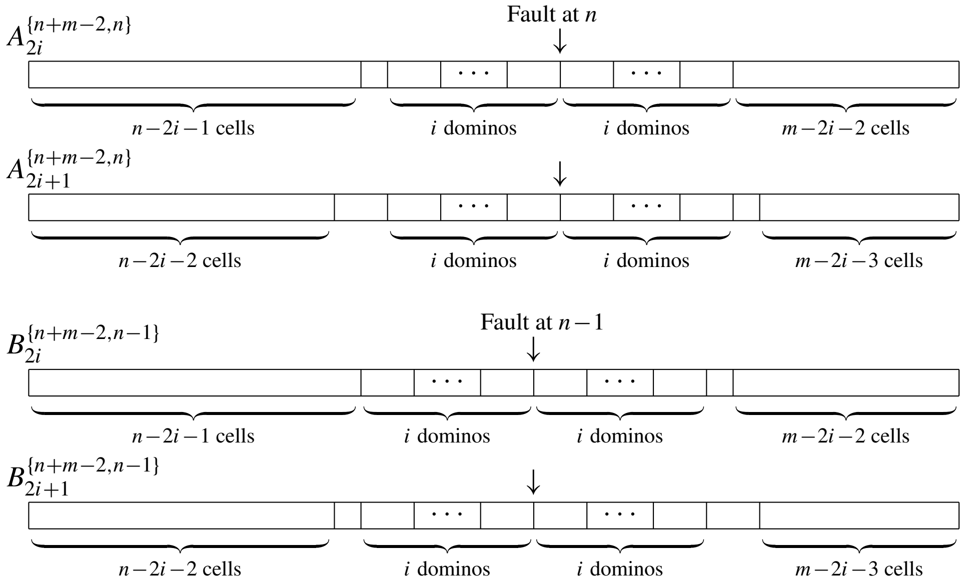

Figure 2. Configurations characterizing membership in various sets.

$n-2i$. For $0 \le i \le \lfloor (m-3)/2 \rfloor$, let $A_{2i+1}^{\{n+m-2,n\}}$ be the set of all tilings of an $(n+m-2)$-board with a fault at n, i dominos on either side of the fault, a domino at cell $n-2i-1$ and a square at cell $n+2i+1$. See Figure 2. For m odd we have

$$A^{\{n+m-2,n\}} = \bigcup_{i=0}^{m-2} A_i^{\{n+m-2,n\}}.$$

If m is even, we need one more set to complete our construction of $A^{\{n+m-2,n\}}$. Let $A_{m-1}^{\{n+m-2,n\}}$ be the set of all tilings of an $(n+m-2)$-board with a fault at n, $m/2-1$ dominos on the right side of the fault and $m/2$ dominos on the left side of the fault. Then

$$A^{\{n+m-2,n\}} = \bigcup_{i=0}^{m-1} A_i^{\{n+m-2,n\}}.$$

Let $B^{\{n+m-2,n-1\}}$ be the set of all tilings of an $(n+m-2)$-board with a fault at $n-1$.

For $0 \le i \le \lfloor (m-2)/2 \rfloor$, let $B_{2i}^{\{n+m-2,n-1\}}$ be the set of all tilings of an $(n+m-2)$-board with a fault at $n-1$, i dominos on either side of the fault and a square at cell $n+2i$. For $0 \le i \le \lfloor (m-3)/2 \rfloor$, let $B_{2i+1}^{\{n+m-2,n-1\}}$ be the set of all tilings of an $(n+m-2)$-board with a fault at $n-1$, i dominos on either side of the fault, a square at cell $n-2i-1$ and a domino at cell $n+2i$. See again Figure 2. For m even we have

$$B^{\{n+m-2,n-1\}} = \bigcup_{i=0}^{m-2} B_i^{\{n+m-2,n-1\}}.$$

If m is odd, we need one more set to complete our construction of $B^{\{n+m-2,n-1\}}$. Let $B_{m-1}^{\{n+m-2,n-1\}}$ be the set of all tilings of an $(n+m-2)$-board with a fault at $n-1$ and $(m-1)/2$ dominos on either side of the fault. Then

$$B^{\{n+m-2,n-1\}} = \bigcup_{i=0}^{m-2} B_i^{\{n+m-2,n-1\}}.$$

Note that $\left|A_i^{\{n+m-2,n\}}\right| = \left|B_i^{\{n+m-2,n-1\}}\right|$ for $0 \le i \le m-2$, since the cardinality of each of these sets is just $f_{n-i-1} f_{m-i-2}$. Thus

$$\left|A^{\{n+m-2,n\}}\right| - \left|B^{\{n+m-2,n-1\}}\right| = \begin{cases} \left|A_{m-1}^{\{n+m-2,n\}}\right| & \text{if } m \text{ is even,} \\ -\left|B_{m-1}^{\{n+m-2\}}\right| & \text{if } m \text{ is odd.} \end{cases}$$

