involve
a journal of mathematics

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2011 vol. 4, no. 2

INVOLVE 4:2 (2011)
https://doi.org/10.2140/involve.2011.4.157

On the size of the resonant set for the products of 2×2 matrices

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(Communicated by Chi-Kwong Li)

For $\theta \in [0, 2\pi)$ and $\lambda > 1$, consider the matrix $h = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and the rotation matrix R_θ. Let $W_n(\theta)$ denote some product of m instances of R_θ and n of h, with the condition $m \leq \epsilon n$ $(0 < \epsilon < 1)$. We analyze the measure of the set of θ for which $\|W_n(\theta)\| \geq \lambda^{\delta n}$ $(0 < \delta < 1)$. This can be regarded as a model problem for the Bochi–Fayad conjecture.

1. Introduction

Avila and Roblin [2009] considered the following problem. Take the two matrices

$$H = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \tag{1}$$

and

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in [0, 2\pi).$$

Fix $\lambda > 1$ and let $m, n \in \mathbb{N}$. Consider words of the form

$$W_n(\theta) = H^{i_1} R_\theta^{j_1} \dots H^{i_k} R_\theta^{j_k},$$

where k is arbitrary and $i_1, \dots, i_k, j_1, \dots, j_k \in \mathbb{N} \cup \{0\}$ are such that

$$i_1 + \cdots + i_k = n, \quad j_1 + \cdots + j_k = m.$$

Assume that m is much smaller than n and take a "generic" angle θ. It is not unreasonable to conjecture that $\|W_n\|$ grows geometrically with n regardless of the combinatorics of the word. Avila and Roblin proved the following theorem, where the norm is given by $\|W\| = |a| + |b| + |c| + |d|$ if $W = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

MSC2010: primary 37H15; secondary 37H05, 37C85.

Keywords: Bochi–Fayad conjecture, resonant set, measure, rotation matrix, Fayad, Krikorian, exponential growth.

Theorem 1. *Assume that* $0 < \delta < 1$ *is fixed. Then there is an* n*-independent set* Ω *such that* $|\Omega| = 2\pi$ *and for any* $\theta \in \Omega$ *there is* $\epsilon > 0$ *so that*

$$\min_{W_n} \|W_n(\theta)\| > \lambda^{\delta n}$$

provided $m < \epsilon n(\ln n \ln\ln n)^{-1}$.

Here the minimum is over all words W_n for n fixed and m as given. This theorem improved earlier results by Fayad and Krikorian [2008]. The special case of the Bochi–Fayad conjecture [Avila and Roblin 2009; Fayad and Krikorian 2008] deals with the similar situation when $m < \epsilon n$ and ϵ is small. One might expect that $|\Omega| \to 2\pi$ as $\epsilon \to 0$ in this case. Proving it seems to be quite hard. We investigate a simpler case. In (1), consider the matrix H when λ is large. Then $\lambda^{-1} \to 0$ as $\lambda \to \infty$ and one might wonder what happens if λ^{-1} is dropped. Thus we consider $h = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right) \notin \mathrm{SL}(2, \mathbb{R})$ instead of H. It turns out that a very precise analysis can be performed for this simpler model problem, as we shall see in the next section. Section 3 provides some numerical evidence and comparison of the model case with the real problem.

2. The model problem

In the previous setting, take $h = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right)$ instead of H and fix $\epsilon \in (0, 1)$. Given n, set

$$f_n(\theta) = \min_{W_n} \|W_n(\theta)\|,$$

where the norm is again given by the sum of absolute values of matrix entries and the minimum is taken over all W_n with $m \le \epsilon n$. Note that we can take the minimum because for a given n there are only finitely many possibilities for W_n.

Finally, we fix $0 < \delta < 1$ and define the *resonant set* $\mathcal{R}$ thus: $\theta \in \mathcal{R}$ if there exists some n such that $f_n(\theta) < \lambda^{\delta n}$. We claim that $|\mathcal{R}| < C\lambda^{-(1-\delta)/\epsilon}$, where C is some constant that can be explicitly computed and $|\mathcal{R}|$ denotes the Lebesgue measure of the set $\mathcal{R}$.

