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P_1 subalgebras of $M_n(\mathbb{C})$

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A linear subspace B of $L(H)$ has the property P_1 if every element of its predual B_* has the form $x + B_\perp$ with $\mathrm{rank}(x) \leq 1$. We prove that if $\dim H \leq 4$ and B is a unital operator subalgebra of $L(H)$ which has the property P_1, then $\dim B \leq \dim H$. We consider whether this is true for arbitrary H.

1. Introduction

The duality between the full algebra $L(H)$ of bounded linear operators on a Hilbert space H and its ideal L_* of trace class operators plays an important role in invariant subspace theory. Indeed, it is easy to use rank one operators in the preannihilator of an operator algebra B to construct nontrivial invariant subspaces for B and conversely (see [Larson 1982]). In his proof that subnormal operators are intransitive, S. Brown [1978] focused attention on a more subtle connection between rank one operators and invariant subspaces. He showed that certain linear subspaces B of $L(H)$ have the following property: every element of its predual B_* has the form $x + B_\perp$ with $\mathrm{rank}(x) \leq 1$, where $B_\perp = \{a \in L_* : \mathrm{Tr}(ba) = 0, \text{ for all } b \in B\}$ is the preannihilator of B. This was called the P_1 *property* in [Larson 1982]. D. Hadwin and E. Nordgren [1982], and independently the third author, observed the connection between this property and *reflexivity*. Although neither property implies the other, if an algebra B has property P_1 and is also reflexive ($B = \mathrm{AlgLat}(B)$) then so are all of its ultra-weakly closed subalgebras.

Azoff obtained many results about linear subspaces of $L(H)$ which have the property P_1. Among them, he proved the following simple, but beautiful, result by using ideas from algebraic geometry. If $\dim H = n \in \mathbb{N}$ and a linear space $S \subset L(H) \equiv M_n(\mathbb{C})$ has the property P_1, then the dimension of S is no larger than $2n-1$. Furthermore, there exists a subspace $S \subset M_n(\mathbb{C})$ which has the property P_1 and $\dim S = 2n-1$. For an expository account of these and related results, we refer

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to [Azoff 1986], where linear spaces with the property P_1 are called *elementary spaces*. For this article the original term P_1 seemed more suitable because we want to work with the more general property P_k in the same context.

In this paper we consider the analogue of Azoff's result for the subcase of *unital operator subalgebras* in $L(H) \equiv M_n(\mathbb{C})$ (an operator algebra is unital if it contains the identity operator of $L(H)$). If B is the diagonal subalgebra of $L(H)$, it is easy to show that B has property P_1 and $\dim B = n$. In Section 5 we show that if $n \leq 4$ and $B \subset M_n(\mathbb{C})$ is a unital subalgebra which has property P_1, then $\dim B \leq n$. It is natural to conjecture that this is also true for arbitrary n. We make this formal:

Question 1. Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra with property P_1. Must $\dim B \leq n$?

Note that if the above conjecture is true, then we can deduce Azoff's result as a corollary. Indeed, if $S \subset L(H) \equiv M_n(\mathbb{C})$ is a linear space with property P_1, then

$$B = \left\{ \begin{pmatrix} \lambda & s \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C}, s \in S \right\} \subset L(H^{(2)}) \equiv M_{2n}(\mathbb{C})$$

is a unital operator algebra with property P_1 [Kraus and Larson 1986; 1985; Azoff 1986]. So $\dim B \leq 2n$ implies $\dim S \leq 2n - 1$.

An algebra $B \subset L(H)$ is called a P_1 *algebra* if A has property P_1. An algebra $B \subset L(H)$ is called a *maximal* P_1 *algebra* if whenever A is a subalgebra of $L(H)$ having property P_1 and $A \supset B$, then $A = B$. We consider a subquestion of Question 1.

Question 2. Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra. If B has property P_1 and $\dim B = n$, is B a maximal P_1 algebra?

In Section 3 and Section 4, we prove that if a unital P_1 subalgebra $B \subset M_n(\mathbb{C})$ is semisimple or singly generated and $\dim B = n$, then B is a maximal P_1 algebra.

