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with an application to a sum of secant powers

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# An observation on generating functions with an application to a sum of secant powers

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(Communicated by Nigel Boston)

Suppose that  $P(x), Q(x) \in \mathbb{Z}[x]$  are two relatively prime polynomials, and that  $P(x)/Q(x) = \sum_{n=0}^{\infty} a_n x^n$  has the property that  $a_n \in \mathbb{Z}$  for all  $n$ . We show that if  $Q(1/\alpha) = 0$ , then  $\alpha$  is an algebraic integer. Then, we show that this result can be used to provide a solution to Problem 11213(b) of the *American Mathematical Monthly* (2006).

## 1. Introduction and statement of results

This paper has two goals. One is to prove this general observation:

**Theorem 1.** *Suppose  $P(x), Q(x) \in \mathbb{Z}[x]$  are relatively prime polynomials with integer coefficients and their quotient is the generating function of an integer series:*

$$\frac{P(x)}{Q(x)} = \sum_{n=0}^{\infty} a_n x^n, \quad \text{with } a_n \in \mathbb{Z} \text{ for all } n.$$

*Then the inverse of any root of  $Q$  is an algebraic integer.*

The second goal is to apply this result to solve a problem from the *American Mathematical Monthly*:

**Problem 11213** [AMM 2006]. *Proposed by Stanley Rabinowitz, Chelmsford, MA.* For positive integers  $n$  and  $m$  with  $n$  odd and greater than 1, let

$$S(n, m) = \sum_{k=1}^{(n-1)/2} \sec^{2m} \left( \frac{k\pi}{n+1} \right).$$

- (a) Show that if  $n$  is one less than a power of 2, then  $S(n, m)$  is a positive integer.
- (b\*) Show that if  $n$  does not have the form of Part (a), then there exists a positive integer  $m$  such that  $S(n, m)$  is not an integer.

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The \* indicates that no solution was known to the *Monthly* editors. (A solution to (a) was provided in [AMM 2008].) We solve part (b) of Problem 11213 by proving the contrapositive:

**Theorem 2.** *Let  $n > 1$  be an odd integer. If, for every positive integer  $m$ , the sum*

$$S(n, m) = \sum_{k=1}^{(n-1)/2} \sec^{2m} \frac{k\pi}{n+1}$$

*has an integer value, then  $n+1$  is a power of 2.*

A similar result to Theorem 1 (but less general) had appeared before in the *Monthly*, as a problem proposed and solved by Michael Larsen:

**Problem E 2993** [AMM 1983; 1986]. Let  $\alpha_1, \alpha_2, \dots, \alpha_n$  a complex numbers such that  $\sum_1^n \alpha_i^m$  is an integer for every positive  $m$ ; then the polynomial  $\prod_1^n (x - \alpha_i)$  has integer coefficients.

Here is an outline of the paper. After recalling the necessary concepts from algebraic number theory in Section 2, we prove in Section 3 two intermediate results:  $S(n, m)$  is always rational, and the generating function of the sequence  $\{S(n, m)\}_{m>0}$  (for fixed odd  $n > 0$ ) has integer coefficients. In Section 4 we prove Theorem 1, from which Theorem 2 follows easily given the intermediate results.

## 2. Background

We review some basic algebraic number theory, which is carefully laid out in [Stewart and Tall 2002], for example. (This citation will be abbreviated as [ST].)

An *algebraic number* is any zero of a polynomial with integer coefficients. An *algebraic integer* is any zero of a monic polynomial with integer coefficients. The set of algebraic numbers is a field, and the set of algebraic integers forms a ring [ST, Theorems 2.1 and 2.9].

For example, if  $p$  is prime,  $\zeta_p = e^{2\pi i/p}$  is an algebraic integer since it is a zero of the polynomial  $x^p - 1$ .

The *minimal polynomial* of an algebraic number  $\alpha$  is the monic polynomial  $p(x)$  with rational coefficients and the smallest possible degree such that  $p(\alpha) = 0$ . All polynomials of which  $\alpha$  is a root are divisible by  $p$ . For example,  $r(x) = x^{p-1} + x^{p-2} + \dots + x + 1 = (x^p - 1)/(x - 1)$  is the minimal polynomial of  $\zeta_p$ .

**Definition.** If  $K$  is a field contained in  $L$ , we say that  $L$  is a field extension of  $K$ , and we denote this by  $L : K$ .