We now make the convention that *there are no zero exponents in the expression of* $W_n(\theta)$. Then, for words having precisely k blocks of rotation matrices, there are four possibilities, differing in which matrix (h or R_θ) begins the word and which matrix ends it:

$$W_n(\theta) = h^{i_1} R_\theta^{j_1} \cdots h^{i_k} R_\theta^{j_k}, \tag{2}$$

$$W_n(\theta) = R_\theta^{j_1} h^{i_1} \cdots R_\theta^{j_k} h^{i_k}, \tag{3}$$

$$W_n(\theta) = R_\theta^{j_1} h^{i_1} \cdots R_\theta^{j_{k-1}} h^{i_{k-1}} R_\theta^{j_k}, \tag{4}$$

$$W_n(\theta) = h^{i_1} R_\theta^{j_1} \cdots h^{i_k} R_\theta^{j_k} h^{i_{k+1}}. \tag{5}$$

For the word in (2), the product has this explicit form:

$$\begin{aligned} W_n(\theta) &= h^{i_1} R_\theta^{j_1} \cdots h^{i_k} R_\theta^{j_k} \\ &= \begin{pmatrix} \lambda^{i_1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos j_1\theta & -\sin j_1\theta \\ \sin j_1\theta & \cos j_1\theta \end{pmatrix} \cdots \begin{pmatrix} \lambda^{i_k} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \cos j_k\theta & -\sin j_k\theta \\ \sin j_k\theta & \cos j_k\theta \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{i_1} \cos j_1\theta & -\lambda^{i_1} \sin j_1\theta \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} \lambda^{i_k} \cos j_k\theta & -\lambda^{i_k} \sin j_k\theta \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^n \cos j_1\theta \cdots \cos j_k\theta & \lambda^n \cos j_1\theta \cdots \cos j_{k-1}\theta \sin j_k\theta \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

Likewise, for (3), we obtain

$$\begin{aligned} W_n(\theta) &= R_\theta^{j_1} h^{i_1} \cdots R_\theta^{j_k} h^{i_k} \\ &= \begin{pmatrix} \cos j_1\theta & -\sin j_1\theta \\ \sin j_1\theta & \cos j_1\theta \end{pmatrix} \begin{pmatrix} \lambda^{i_1} & 0 \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} \cos j_k\theta & -\sin j_k\theta \\ \sin j_k\theta & \cos j_k\theta \end{pmatrix} \begin{pmatrix} \lambda^{i_k} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^{i_1} \cos j_1\theta & 0 \\ \lambda^{i_1} \sin j_1\theta & 0 \end{pmatrix} \cdots \begin{pmatrix} \lambda^{i_k} \cos j_k\theta & 0 \\ \lambda^{i_k} \sin j_k\theta & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^n \cos j_1\theta \cdots \cos j_k\theta & 0 \\ \lambda^n \sin j_1\theta \cos j_2\theta \cdots \cos j_k\theta & 0 \end{pmatrix}. \end{aligned} \tag{7}$$

Using the result in (7), we get for the word (4)

$$\begin{aligned} &W_n(\theta) \\ &= (R_\theta^{j_1} h^{i_1} \cdots R_\theta^{j_{k-1}} h^{i_{k-1}}) R_\theta^{j_k} \\ &= \begin{pmatrix} \lambda^{i_1+\cdots+i_{k-1}} \cos j_1\theta \cos j_2\theta \cdots \cos j_{k-1}\theta & 0 \\ \lambda^{i_1+\cdots+i_{k-1}} \sin j_1\theta \cos j_2\theta \cdots \cos j_{k-1}\theta & 0 \end{pmatrix} \begin{pmatrix} \cos j_k\theta & -\sin j_k\theta \\ \sin j_k\theta & \cos j_k\theta \end{pmatrix} \\ &= \begin{pmatrix} \lambda^n \cos j_1\theta \cos j_2\theta \cdots \cos j_k\theta & -\lambda^n \cos j_1\theta \cos j_2\theta \cdots \cos j_{k-1}\theta \sin j_k\theta \\ \lambda^n \sin j_1\theta \cos j_2\theta \cdots \cos j_k\theta & -\lambda^n \sin j_1\theta \cos j_2\theta \cdots \cos j_{k-1}\theta \sin j_k\theta \end{pmatrix}. \end{aligned} \tag{8}$$