In [Larson 1982], the third author showed that if a weakly closed operator algebra B has property P_1, then B is 3-reflexive [Azoff 1973], that is, its three-fold ampliation $B^{(3)}$ is reflexive. (This result also holds for linear subspaces with the same proof). He raised the following problem: Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra with property P_1. Is B 2-reflexive? Note that this question also makes sense for linear subspaces. Azoff [1986] showed that the answer to the above question is affirmative for $n = 3$ (for all linear subspaces of $M_3(\mathbb{C})$ with property P_1). Very little additional progress has been made on this problem since the mid 1980's. The purpose of the research project resulting in this article was to push further on this problem. In Section 6 of this paper, we will show that the answer to the above question for unital algebras is also affirmative for $n = 4$. The proof requires a detailed analysis of several subcases undertaken in the preceding sections.

We would like to pose the following subquestion.

Question 3. Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra with property P_1 and $\dim B = n$. Is B 2-reflexive?

Throughout this paper, we will use the following notation. If H is a Hilbert space and n is a positive integer, then $H^{(n)}$ denotes the direct sum of n copies of H, that is, the Hilbert space $H \oplus \cdots \oplus H$. If a is an operator on H, then $a^{(n)}$ denotes the direct sum of n copies of a (regarded as an operator on $H^{(n)}$). However, we will use I_n instead of $I^{(n)}$ to denote the identity operator on $H^{(n)}$. If B is a set of operators on H, then $B^{(n)} = \{b^{(n)} : \ b \in B\}$.

This paper focuses on problems concerning operator algebras and linear subspaces of operators in finite dimensions. All of our results and proofs are given for finite dimensions. However, many of the definitions are given in the mathematics literature for infinite (as well as finite) dimensions, where the Hilbert space is assumed to be separable. The Hahn–Banach theorem and the Riesz representation theorem, the definitions of reflexive algebras and subspaces, the properties P_1 and P_k, are all given in the literature for infinite dimensions, but we will only use them here in the context of finite dimensions. In cases where proofs of known results are given for the sake of exposition, we will usually just give the proofs for finite dimensions. However, we will adopt the convention that if the statement of a result or definition in this article does not specify finite dimensions, then the reference we cite actually gives the infinite dimensional proof, or, if no reference is cited, then the proof we provide is in fact valid for infinite dimensions.

We will use some standard notation: If $A \in L(H)$, it is common to use $\mathrm{Alg}(A)$ to denote the algebra generated by A and I and $\mathrm{Alg}_0(A)$ to denote the algebra generated by A alone. If L is a lattice of subspaces, then it is also common to use $\mathrm{Alg}(L)$ to denote the algebra of operators that holds each element of L invariant. The meaning of the use of $\mathrm{Alg}(\cdot)$ will be clear from context so there will be no ambiguity.

2. Preliminaries

Let H be a Hilbert space with $\dim H = n$. Then $L(H) \equiv M_n(\mathbb{C})$. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of H. If $a \in L(H) \equiv M_n(\mathbb{C})$ is an arbitrary operator, then the trace of a is defined as

$$\mathrm{Tr}(a) = \sum_{i=1}^{n} \langle a e_i, e_i \rangle.$$

It is easy to show that $\mathrm{Tr}(a)$ does not depend on the choice of $\{e_i\}_{i=1}^n$. Moreover, the trace has the important property that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for all $a, b \in L(H) \equiv M_n(\mathbb{C})$. In this case, the space of trace class operators on H, denoted L_*, can be identified algebraically with $M_n(\mathbb{C})$, and is equipped with the trace class norm

$$\|a\|_1 = \mathrm{Tr}((a^*a)^{1/2}).$$

Recall that the dual of a linear space is the space of all (continuous) linear functionals on the space. In the case of $L_* = M_n(\mathbb{C})$, every linear functional on L_* has the form $a \to \mathrm{Tr}(ab)$ for some $b \in L(H) \equiv M_n(\mathbb{C})$. In this way, $L(H)$ is identified as the dual space of L_*, and L_* is called the predual of $L(H)$. If $S \subset L(H)$ is a linear subspace, then as a linear space itself S can be identified as the dual of the quotient linear space $L_*/S_\perp$, where $S_\perp = \{a \in L_* | \mathrm{Tr}(ba) = 0 \text{ for all } b \in S\}$ is the preannihilator of S. Here, as usual, the quotient space $L_*/S_\perp$ means the set of all cosets of L_*, $\{x + S_\perp | x \in L_*\}$. We also write $x + S_\perp$ as $[x]$. We write $S_* = L_*/S_\perp$. The duality between S and S_* is that if $[x] \in S_*$ for some $x \in L_*$, and associate the linear functional on S given by