Noting that

$$\left|A^{\{n+m-2,n\}}\right| = f_n f_{m-2}, \quad \left|B^{\{n+m-2,n-1\}}\right| = f_{n-1} f_{m-1},$$
$$\left|A_{m-1}^{\{n+m-1,n\}}\right| = \left|B_{m-1}^{\{n+m-2\}}\right| = f_{n-m},$$

we see that

$$\begin{aligned} f_n f_{m-2} - f_{n-1} f_{m-1} &= \left|A^{\{n+m-2,n\}}\right| - \left|B^{\{n+m-2,n-1\}}\right| \\ &= \begin{cases} \left|A_{m-1}^{\{n+m-2,n\}}\right| & \text{if } m \text{ is even,} \\ -\left|B_{m-1}^{\{n+m-2\}}\right| & \text{if } m \text{ is odd,} \end{cases} \\ &= (-1)^m f_{n-m}. \end{aligned}$$

□

We present four corollaries helpful in proving the Zeckendorf representation of $L_m L_n$. In each of them, an application of Lemma 1.2 is used.

Corollary 3.2. *For integers k and n such that $n > 2k > 1$,*

$$f_n f_{2k-2} - (f_{n+2k} - f_n f_{2k}) = f_{n-2k}.$$

Proof. Let $m \to 2k$ in Lemma 3.1 and note that

$$f_{n-1} f_{2k-1} = f_{n+2k} - f_n f_{2k}.$$

□

Corollary 3.3. *For integers k and n such that $n-2 > 2k > 1$,*

$$f_{n-2} f_{2k-2} - (f_{n+2k-2} - f_{n-2} f_{2k}) = f_{n-2k-2}.$$

Proof. Let $m \to 2k$ and $n \to n-2$ in Lemma 3.1 and note that

$$f_{n-3} f_{2k-1} = f_{n+2k-2} - f_{n-2} f_{2k}.$$

□

Corollary 3.4. *For integers k and n such that $n-1 > 2k+2 > 1$,*

$$(f_{n+2k+1} - f_n f_{2k+1}) - f_{n-2} f_{2k+1} = f_{n-2k-3}.$$

Proof. Let $m \to 2k+2$ and $n \to n-1$ in Lemma 3.1 and note that

$$f_{n-1} f_{2k} = f_{n+2k+1} - f_n f_{2k+1}. \qquad \square$$

Corollary 3.5. *For integers k and n such that $n-1 > 2k > 1$,*

$$(f_{n+2k-1} - f_n f_{2k-1}) - f_{n-2} f_{2k-1} = f_{n-2k-1}.$$

Proof. Let $m \to 2k$ and $n \to n-1$ in Lemma 3.1 and note that

$$f_{n-1} f_{2k-2} = f_{n+2k-1} - f_n f_{2k-1}. \qquad \square$$

Theorem 3.6. *For integers k and n such that $n-2 > 2k > 1$,*

$$L_{2k} L_n = f_{n+2k} + f_{n+2k-2} + f_{n-2k} + f_{n-2k-2}.$$

Proof. By Lemma 1.3 we know that

$$L_{2k} L_n = f_n f_{2k} + f_n f_{2k-2} + f_{n-2} f_{2k} + f_{n-2} f_{2k-2}.$$

Rearranging terms we see that our theorem can be rewritten as

$$f_n f_{2k-2} - (f_{n+2k} - f_n f_{2k}) + f_{n-2} f_{2k-2} - (f_{n+2k-2} - f_{n-2} f_{2k}) = f_{n-2k} + f_{n-2k-2}.$$

Applying Corollaries 3.2 and 3.3 concludes our proof. $\square$

Before moving on to the case $L_n L_{2k+1}$, we need another lemma:

Lemma 3.7. *For integers k and n such that $n+2 > 2k-1 > 0$,*

$$\begin{aligned} f_{n-2k-4} + f_{n-2k-1} + f_{n+2k+1} + \sum_{j=1}^{2k-1} f_{n-2k+2j} \\ = f_{n+2k+1} + f_{n+2k-1} - f_{n-2k-1} - f_{n-2k-3}. \end{aligned}$$

Proof. We will first turn our eye to the summation on the left-hand side of our identity. Applying Lemma 1.1 we can collapse this sum to two terms:

$$\begin{aligned} \sum_{j=1}^{2k-1} f_{n-2k+2j} &= (f_0 + f_2 + \cdots + f_{n+2k-2}) - (f_0 + f_2 + \cdots + f_{n-2k}) \\ &= f_{n+2k-1} - f_{n-2k+1}. \end{aligned}$$

It is left to show that

$$\begin{aligned} f_{n-2k-4} + f_{n-2k-1} + f_{n+2k+1} + f_{n+2k-1} - f_{n-2k+1} \\ = f_{n+2k+1} + f_{n+2k-1} - f_{n-2k-1} - f_{n-2k-3}, \end{aligned}$$

or, equivalently,

$$f_{n-2k-4} + f_{n-2k-3} + f_{n-2k-1} = f_{n-2k+1} - f_{n-2k-1}. \tag{3-1}$$

We do this by showing that both sides of our identity count the total number of ways of tiling an $(n-2k)$-board.

On the left-hand side of (3-1) we have all the tilings of an $(n-2k-4)$-board, an $(n-2k-3)$-board and an $(n-2k-1)$-board. To each of the tilings of length $n-2k-4$ add two dominos at the end of the board. To those of length $n-2k-3$ add a square followed by a domino at the end of the board. To the tilings of length $n-2k-1$ add a square at the end of the board. This constructs all tilings of length $n-2k$.

On the right-hand side of (3-1) we have all the tilings of an $(n-2k+1)$-board and an $(n-2k-1)$-board. If we append a domino to all of our tilings of length $n-2k-1$, we see that our right-hand side can be interpreted as all tilings of length $n-2k+1$ that do not end in a domino. Thus, we are counting all tilings of length $n-2k+1$ that end in a square. Removing the square in each of the tilings leaves us with all tilings of length $n-2k$. □

Theorem 3.8. *For integers k and n such that $n-3>2k>1$,*

$$L_{2k+1}L_n = f_{n-2k-4} + f_{n-2k-1} + f_{n+2k+1} + \sum_{j=1}^{2k-1} f_{n-2k+2j}.$$

Proof. Applying Lemmas 1.3 and 3.7, we can rewrite this equality as

$$\begin{aligned} f_n f_{2k+1} + f_{n-2} f_{2k+1} + f_n f_{2k-1} + f_{n-2} f_{2k-1} \\ = f_{n+2k+1} + f_{n+2k-1} - f_{n-2k-1} - f_{n-2k-3}. \end{aligned}$$

Rearranging terms, we see that this is equivalent to

$$\begin{aligned} f_{n-2k-3} + f_{n-2k-1} \\ = (f_{n+2k+1} - f_n f_{2k+1}) - f_{n-2} f_{2k+1} + (f_{n+2k-1} - f_n f_{2k-1}) - f_{n-2} f_{2k-1}. \end{aligned}$$

Applying Corollaries 3.4 and 3.5 concludes our proof. □

4. Answering an open problem and new Zeckendorf representations

In [Artz and Rowell 2009], the following theorem was given and an open problem was posed to find a combinatorial proof. The following proof gives an answer to the open question.

Theorem 4.1. *For integers m and n such that $n>m>0$,*

$$(f_{m+1} + f_{m-1}) f_n = f_{n+m+1} - (-1)^m f_{n-m-1}.$$

Proof. Let $m \to 2k+1$ in Lemma 3.1. Then

$$f_n f_{2k-1} - f_{n-1} f_{2k} = -f_{n-2k-1}.$$

Applying Lemma 1.2, we see that this is equivalent to

$$f_n f_{2k-1} - (f_{n+2k+1} - f_n f_{2k+1}) = -f_{n-2k-1}.$$

Rearranging terms we see that this proves the case m odd of our theorem. Similarly, we use Corollary 3.2 to prove the case m even. □

Filipponi and Hart introduced Zeckendorf representations of mixed triple products including both Fibonacci and Lucas numbers, namely of the form $f_m^2 L_n$ and $L_m^2 f_n$. We extend their work and present the Zeckendorf representations of a mixed products including three variables. In each of the following identities we assume that our variables take on appropriate integer values.