Finally, using (6), we get for the word (5) simply

$$\begin{aligned} W_n(\theta) &= (h^{i_1} R_\theta^{j_1} \cdots h^{i_{k-1}} R_\theta^{j_{k-1}} h^{i_k} R_\theta^{j_k}) h^{i_{k+1}} \\ &= \begin{pmatrix} \lambda^{i_1+\cdots+i_k} \cos j_1\theta \cdots \cos j_k\theta & \lambda^{i_1+\cdots+i_k} \cos j_1\theta \cdots \cos j_{k-1}\theta \sin j_k\theta \\ 0 & 0 \end{pmatrix} \\ &\qquad \times \begin{pmatrix} \lambda^{i_{k+1}} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \lambda^n \cos j_1\theta \cdots \cos j_k\theta & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{9}$$

Therefore $\|W_n(\theta)\|$ is given by

$$\begin{aligned}
&\lambda^n|\cos j_1\theta\cdots\cos j_{k-1}\theta|\big(|\cos j_k\theta|+|\sin j_k\theta|\big) && \text{for } W_n \text{ of type (2)},\\
&\lambda^n|\cos j_2\theta\cdots\cos j_k\theta|\big(|\cos j_1\theta|+|\sin j_1\theta|\big) && \text{for } W_n \text{ of type (3)},\\
&\lambda^n|\cos j_2\theta\cdots\cos j_{k-1}\theta|\big(|\cos j_1\theta|+|\sin j_1\theta|\big) && \\
&\qquad\qquad\times\big(|\cos j_k\theta|+|\sin j_k\theta|\big) && \text{for } W_n \text{ of type (4)},\\
&\lambda^n|\cos j_1\theta\cdots\cos j_k\theta| && \text{for } W_n \text{ of type (5)}.
\end{aligned}$$

Remark. This shows that, among words with k rotation blocks, $\min_{W_n}\|W_n(\theta)\|$ is reached by words of type (5).

Theorem 2. *Let*

$$\begin{aligned}
S_\alpha &= \big\{\theta\in[0,2\pi)\;\big|\;|\cos\alpha\theta|<\lambda^{-(1-\delta)\alpha/\epsilon-1}\big\},\\
\tilde S_\alpha &= \big\{\theta\in[0,2\pi)\;\big|\;|\cos\alpha\theta|<\lambda^{-(1-\delta)\alpha/\epsilon}\big\}.
\end{aligned}$$

Then the resonant set $\mathcal{R}$ satisfies

$$\bigcup_{\alpha\in\mathbb{N}} S_\alpha \subseteq \mathcal{R} \subseteq \bigcup_{\alpha\in\mathbb{N}} \tilde S_\alpha.$$

Proof. Suppose $\theta\in\bigcup_{\alpha\in\mathbb{N}} S_\alpha$. Then $\theta\in S_\alpha$ for some $\alpha\in\mathbb{N}$ and

$$|\cos\alpha\theta|<\lambda^{-(1-\delta)\alpha/\epsilon-1}.$$

Let $n=[\alpha/\epsilon]+1$. Then, $n-1\le\alpha/\epsilon<n$ and $\alpha<\epsilon n$. Consider the word $\omega_n(\theta)=h^{i_1}R_\theta^\alpha h^{i_2}$ where $i_1+i_2=n$. Since $m=\alpha$, we have $m\le\epsilon n$. Then

$$f_n(\theta)=\min_{W_n(\theta)}\|W_n(\theta)\|\le\|\omega_n(\theta)\|=\lambda^n|\cos\alpha\theta|<\lambda^n\cdot\lambda^{-(1-\delta)\alpha/\epsilon-1}\le\lambda^{\delta n}.$$

Therefore $\theta\in\mathcal{R}$.