If  $K$  is a field and  $\alpha$  is an algebraic number let  $K(\alpha)$  denote the smallest field containing all the elements of  $K$  and  $\alpha$ . One way to think about field extensions is that if  $L : K$  is a field extension, then  $L$  has a natural structure as a vector space over

$K$ . The dimension of this vector space, which is called the *degree*, is represented with  $[L : K]$ . If  $[L : K]$  is a number the field extension is called finite. If  $H, K$ , and  $L$  are fields such that  $K$  is a subset of  $L$  and  $H$  is a subset of  $K$ , then

$$[L : H] = [L : K][K : H] \tag{1}$$

[ST, Theorem 1.10].

In algebraic number theory field extensions of the form  $\mathbb{Q}(\alpha)$  are of interest. If  $\alpha$  is an algebraic number, then  $[\mathbb{Q}(\alpha) : \mathbb{Q}]$  equals the degree of the minimal polynomial of  $\alpha$  [ST, Theorem 1.1]. A field  $K$  is called an *algebraic number field* if  $[K : \mathbb{Q}]$  is finite. If  $K = \mathbb{Q}(\alpha)$  and  $\alpha$  is an algebraic number, then the ring of algebraic integers in  $K$  is finitely generated as an abelian group [ST, Theorem 2.16].

**Definition.** If  $K = \mathbb{Q}(\alpha)$  is an algebraic number field of degree  $n$ , then there are  $n$  distinct monomorphisms  $\sigma_1, \dots, \sigma_n$  from  $K$  to  $\mathbb{C}$ . The *conjugates* of an element  $\beta \in K$  are the numbers  $\sigma_i(\beta)$  for all  $i$  between 1 and  $n$ .

The conjugates of an algebraic number  $\alpha$  are the zeros of the minimal polynomial of  $\alpha$ . For example, if  $\alpha = \zeta_n = e^{2\pi i/n}$ , where  $n > 0$  is an integer, then  $\alpha$  has  $\phi(n)$  conjugates in  $\mathbb{Q}(\alpha)$ , where  $\phi$  is the Möbius function. The conjugates of  $\zeta_n$  are all the elements in the set

$$\{e^{2\pi ik/n} : (k, n) = 1\}.$$

This information can be found in [Milne 2009, page 93].

**Definition.** Let  $K = \mathbb{Q}(\alpha)$  be an algebraic number field, and consider  $\beta \in K$ . The *trace* of  $\beta$  in  $K$ , denoted by  $\text{Tr}_K \beta$ , is the sum of all the conjugates of  $\beta$ . The *norm* of  $\beta$  in  $K$ , denoted by  $N_K(\beta)$ , is the product of all of the conjugates of  $\beta$ .

Thus  $\text{Tr}_K \zeta_p = -1$  and  $N_K(\zeta_p) = (-1)^{p-1}$  for  $p$  prime, where  $K = \mathbb{Q}(\zeta_p)$ . If one notes that

$$\frac{\zeta_n + \zeta_n^{-1}}{2} = \cos \frac{2\pi}{n}$$

and applies (1) one can see that the conjugates of  $\alpha = \cos \frac{2\pi}{n}$  in  $\mathbb{Q}(\alpha)$  are all the elements in the set

$$\left\{ \cos \frac{2\pi k}{n} : (k, n) = 1, 0 < k < n/2 \right\}. \tag{2}$$

A formal proof of this can be found in [Milne 2009, pages 95–96]. Also, as a consequence of Theorem 2.6(a), Lemma 2.13, and Lemma 1.7 of [ST], if  $\alpha$  is an algebraic number its trace is rational; and as a consequence of Lemma 2.14 of the same reference, if  $\alpha$  is an algebraic integer its norm is an integer.

### 3. Intermediate results

**Lemma 3.** *If  $n > 1$  is odd and  $m \geq 1$ , the sum  $S(n, m)$  of Theorem 2 is a rational number.*

*Proof.* We make use of the trigonometric identity  $\sec^2 x = \frac{2}{\cos(2x)+1}$  to write  $\sec^{2m} x = f(\cos 2x)$ , where

$$f(x) := \left( \frac{2}{x+1} \right)^m.$$

Then, dropping  $m$  from the notation and introducing  $N = n + 1$  for convenience, we can rewrite our sum as

$$\sum_{0 < k < N/2} s(k), \quad \text{where } s(k) := f\left(\cos \frac{2\pi k}{N}\right). \quad (3)$$

We assume at first that  $N/2$  is an odd prime. All the  $s(k)$  then lie in the extension  $K = \mathbb{Q}(\cos 2\pi/N)$ , as follows from the characterization (2) (with  $n$  in that formula equal to  $N$  here). More precisely, if  $k$  is odd,  $\cos 2\pi k/N$  is a conjugate of  $\cos 2\pi/N$  in  $K$ . If  $k$  is even,  $\cos 2\pi k/N$  equals  $-\cos 2\pi k'/N$ , for  $k' = N/2 - k$  odd; therefore it is a conjugate of  $-\cos 2\pi/N$ . Either way,  $\cos 2\pi k/N$  lies in  $K$ , and therefore so does  $s(k)$ , since  $f$  is a rational function.