The remainder of this section was motivated almost entirely by the even case of Theorem 4.1. For sufficiently large values of n, we can ensure that our Zeckendorf representations do not overlap.

Theorem 4.2. *For $n > 2j > m$ and $n > 2j + m$,*

$$f_m L_{2j} f_n = \begin{cases} f_{n+2j-m} + f_{n-2j-m} + \sum_{i=1}^{m/2} f_{n+2j+m-4i-1} + \sum_{i=1}^{m/2} f_{n-2j+m-4i-1} & \text{for } m \text{ even,} \\ \sum_{i=1}^{(m+1)/2} f_{n+2j-m-3+4i} + \sum_{i=1}^{(m+1)/2} f_{n-2j-m-3+4i} & \text{for } m \text{ odd.} \end{cases}$$

Proof. We begin with the first case, say $m = 2k$ for some positive integer k. Applying Theorem 4.1 with $m \to 2j$, followed by Theorem 1.6 with $n \to n + 2j$ and $n \to n - 2j$, we get

$$\begin{aligned} f_{2k} L_{2j} f_n &= f_{2k}(f_{n+2j} + f_{n-2j}) \\ &= f_{n+2j-2k} + f_{n-2j-2k} + \sum_{i=1}^{k-1} f_{n+2j+2k-4i-1} + \sum_{i=1}^{k-1} f_{n-2j+2k-4i-1}. \end{aligned}$$

Next let $m = 2k + 1$ instead. Apply Theorem 4.1 with $m \to 2j$, followed by Theorem 1.5 with $n \to n + 2j$ and $n \to n - 2j$ to see that

$$\begin{aligned} f_{2k+1} L_{2j} f_n &= f_{2k+1} f_{n+2j} + f_{2k+1} f_{n-2j} \\ &= \sum_{i=1}^{k+1} f_{n+2j-2k-4+4i} + \sum_{i=1}^{k+1} f_{n-2j-2k-4+4i}. \end{aligned}$$ □

Noting that $L_m = f_{m-2} + f_m$, it is easy to extend our previous theorem to the following:

Theorem 4.3. *For $n > 2j > m$ and $n > 2j + m$*

$$L_m L_{2j} f_n = \begin{cases} f_{n-2j-m} + f_{n-2j+m} + f_{n+2j-m} + f_{n+2j+m} & \text{for } m \text{ even,} \\ \sum_{i=1}^{m-1} f_{n+2j-m-1+2i} + \sum_{i=1}^{m} f_{n-2j-m-1+2i} & \text{for } m \text{ odd.} \end{cases}$$

Proof. Let $m = 2k$ for some positive integer k. Applying Theorem 4.1 with $m \to 2j$, followed by Theorem 4.1 twice more with $m \to n+2j$ and $m \to n-2j$, we see that

$$L_{2k}L_{2j}f_n = L_{2k}(f_{n+2j} + f_{n-2j}) = f_{n-2j-2k} + f_{n-2j+2k} + f_{n+2j-2k} + f_{n+2j+2k}.$$

If instead $m = 2k+1$, rewriting L_{2k+1} as $f_{2k+1} + f_{2k-1}$ and applying Theorem 4.2 twice yields our result. □

We next consider the Zeckendorf representation of a Lucas triple product.

Lemma 4.4. *For* $k > 1$,

$$2\sum_{i=1}^{k} f_{n+2i-2} = f_{n-2} + f_{n+2k} + \sum_{i=1}^{k-2} f_{n+2i}.$$

Proof. Noting $2f_m = f_{m-2} + f_{m+1}$ [Benjamin and Quinn 2003, Identity 16, page 13], we see that

$$\begin{aligned}2\sum_{i=1}^{k} f_{n+2i-2} &= \sum_{i=1}^{k} 2f_{n+2i-2} = \sum_{i=1}^{k} f_{n+2i-4} + f_{n+2i-1} = \sum_{i=1}^{k} f_{n-4+2i} + \sum_{i=1}^{k} f_{n+2i-1}\\ &= f_{n-2} + \sum_{i=1}^{k-1} f_{n+2i-2} + \sum_{i=1}^{k-1} f_{n+2i-1} + f_{n+2k-1}.\end{aligned}$$