$$b \to \mathrm{Tr}(bx), \qquad \text{for all } b \in S.$$

This is well defined by the definition of $S_\perp$. In order to obtain S as exactly the dual of the space S_*, one needs to apply a version of the Hahn–Banach theorem [Han et al. 2007]. We say a linear subspace S of $L(H) \equiv M_n(\mathbb{C})$ has property P_1 if every element of its predual B_* has the form $x + B_\perp$ with $\mathrm{rank}(x) \leq 1$.

Let $B \subset L(H) \equiv M_n(\mathbb{C})$ be a unital operator subalgebra. If $z \in L(H)$ is an invertible operator, elementary computations yield $(zBz^{-1})_\perp = z^{-1}B_\perp z$ and $(zBz^{-1})_* = z^{-1}B_* z$, where the multiplication action of z on the quotient space B_* is given by

$$z^{-1}(x + B_\perp)z = z^{-1}xz + z^{-1}B_\perp z = z^{-1}xz + (zBz^{-1})_*.$$

From this it is easy to see that if B has property P_1, then so does zBz^{-1}. It is also true that B has property P_1 if and only if its adjoint algebra $B^* = \{b^* | b \in B\}$ has property P_1.

Lemma 2.1 [Larson 1982]. *An algebra B has property* P_1 *if and only if every element $b^* \in B^*$ has the form $x + B_\perp$ with* $\mathrm{rank}(x) \leq 1$.

Proof. Only if is trivial. Suppose every element $b^* \in B^*$ has the form $x + B_\perp$ with $\mathrm{rank}(x) \leq 1$. Note that for each $b \in B$ and each $b_\perp \in B_\perp$, $\mathrm{Tr}(bb_\perp) = 0$. This implies that $L(H) = B^* \oplus B_\perp$ with respect to the inner product $\langle x, y \rangle = \mathrm{Tr}(y^*x)$. So for each $a \in L(H)$, $a = b^* + b_\perp$ for some $b^* \in B^*$ and $b_\perp \in B_\perp$. Therefore, $a = x + B_\perp$ with $\mathrm{rank}(x) \leq 1$ by the assumption of the lemma. □

Lemma 2.2. *Let B be a subalgebra of $L(H)$. If B has property* P_1 *and $p \in B$ is a projection, then $pBp \subset L(pH)$ also has property* P_1.

Proof. Suppose $z \in B_\perp$ and $b \in B$. Then $\mathrm{Tr}(pbppzp) = \mathrm{Tr}(pbpz) = 0$. So $pzp \in (pBp)_\perp$. For each $a \in L(H)$, there exists a $b_\perp \in B_\perp$ such that the rank of

$a+b_\perp$ is at most 1. So the rank of $pap + pb_\perp p = p(a+b_\perp)p$ is at most 1. This proves the lemma. $\square$

Recall that a vector $\xi \in \mathcal{H}$ is a separating vector of B if $b\xi = 0$ for some $b \in B$ then $b = 0$. We say that B has the separating vector property if it has a separating vector. A direct sum of subspaces with the separating vector property has the separating vector property (take the direct sum of the separating vectors). If B is similar to a subspace with a separating vector, then B has a separating vector. (If $B = TCT^{-1}$, and x separates C, then Tx separates B).

Lemma 2.3. *If* $\mathrm{Alg}(A, I)$ *is a singly generated unital subalgebra of* $L(H)$ *with* H *finite dimensional*, *then* B *has a separating vector.*

Consider a Jordan block B. The vector

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

separates B. Since any matrix is similar to a finite direct sum of Jordan blocks, and each Jordan block has a separating vector, the result follows.

The following result is the finite-dimensional special case of Proposition 1.2 of [Herrero et al. 1991].

Theorem 2.4. *If* B *is a subalgebra of* $L(H)$, *with* H *finite dimensional*, *such that either* B *or* B^* *has a separating vector*, *then* B *has property* P_1.

Property P_k, a generalization of property P_1, was also introduced by the third author in [Larson 1982]. Recall that an algebra B has *property* P_k if every element of its predual B_* has the form $x + B_\perp$ with $\mathrm{rank}(x) \le k$.