The operation of taking conjugates commutes with applying  $f$  (monomorphisms preserve sums, products and inverses, and fix the numbers 1 and 2). Putting this together with the previous paragraph, we conclude that half of the  $s(k)$  (those where  $k$  is odd) make up the conjugates in  $K$  of  $s(1)$ , while the other half make up the conjugates of  $s(2)$  (taking  $k = 2$  as a representative of the even  $k$ 's). It follows that

$$\sum_{k=1}^{N/2-1} s(k) = \text{Tr}_K s(1) + \text{Tr}_K s(2) = \text{Tr}_K f\left(\cos \frac{2\pi k}{N}\right) + \text{Tr}_K f\left(\cos \frac{2 \times 2\pi k}{N}\right).$$

Thus  $S(n, m)$  is the sum of two traces of algebraic numbers, and so rational.

Now let  $N/2$  be arbitrary. Our strategy is the same: we partition the values of  $k$  according to their gcd with  $N$ . Let  $d_1, \dots, d_l$  be all the divisors of  $N$  apart from  $N$  and  $N/2$ , and define

$$D_i := \{k : \gcd(k, N) = d_i, 0 < k < N/2\} = \{d_i j : \gcd(j, N/d_i) = 1, 0 < j < N/(2d_i)\}.$$

The  $D_i$  are disjoint, and together they account for all the  $k$  in the sum (3). Moreover,

$$\sum_{k \in D_i} s(k) = \sum_{\substack{j : \gcd(j, N/d_i) = 1 \\ 0 < j < N/(2d_i)}} f\left(\cos \frac{2\pi j}{N/d_i}\right) = \text{Tr}_{\mathbb{Q}(\cos \frac{2\pi}{N/d_i})} f\left(\cos \frac{2\pi}{N/d_i}\right),$$

where the last equality follows from the same reasoning used earlier for  $k$  odd (with  $N$  replaced by  $N/d_i$ ). We have expressed  $S(n, m)$  as a sum of traces of algebraic numbers, which means it is rational.  $\square$

This result allows us to prove that the generating function for the sequence  $\{S(n, m)\}_{m>0}$  (for fixed odd  $n > 0$ ) is a rational function.

**Lemma 4.** *If  $n > 1$  is odd,  $m \geq 1$ , and*

$$F_n(x) = \sum_{m=0}^{\infty} S(n, m)x^m,$$

*then there exist  $P(x), Q(x) \in \mathbb{Z}[x]$  such that  $F_n(x) = P(x)/Q(x)$ .*

*Proof.* Using the formula for the sum of a geometric series, we write

$$F_n(x) = \sum_{m=0}^{\infty} \left( \sum_{k=1}^{(n-1)/2} \sec^{2m} \frac{k\pi}{n+1} \right) x^m = \sum_{k=1}^{(n-1)/2} \frac{1}{1 - x \sec^2 \frac{k\pi}{n+1}},$$

so that

$$Q(x) = \prod_{k=1}^{(n-1)/2} \left( 1 - x \sec^2 \frac{k\pi}{n+1} \right).$$

We will show that  $Q(x)$  is a polynomial with rational coefficients. Set

$$b_k := \sec^2 \frac{k\pi}{n+1},$$

where  $1 \leq k \leq (n-1)/2$ . Let  $s_i$  be the sum of the products of each  $i$ -element subset of the set  $\{b_1, b_2, \dots, b_{(n-1)/2}\}$  (in other words,  $s_i$  is the  $i$ -th elementary symmetric polynomial applied to the  $b_i$ ). The coefficient of  $x^i$  in  $Q(x)$  is  $(-1)^i s_i$ . Also, let

$$p_r := \sum_{k=1}^n b_k^r.$$

The Newton–Girard formulas tell us that

$$p_i - s_1 p_{i-1} + s_2 p_{i-2} + \dots + (-1)^{i-1} s_{i-1} p_1 + (-1)^i i s_i = 0,$$

for all  $1 \leq i \leq (n-1)/2$ . It is clear that  $p_i$  is rational for all  $i$  by Lemma 4. An easy induction argument implies that  $s_i$  is rational for all  $i$ . Since the coefficients of  $Q(x)$  can be expressed in terms of the  $s_i$ , we see that  $Q(x)$  has rational coefficients. Thus  $P(x) = F_n(x)Q(x)$  has rational coefficients. The desired result follows.  $\square$