Now suppose $\theta\notin\bigcup_{\alpha\in\mathbb{N}}\tilde S_\alpha$. Then $|\cos\alpha\theta|\ge\lambda^{-(1-\delta)\alpha/\epsilon}$ for all $\alpha\in\mathbb{N}$. Choose an arbitrary $n\in\mathbb{N}$. Then

$$\begin{aligned}
f_n(\theta)&=\min_{W_n(\theta)}\|W_n(\theta)\|=\|\omega_n(\theta)\| && \text{(for some word } \omega_n(\theta))\\
&=\lambda^n|\cos j_1\theta\cdots\cos j_k\theta| && \text{(by the remark above)}\\
&=\lambda^n|\cos\alpha_1\theta^{m_1}\cdots\cos\alpha_l\theta^{m_l}|,
\end{aligned}$$

where $\alpha_1<\cdots<\alpha_l$ and $m_1\alpha_1+\cdots+m_l\alpha_l=m\le\epsilon n$. Then

$$\begin{aligned}
f_n(\theta)&=\lambda^n|\cos\alpha_1\theta^{m_1}\cdots\cos\alpha_l\theta^{m_l}|\\
&\ge\lambda^n\cdot\lambda^{-(1-\delta)(m_1\alpha_1+\cdots+m_l\alpha_l)/\epsilon}=\lambda^n\cdot\lambda^{-m(1-\delta)/\epsilon}\ge\lambda^{\delta n},
\end{aligned}$$

and therefore $\theta\notin\mathcal{R}$. □

We claim that $\mathcal{R}$ is a dense open set. To show that $\mathcal{R}$ is open, we show that for each n, f_n is continuous. For each n,

$$R_n = \{\theta \in [0, 2\pi) \mid f_n(\theta) < \lambda^{\delta n}\} = f_n^{-1}((-\infty, \lambda^{\delta n})),$$

which is open as the preimage of a continuous function of an open set. Note that $\mathcal{R} = \bigcup_{n=1}^{\infty} R_n$, a union of open sets, so $\mathcal{R}$ is open.

To show that f_n is continuous, we note that f_n is the minimum of a finite number of continuous functions (the norms of a finite number of words). Denote these functions by $F_1, F_2, \ldots, F_M$, $M \in \mathbb{N}$. Fix arbitrary $\theta \in [0, 2\pi)$, fix $\zeta > 0$, and let $\eta > 0$ be such that whenever $|\theta - \tilde{\theta}| < \eta$, $|F_k(\theta) - F_k(\tilde{\theta})| < \zeta$ for all $k = 1, \ldots, M$. Consider arbitrary $\tilde{\theta} \in (\theta - \eta, \theta + \eta)$. For some i, j,

$$f_n(\theta) = F_i(\theta) \quad \text{and} \quad f_n(\tilde{\theta}) = F_j(\tilde{\theta}).$$

By the definition of f_n,

$$F_i(\theta) \le F_j(\theta) \quad \text{and} \quad F_j(\tilde{\theta}) \le F_i(\tilde{\theta}).$$

Notice that if $F_i(\theta) = F_j(\tilde{\theta})$, then $|f_n(\theta) - f_n(\tilde{\theta})| = 0 < \zeta$ and we are done. Suppose that $F_i(\theta) > F_j(\tilde{\theta})$. Then $|f_n(\theta) - f_n(\tilde{\theta})| = F_i(\theta) - F_j(\tilde{\theta}) \le F_j(\theta) - F_j(\tilde{\theta}) < \zeta$. Otherwise, if $F_i(\theta) < F_j(\tilde{\theta})$, then

$$|f_n(\theta) - f_n(\tilde{\theta})| = F_j(\tilde{\theta}) - F_i(\theta) \le F_i(\tilde{\theta}) - F_i(\theta) < \zeta.$$