Lemma 2.5 [Larson 1982]. *Let* B *be a subalgebra of* $L(H)$. *Then* B *has property* P_k *if and only if* $B^{(k)} = \{b^{(k)} | b \in B\} \subset L(H^{(k)})$ *has property* P_1.

Proof. "$\Rightarrow$". By Lemma 2.1, we need to show that each operator $(b^*)^{(k)}$, $b \in B$, can be written as $f + B_\perp$ with $\mathrm{rank}(f) \le 1$. Note that

$$B_\perp^{(k)} = \left\{(x_{ij})_{k\times k} | x_{11} + \cdots + x_{kk} \in B_\perp\right\} \supset \left\{(x_{ij})_{k\times k} | x_{11} \cdots , x_{kk} \in B_\perp\right\}.$$

By the assumption, B has property P_k. So there exists a $b_\perp \in B_\perp$ such that the rank of $b^* + b_\perp$ is at most k. We can write $b^* + b_\perp = \xi_1 \otimes \eta_1 + \cdots + \xi_k \otimes \eta_k$, where $\xi_i \otimes \eta_i$ is the rank one operator defined by $\xi_i \otimes \eta_i(\xi) = \langle \xi, \eta_i \rangle \xi_i$. Let

$z_{ii} = k\xi_i \otimes \eta_i - \sum_{1\le r\le k} \xi_r \otimes \eta_r$, $1 \le i \le k$, and let

$$z = \begin{pmatrix} z_{11} & k\xi_2 \otimes \eta_2 & \cdots & k\xi_k \otimes \eta_k \\ k\xi_1 \otimes \eta_1 & z_{22} & \cdots & k\xi_k \otimes \eta_k \\ \cdots & \cdots & \cdots & \cdots \\ k\xi_1 \otimes \eta_1 & k\xi_2 \otimes \eta_2 & \cdots & z_{kk} \end{pmatrix}.$$

Then $z \in B_\perp^{(k)}$ and

$$(b^*)^{(k)} + (b_\perp)^{(k)} + z = k \begin{pmatrix} \xi_1 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_k \otimes \eta_k \\ \xi_1 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_k \otimes \eta_k \\ \cdots & \cdots & \cdots & \cdots \\ \xi_1 \otimes \eta_1 & \xi_2 \otimes \eta_2 & \cdots & \xi_k \otimes \eta_k \end{pmatrix}$$

is a rank 1 matrix.

"$\Rightarrow$". By the assumption, for each $a \in L(H)$ there exists $z \in B_\perp^{(n)}$ such that the rank of $a^{(n)} + z$ is at most 1. Write $z = (z_{ij})_{k\times k}$. Then $z_{11} + \cdots + z_{kk} \in B_\perp$ and the rank of $a + z_{ii}$ is at most 1. So the rank of

$$a + \frac{1}{k}(z_{11} + \cdots + z_{kk}) = \frac{1}{k}((a + z_{11}) + \cdots + (a + z_{kk}))$$

is at most k. $\square$

Corollary 2.6. *If B is a subalgebra of $L(H)$ and* $\dim H = k$, *then $B^{(k)} \subset L(H^{(k)})$ has property* P_1.

3. Semi-simple maximal $\mathbf{P_1}$ algebras

Suppose B is a subalgebra of $M_n(\mathbb{C})$ which has property P_1. Recall that B is a maximal P_1 algebra of $M_n(\mathbb{C})$ if whenever A is a subalgebra of $M_n(\mathbb{C})$ having property P_1 and $A \supseteq B$, then $A = B$. The main result of this section is the following theorem.

Theorem 3.1. *Let $B \subseteq M_n(\mathbb{C})$ be a unital semisimple algebra. If B has property* P_1, *then* $\dim B \le n$. *Furthermore, if* $\dim B = n$, *then B is a maximal* P_1 *algebra.*

To prove this theorem, we will need the following lemmas:

Lemma 3.2. *Let $B \subseteq L(H) = M_n(\mathbb{C})$ be a semisimple algebra. If B has property* P_1, *then* $\dim B \le n$.