Finally, noting that $f_m = f_{m-1} + f_{m-2}$, we see that

$$\begin{aligned}2\sum_{i=1}^{k} f_{n+2i-2} &= f_{n-2} + \sum_{i=1}^{k-1} f_{n+2i-2} + \sum_{i=1}^{k-1} f_{n+2i-1} + f_{n+2k-1}\\ &= f_{n-2} + \sum_{i=1}^{k-1} f_{n+2i} + f_{n+2k-1} = f_{n-2} + \sum_{i=1}^{k-2} f_{n+2i} + f_{n+2k-2} + f_{n+2k-1}\\ &= f_{n-2} + \sum_{i=1}^{k-2} f_{n+2i} + f_{n+2k}.\end{aligned}$$ □

Theorem 4.5. *For* $n > 2j > m$ *and* $n > 2j + m + 2$

$$L_mL_{2j}L_n = \begin{cases} f_{n-2j-m} + f_{n-2j+m} + f_{n+2j-m} + f_{n+2j+m} + f_{n-2j-m-2} \\ \quad + f_{n-2j+m-2} + f_{n+2j-m-2} + f_{n+2j+m-2} & \text{for } m \text{ even,} \\ f_{n+2j-m-3} + f_{n+2j-m} + \sum_{i=1}^{m} f_{n+2j-m+2i+1} \\ \quad + f_{n-2j-m-3} + f_{n-2j-m} + \sum_{i=1}^{m} f_{n-2j-m+2i+1} & \text{for } m \text{ odd.} \end{cases}$$

Proof. Let $m = 2k$ for some positive integer k. Rewriting L_n as $f_n + f_{n+2}$ and applying Theorem 4.3 twice yields the result for m even.

Let $m = 2k+1$. Rewriting L_n as $f_n + f_{n-2}$ and applying Theorem 4.3 twice we see that

$$L_{2k+1}L_{2j}L_n = f_{n+2j-2k-2} + f_{n+2j+2k} \\ +2\sum_{i=1}^{2k} f_{n+2j-2k-2+2i} + f_{n-2j-2k-2} + f_{n-2j+2k} + 2\sum_{i=1}^{2k} f_{n-2j-2k-2+2i}.$$

Applying Lemma 4.4 to each of our series with $n \to n+2j-2k$, $k \to 2k$ and $n \to n-2j-2k$, $k \to 2k$, respectively, yields,

$$L_{2k+1}L_{2j}L_n = 2f_{n+2j-2k-2} + f_{n+2j+2k} + f_{n+2j+2k+2} + \sum_{i=1}^{2k-1} f_{n+2j-2k+2i} \\ +2f_{n-2j-2k-2} + f_{n-2j+2k} + f_{n-2j+2k+2} + \sum_{i=1}^{2k-1} f_{n-2j-2k+2i}.$$

Finally, we will apply Theorem 1.6 with $2k \to 2$ and with $n \to n+2j-2k-2$ and $n \to n-2j-2k-2$, respectively. □

We present our last Zeckendorf representation of a triple product,

Theorem 4.6. *For $n > m > 2j$ and $n > m+2j$,*

$$L_{2j}f_mf_n = \begin{cases} f_{n-m+2j-1} + f_{n+m-2j} + \sum_{i=1}^{j} f_{n-m-2j-3+4i} + \sum_{i=1}^{j} f_{n+m-2j-1+4i} \\ \qquad + \sum_{i=1}^{m/2-j} f_{n-m+2j+4i} \quad \text{for } m \text{ odd}, \\ f_{n-m-2j} + f_{n+m-2j} + \sum_{i=1}^{j} f_{n-m-2j-1+4i} + \sum_{i=1}^{j} f_{n+m-2j-1+4i} \\ \qquad + \sum_{i=1}^{m/2-j} f_{n-m+2j-2+4i} \quad \text{for } m \text{ even}. \end{cases}$$