**Lemma 5.** *Suppose that  $a$  and  $b$  are algebraic numbers, and*

$$F(x) = \frac{a}{1 - bx} = \sum_{n=0}^{\infty} a_n x^n.$$

*If  $a_n$  is an algebraic integer for all  $n$ , then  $b$  is an algebraic integer.*

*Proof.* The assumption implies that  $a_n = ab^n$ . We know that  $ab^n$  is an algebraic integer for all  $n$ , and so lies in the ring of algebraic integers of the field  $K = \mathbb{Q}(b)$ . This ring is finitely generated as an abelian group. Suppose that it is generated by  $\{v_1, v_2, \dots, v_l\}$ . Then  $b^n$  must be in the finitely generated abelian group generated by  $\{v_1/a, \dots, v_l/a\}$  for all  $n$ . Lemma 2.8 of [ST] states that a complex number  $\theta$  is an algebraic integer if and only if the additive group generated by all powers  $1, \theta, \theta^2, \dots$  is finitely generated. Thus,  $b$  is an algebraic integer.  $\square$

Now, we wish to expand upon the ideas presented in Lemma 5.

**Definition.** A sequence  $\{a_n\}$  of algebraic numbers has a *bounded denominator* if there exists a positive integer  $m$  such that  $ma_n$  is an algebraic integer for all  $n$ .

**Lemma 6.** *Let*

$$F(x) = \sum_{n=0}^{\infty} a_n x^n,$$

where  $\{a_n\}$  is a sequence with bounded denominator. Suppose  $p(x)$  is a polynomial whose coefficients are algebraic numbers and let

$$F(x)p(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then, the sequence  $\{b_n\}$  has bounded denominator.

*Proof.* This follows from the fact that the algebraic numbers form a subfield of the complex numbers and the fact that given an algebraic number  $a$  there exists a positive integer  $n$  such that  $na$  is an algebraic integer.  $\square$

**Lemma 7.** *Let  $\zeta_{4p} = e^{2\pi i/4p}$ , where  $p$  is an odd prime. Then*

$$N_{\mathbb{Q}(\zeta_{4p})}(\zeta_{4p} + \zeta_{4p}^{-1}) = p^2.$$

*Proof.* First note that

$$\zeta_{4p} + \zeta_{4p}^{-1} = 2 \cos \frac{\pi}{2p},$$

and recall the characterization of the conjugates of  $\cos 2\pi/n$  given in (2). We have

$$N_{\mathbb{Q}(\zeta_{4p})}(\zeta_{4p} + \zeta_{4p}^{-1}) = \prod_{\substack{(k,4p)=1 \\ 1 \leq k \leq 4p}} (e^{\frac{2\pi ik}{4p}} + e^{\frac{-2\pi ik}{4p}}) = \zeta_{4p}^{-\phi(4p)2p} \prod_{\substack{(k,4p)=1 \\ 1 \leq k \leq 4p}} (e^{\frac{4\pi ik}{4p}} + 1).$$

Now, we know that

$$N_{\mathbb{Q}(\zeta_{2p})}(\zeta_{2p} + 1) = \prod_{\substack{(k,2p)=1 \\ 1 \leq k \leq 2p}} (e^{\frac{2\pi ik}{2p}} + 1).$$



This implies

$$\zeta_{4p}^{-\phi(4p)2p} \prod_{\substack{(k,4p)=1 \\ 1 \leq k \leq 4p}} (e^{\frac{4\pi ik}{4p}} + 1) = N_{\mathbb{Q}(\zeta_{2p})}(\zeta_{2p} + 1)^2 (e^{-2\pi i}).$$

Now, the minimal polynomial of  $\zeta_{2p}$  is the same as that of  $-\zeta_p$ . Furthermore,

$$r(x) = x^{p-1} + x^{p-2} + \dots + x + 1 = \prod_{k=1}^{p-1} (x - \zeta_p^k).$$

So,  $N_{\mathbb{Q}(\zeta_p)}(1 - \zeta_p) = r(1) = p$  since the minimal polynomial of  $\zeta_p$  is  $r(x)$ . Thus,

$$N_{\mathbb{Q}(\zeta_{2p})}(\zeta_{2p} + 1)^2 (e^{-2\pi i}) = N_{\mathbb{Q}(\zeta_p)}(1 - \zeta_p)^2 = p^2,$$

as desired. □

**Lemma 8.** *If, for all  $k$  satisfying  $1 \leq k \leq (n - 1)/2$ , the value of  $\sec^2 \frac{k\pi}{n+1}$  is an algebraic integer, then  $n + 1$  is a power of two.*