To see that $\mathcal{R}$ is dense, let I be any open interval in $[0, 2\pi)$. The collection of points $R_\alpha = \{\pi/2\alpha + (\pi/\alpha)k : k \in \{1, \ldots, 2\alpha - 1\}\}$ is in S_α; indeed, for any $\phi \in R_\alpha$, $\cos \alpha\phi = 0 < \lambda^{-(1-\delta)\alpha/\epsilon - 1}$. If we choose $\alpha > |I|/\pi$, then there must be some element ϕ of R_α in I. Since $\phi \in \bigcup_{\alpha \in \mathbb{N}} S_\alpha \subseteq \mathcal{R} \subseteq \bigcup_{\alpha \in \mathbb{N}} \tilde{S}_\alpha$, we see that every open interval in $[0, 2\pi)$ contains a point in $\mathcal{R}$.

Now we are ready to estimate the size of $\mathcal{R}$. Consider $\tilde{S}_\alpha$ for arbitrary $\alpha \in \mathbb{N}$. The measure of this set is

$$\begin{aligned} |\tilde{S}_\alpha| &= 4\alpha\left(\frac{\pi}{2\alpha} - \frac{1}{\alpha}\cos^{-1}(\lambda^{-(1-\delta)\alpha/\epsilon})\right) \\ &= 2\pi - 4\cos^{-1}(\lambda^{-(1-\delta)\alpha/\epsilon}) \\ &\approx 2\pi - 2\pi + 4\lambda^{-(1-\delta)\alpha/\epsilon} \\ &= 4\lambda^{-(1-\delta)\alpha/\epsilon}. \end{aligned}$$

Then our estimate for the size of $\mathcal{R}$ is

$$|\mathcal{R}| \le \left|\bigcup_{\alpha \in \mathbb{N}} \tilde{S}_\alpha\right| \le \sum_{\alpha=1}^{\infty} |\tilde{S}_\alpha| \approx 4 \sum_{\alpha=1}^{\infty} \lambda^{-(1-\delta)\alpha/\epsilon} = \frac{4\lambda^{-(1-\delta)/\epsilon}}{1 - \lambda^{-(1-\delta)/\epsilon}} \approx 4\lambda^{-(1-\delta)/\epsilon},$$

as $\epsilon \sim 0$ and λ, δ are fixed.

3. Some numerical evidence

We provide some numerical and graphical evidence of what was proved. We see how the graphs of the model case and the real case compare for fixed n and m. In addition, it is shown graphically that changing the multiplicities of H affects the word's norm, but in the model case the word's norm is invariant under these changes. The graphs in this section were plotted with Maple 14.

Based on the similarities between the pictures of the real case and the model case, we conjecture that the resonant set in the model case is, in some sense, the limiting set of the resonant set in the real case as λ grows large, since the λ^{-1} term goes to 0 as λ goes to infinity. Of course, since here we take λ relatively small ($\lambda = 2$) for graphing convenience, this is a rough conjecture; in fact, proving it seems to be rather difficult.

Figure 1 shows a red curve and a blue curve. The blue curve is the graph of $\|h^{i_1} R_\theta^2 h^{i_2} R_\theta^3 h^{i_3}\|$ where $i_1, i_2, i_3 \in \mathbb{N}$ and $i_1+i_2+i_3 = 15$. Recall that h is the matrix we use in the model case, where λ^{-1} is replaced by 0. With these combinatorics, varying i_1, i_2, i_3 does not change the graph, as long as their sum is 15. Specifically, Figure 1 is the model case of $n = 15$, $m = 5$ ($j_1 = 2$, $j_2 = 3$), and $\lambda = 2$.