Proof. Let $m = 2k$ for some positive integer k. Applying Theorem 1.6 we see that

$$L_{2j}f_{2k}f_n = L_{2j}\big(f_{n-2k} + \sum_{i=1}^{k} f_{n-2k-1+4i}\big).$$

Now distribute L_{2j} and apply Theorem 4.1 to each term. Rearranging terms we see that

$$\begin{aligned} L_{2j}f_{2k}f_n &= f_{n-2k-2j} + f_{n-2k+2j} + \sum_{i=1}^{k} (f_{n-2k-2j-1+4i} + f_{n-2k+2j-1+4i}) \\ &= f_{n-2k-2j} + f_{n-2k+2j} + \sum_{i=1}^{j} f_{n-2k-2j-1+4i} + 2\sum_{i=1}^{k-j} f_{n-2k+2j-1+4i} \\ &\qquad + \sum_{i=1}^{j} f_{n+2k-2j-1+4i}. \end{aligned}$$

We can now apply Theorem 1.6, with $2k \to 2$. Recalling that $f_n = f_{n-1} + f_{n-2}$, we obtain

$$
\begin{aligned}
L_{2j} f_{2k} f_n &= f_{n-2k-2j} + f_{n-2k+2j} + \sum_{i=1}^{j} f_{n-2k-2j-1+4i} + \sum_{i=1}^{j} f_{n+2k-2j-1+4i} \\
&\qquad + \sum_{i=1}^{k-j} (f_{n-2k+2j+4i} + f_{n-2k+2j-3+4i}) \\
&= f_{n-2k-2j} + f_{n-2k+2j} + \sum_{i=1}^{j} f_{n-2k-2j-1+4i} + \sum_{i=1}^{j} f_{n+2k-2j-1+4i} \\
&\qquad + f_{n-2k+2j+1} + f_{n+2k-2j} + \sum_{i=1}^{k-j-1} (f_{n-2k+2j+4i} + f_{n-2k+2j+1+4i}) \\
&= f_{n-2k-2j} + f_{n+2k-2j} + \sum_{i=1}^{j} f_{n-2k-2j+4i-1} + \sum_{i=1}^{j} f_{n+2k-2j+4i-1} \\
&\qquad + \sum_{i=1}^{k-j} f_{n-2k+2j+4i-2}.
\end{aligned}
$$

We turn to the case m odd, $m = 2k+1$. Applying Theorem 1.5 we can see that

$$
L_{2j} f_{2k+1} f_n = L_{2j} \Big(\sum_{i=1}^{k+1} f_{n-2k-4+4i} \Big).
$$

Now distribute L_{2j} and apply Theorem 4.1 to each term. Rewriting terms reveals

$$
L_{2j} f_{2k+1} f_n = \sum_{i=1}^{j} f_{n-2k-2j-4+4i} + \sum_{i=1}^{j} f_{n+2k-2j+4i} + 2 \sum_{i=1}^{k-j+1} f_{n-2k+2j-4+4i}.
$$

We now apply Theorem 1.6 with $2k \to 2$, recalling the recursion relation of the Fibonacci sequence, which shows

$$
\begin{aligned}
L_{2j} f_{2k+1} f_n &= \sum_{i=1}^{j} f_{n-2k-2j-4+4i} + \sum_{i=1}^{j} f_{n+2k-2j+4i} \\
&\qquad + \sum_{i=1}^{k-j+1} f_{n-2k+2j-3+4i} + f_{n-2k+2j-6+4i} \\
&= f_{n-2k+2j-2} + f_{n+2k-2j+1} + \sum_{i=1}^{j} f_{n-2k-2j-4+4i} + \sum_{i=1}^{j} f_{n+2k-2j+4i} \\
&\qquad + \sum_{i=1}^{k-j} f_{n-2k+2j-1+4i}. \qquad \square
\end{aligned}
$$

5. Conclusions and future work

Having proved the Zeckendorf representation of $2 f_n f_m$, we can see that we can prove individual cases of $k f_n f_m$ using similar methods. Further, Lemma 3.1 seems to hold the key to many interesting Zeckendorf representations involving Lucas numbers. We find it especially intriguing that it led to mixed products of three variables involving even Lucas numbers. We did, however, have little luck finding closed form Zeckendorf representation of $f_p L_m f_n$ where m is odd.

The Zeckendorf representations in Section 4 are proved using many combinatorial mappings of our boards and bracelets to produce their Zeckendorf representations. We believe much insight into the problem could be found by proving each with a single mapping.

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