*Proof.* Assume that  $n + 1$  is not a power of two. Let  $p$  be an odd prime factor of  $2(n + 1)$ . Since  $n$  is odd,  $2(n + 1)$  is a multiple of 4 and so  $4p$  divides  $2(n + 1)$ . Let  $k = 2(n + 1)/(4p)$ , so  $2(n + 1)/k = 4p$ . Then

$$\sec^2 \frac{k\pi}{n + 1} = \left( \frac{2}{\zeta_{2(n+1)}^k + \zeta_{2(n+1)}^{-k}} \right)^2 = \left( \frac{2}{\zeta_{4p} + \zeta_{4p}^{-1}} \right)^2.$$

Now, from the previous lemma,  $N_{\mathbb{Q}(\zeta_{4p})}(\zeta_{4p} + \zeta_{4p}^{-1}) = p^2$ . This implies

$$N_{\mathbb{Q}(\zeta_{4p})} \left( \frac{2}{\zeta_{4p} + \zeta_{4p}^{-1}} \right)^2 = \frac{2^{2\phi(4p)}}{p^4}.$$

Then, since  $p$  is an odd prime we know that  $2^{2\phi(4p)}/p^4$  is not an integer. This means that with the chosen  $k$ ,  $\sec^2 k\pi/(n + 1)$  is not an algebraic integer. This proves the desired result. □

#### 4. Proof of the theorems

*Proof of Theorem 2.* This is a more general version of Lemma 5. Let  $\alpha_1, \alpha_2, \dots, \alpha_n$  be all the numbers whose reciprocals are zeros of  $Q(x)$ . Then  $F(x)$  has a partial fraction expansion whose terms are of the form

$$\frac{A_{i,l}}{(1 - \alpha_i x)^l},$$

plus a polynomial part. Write

$$Q(x) = \prod_{i=1}^n (1 - \alpha_i x)^{k_i}.$$

Let  $j$  be the largest positive integer such that in the partial fraction decomposition of  $F(x)$  the term  $A_{i,j}/(1 - \alpha_i x)^j$  is nonzero. Clearly  $j > 0$ , since  $P(x)$  and  $Q(x)$  are relatively prime. Now, let

$$Q_i(x) = \frac{Q(x)}{(1 - \alpha_i x)^{k_i - j + 1}}.$$

The highest power of  $(1 - \alpha_i x)$  that divides  $Q_i(x)$  is clearly  $j - 1$ .

We have

$$F(x)Q_i(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Then, by Lemma 6,  $\{b_n\}$  has a bounded denominator. Now, we will consider the effect of multiplying  $F(x)$  and  $Q_i(x)$  by considering what happens to each term in the partial fraction expansion of  $F(x)$ . With the exception of the term

$$\frac{A_{i,j}}{(1 - \alpha_i x)^j},$$

$Q_i(x)$  times a term in the partial fraction expansion of  $F(x)$  is a polynomial of finite degree. Now, one can see that

$$Q_i(x) \frac{A_{i,j}}{(1 - \alpha_i x)^j} = \frac{Q_i(x)}{(1 - \alpha_i x)^{j-1}} \frac{A_{i,j}}{(1 - \alpha_i x)}.$$

It is clear that  $Q_i(x)/(1 - \alpha_i x)^{j-1}$  is a polynomial. Thus,

$$F(x)Q_i(x) = q(x) + \frac{D_i}{1 - \alpha_i x},$$

where  $q(x)$  is a polynomial and  $D_i$  is some algebraic number. So, we can say that for sufficiently large  $n$ ,  $b_n = D_i \alpha_i^n$  where  $D_i$  and  $b_n$  are algebraic numbers. Then, by Lemma 5,  $\alpha_i$  is an algebraic integer.  $\square$

*Proof of Theorem 1.* Suppose  $S(n, m)$  is an integer for all  $m > 0$ . By Lemma 4,

$$F_n(x) = \sum_{m=0}^{\infty} \left( \sum_{k=1}^{(n-1)/2} \sec^{2m} \frac{k\pi}{n+1} \right) x^m = \sum_{k=1}^{(n-1)/2} \left( \frac{1}{1 - x \sec(\frac{k\pi}{n+1})} \right)$$

is a rational function. Hence,  $F_n(x) = P(x)/Q(x)$  where  $P(x), Q(x) \in \mathbb{Q}[x]$ . Theorem 1 now implies that  $\sec^2(k\pi/(n+1))$  is an algebraic integer for all  $k$  with  $1 \leq k \leq (n-1)/2$ . According to Lemma 8, this means  $n+1$  is a power of two.  $\square$

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