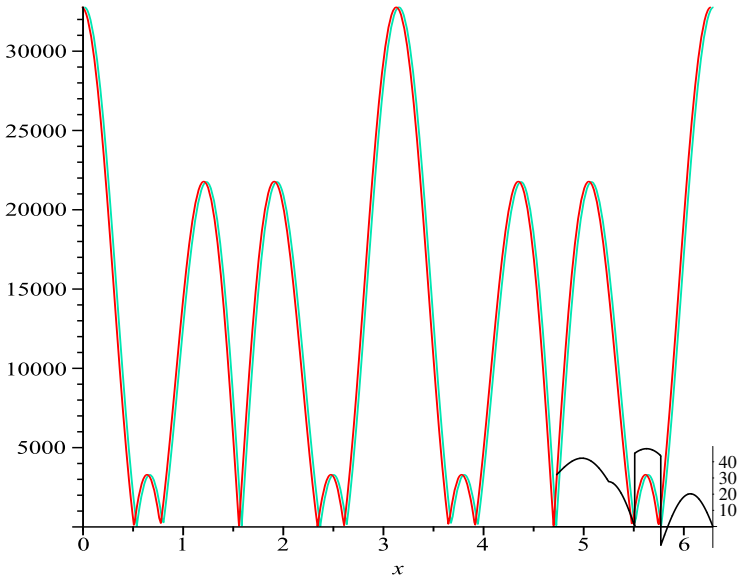

Figure 1. Graphs of the functions $\|H^5 R_\theta^2 H^5 R_\theta^3 H^5\|$ (red curve) and $\|h^5 R_\theta^2 h^5 R_\theta^3 h^5\|$ (blue curve) when $\lambda = 2$. (The curves have been slightly offset horizontally; otherwise they would coincide at this resolution.) The thin black curve on the bottom right is the difference between the first and second functions, with the y-axis expanded 1000 times.

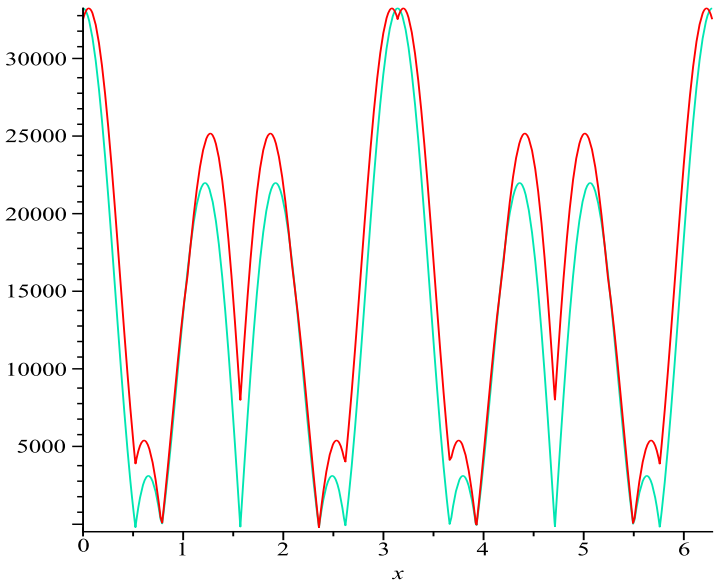

Figure 2. Graphs of the functions $\|H^5R_\theta^2H^9R_\theta^3H^1\|$ (red curve) and $\|h^5R_\theta^2h^9R_\theta^3h^1\|$ (blue curve) when $\lambda = 2$.

The red curve in Figure 1 shows the case where we replace h with H and set $i_1 = i_2 = i_3 = 5$. Explicitly, we are graphing $\|H^5R_\theta^2H^5R_\theta^3H^5\|$.

In Figure 2 we change only two parameters relative to Figure 1: the lengths of the last two blocks of H's (or h's) are now $i_2 = 9$, and $i_3 = 1$. As already seen, the h curve (blue) remains the same, but the H curve (red) — that is, the graph of the function $\|H^5R_\theta^2H^9R_\theta^3H^1\|$ — changes significantly as a result of changing the order of multiplication in the word.

By comparing the red curves in Figures 1 and 2, we observe that a greater disparity between the multiplicities of H (the i_k's) is correlated with a smaller resonant set (the set of points θ between 0 and 2π such that the norm of the word is within a certain distance of zero). The slope of the word's norm is steeper in Figure 2 than it is in Figure 1 and the peaks in Figure 2 are associated with larger values of the word's norm than in the case depicted by Figure 1. Both conditions lead to fewer points θ that are mapped to a norm of the word that is close to zero.

Figure 3 shows the graph of $\|H^1R_\theta^2H^1R_\theta^3H^{13}\|$. Comparing this graph with Figure 2 provides further evidence that a greater disparity between the multiplicities of H results in a smaller resonant set.

To further justify our use of the model case, consider the Figure 4, which treats the case of a word of the form (4). Specifically, the blue curve shows the function $\|R_\theta h^{i_1}R_\theta h^{i_2}R_\theta\|$, where $i_1+i_2 = 15$ and $\lambda = 2$. We take $i_1 = 7$ and $i_2 = 8$ and replace

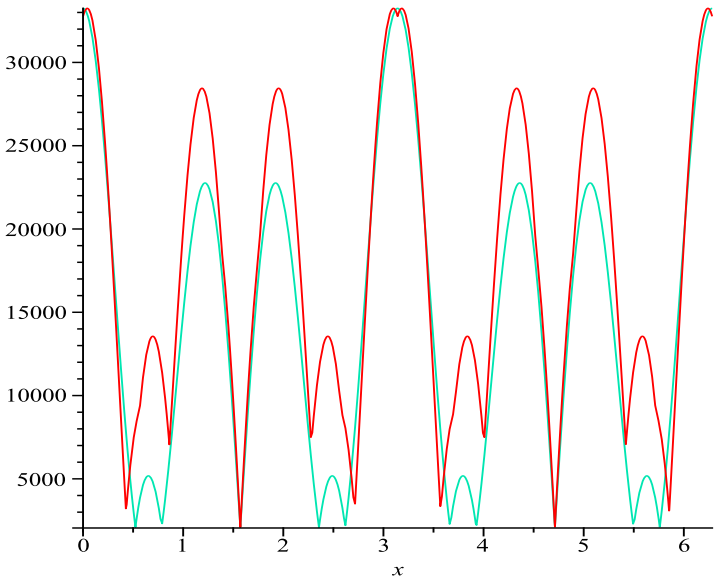

Figure 3. Graphs of the functions $\|H^1R_\theta^2H^1R_\theta^3H^{13}\|$ (red curve) and $\|h^1R_\theta^2h^1R_\theta^3h^{13}\|$ (blue curve) when $\lambda = 2$.

h by H to obtain the red curve. As in Figure 1, the two curves are indistinguishable to within the plot's resolution.

Note that the blue curve in Figure 4 is not comparable to that of Figures 1–3. Both show the model case, but with different combinatorics on the word: expression (5) for the earlier figures, and (4) for Figure 4. Both graphs still have a small resonant set.

Figure 5 shows the graphs of $\|R_\theta H^{14}R_\theta H R_\theta\|$ (red) and of $\|R_\theta h^{14}R_\theta h R_\theta\|$ (blue); the latter of course is the same as the blue curve of Figure 4. Comparing Figure 4 with Figure 5, again we see that a greater disparity between the multiplicities of H results in a smaller resonant set.

4. Conclusion

We hope that our model problem is a viable approximation for what happens when the matrix H is used. The next step might be to express the the Bochi–Fayad problem in terms of the model problem. One way to do this might be to write $H = h + e$, where e is the matrix

$$e = \begin{pmatrix} 0 & 0 \\ 0 & \lambda^{-1} \end{pmatrix},$$

and then express a product of H's and R_θ's in terms of a product of h's and R_θ's, and some other, hopefully small, error terms. The numerical evidence above suggests

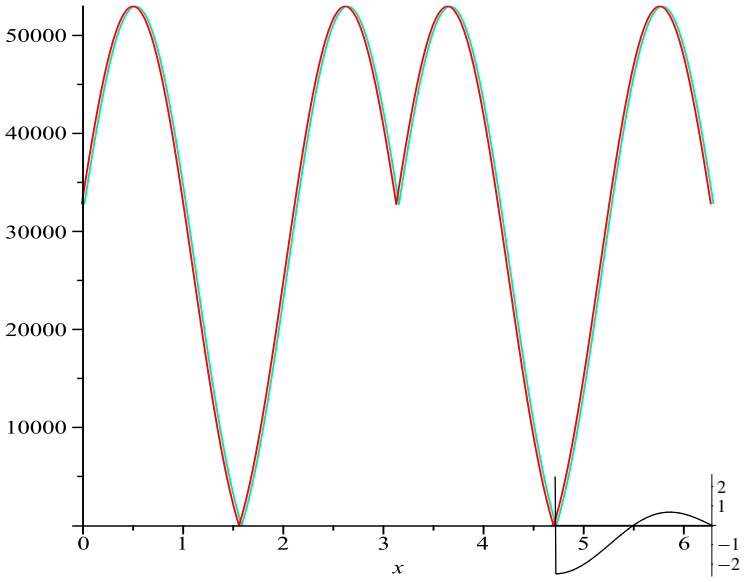

Figure 4. Graphs of the functions $\|R_\theta H^7 R_\theta H^8 R_\theta\|$ (red curve) and $\|R_\theta h^7 R_\theta h^8 R_\theta\|$ (blue curve) when $\lambda = 2$. (The curves have been slightly offset horizontally; otherwise they would coincide at this resolution.) The thin black curve on the bottom right is the difference between the first and second functions, with the y-axis expanded 2500 times.

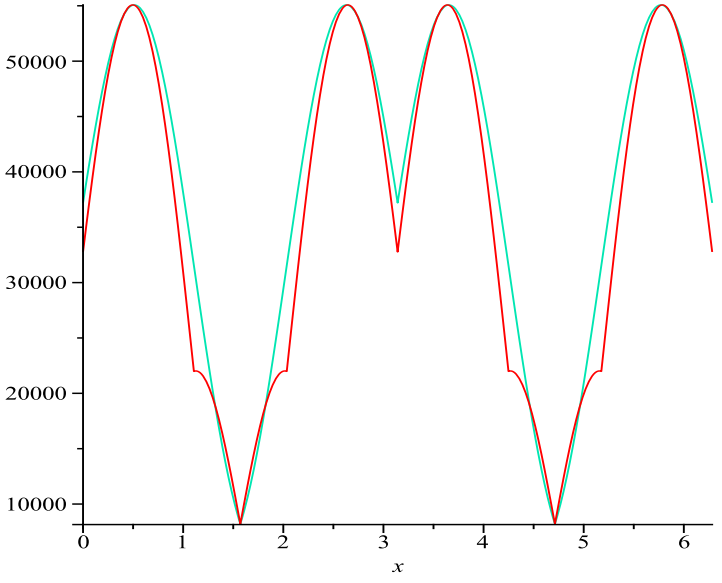

Figure 5. Graphs of $\|R_\theta h^{14} R_\theta h R_\theta\|$ and $\|R_\theta H^{14} R_\theta H R_\theta\|$ when $\lambda = 2$.

that the resonant set for words using the matrix H would actually be smaller than that for words using the matrix h, especially if the distribution of H's is in some sense irregular. This behavior might become more apparent when λ is taken much larger, and ϵ much smaller, than in the experiments above.

Acknowledgements

We would like to thank Professors Serguei Denissov and Alexander Kiselev for their constant support throughout the research and writing processes. We are grateful to them for suggesting the topic and providing us with the opportunity to apply the concepts we have learned throughout our mathematical careers. We thank them both for their excellent advice and encouragement.

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Received: 2010-12-10 Revised: 2011-02-17 Accepted: 2011-04-03

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://msp.org/

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Typeset in LaTeX

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