Proof. We will use induction on n. The case $n = 1$ is clear. Suppose this is true for $n \le k$ and let $B \subset M_{k+1}(\mathbb{C})$ be a semisimple algebra. We need to show $\dim B \le k+1$. Suppose B has a nontrivial central projection, p, $0 < p < 1$. Then, $B = pBp \oplus (1-p)B(1-p)$. By Lemma 2.1,

$$pBp \subset L(pH) \quad \text{and} \quad (1-p)B(1-p) \subset L((1-p)H),$$

are both semisimple algebras with property P_1. By the assumption of induction $\dim pBp \le \dim(pH)$ and $\dim(1-p)B(1-p) \le \dim(1-p)H$. Therefore,

$$\begin{aligned}\dim B &= \dim(pBp) + \dim((1-p)B(1-p)) \\ &\le \dim pH + \dim(1-p)H \\ &= \dim H = k+1.\end{aligned}$$

Suppose B does not have a nontrivial central projection. Then, $B \cong M_r(\mathbb{C})$. Since B has property P_1, $r^2 \le n+1$ by Lemma 2.5. So $r \le n+1$. □

Lemma 3.3. *Suppose* $0 \ne a \in M_n(\mathbb{C})$. *Then there exists a finite set of operators* $b_1, \dots, b_k, c_1, \dots, c_k$, *such that* $\sum_{i=1}^k b_i a c_i = I_n$.

Proof. Note that $M_n(\mathbb{C}) a M_n(\mathbb{C})$ is a two sided ideal of $M_n(\mathbb{C})$ and

$$M_n(\mathbb{C}) a M_n(\mathbb{C}) \ne 0.$$

Since $M_n(\mathbb{C})$ is a simple algebra, $M_n(\mathbb{C}) a M_n(\mathbb{C}) = M_n(\mathbb{C})$, which implies the lemma. □

The following well known lemma will be very helpful.

Lemma 3.4. *There are finitely many unitary matrices* $u_1, u_2, \dots, u_k \in M_n(\mathbb{C})$ *such that* $\frac{1}{k}\sum_{i=1}^k u_i a u_i^* = (\mathrm{Tr}(a)/n) I_n$ *for all* $a \in M_n(\mathbb{C})$.

The following lemma is a special case of Lemma 3.6. However, we include its proof to illustrate our idea.

Lemma 3.5. *Suppose B is a unital subalgebra of* $M_4(\mathbb{C})$ *and* $B \cong M_2(\mathbb{C})$, *then B is a maximal* P_1 *algebra.*

Proof. We may write $M_4(\mathbb{C})$ as $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ and assume $B = M_2(\mathbb{C}) \otimes I_2$. Note that with respect to the matrix units of $I_2 \otimes M_2(\mathbb{C})$, each element of $B = M_2(\mathbb{C}) \otimes I_2$ has the following form $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, $a \in M_2(\mathbb{C})$. By Corollary 2.6, B has property P_1. Assume $B \subsetneq R \subseteq M_4(\mathbb{C})$ and R is an algebra with property P_1. We can write $R = R_1 + J$, where $R_1 \supset B$ is the semisimple part and J is the radical of R. Since R has property P_1, R_1 has property P_1. By Lemma 3.2, $\dim R_1 \le 4$. Since $\dim B = 4$, we have $R_1 = B$.

Suppose $0 \ne x = (x_{ij})_{1 \le i,j \le 2} \in J$ with respect to the matrix units $I_2 \otimes M_2(\mathbb{C})$. Without loss of generality, we may assume $x_{11} \ne 0$. By Lemma 3.3, there are sets of operators $b_1, \dots, b_k, c_1, \dots, c_k \in M_2(\mathbb{C})$, such that

$$\sum_{i=1}^k b_i x_{11} c_i = I_2. \tag{1}$$

Let $y = (y_{ij})_{1\le i,j\le 2} = \sum_{i=1}^{k}(b_i \otimes I_2)x(c_i \otimes I_2) \in J$. By (1), we have $y_{11} = I_2$. Choose unitary matrices $u_1, \dots, u_k$ as in Lemma 3.4. Let

$$z = (z_{ij}) = \sum_{i=1}^{k}(u_i \otimes I_2)y(u_i^* \otimes I_2) \in J.$$

Then, $z_{11} = I_2$ and $z_{ij} = \lambda_{ij} I_2$ for some $\lambda_{ij} \in \mathbb{C}$, $1 \le i, j \le 2$. So, $z \in I_2 \otimes M_2(\mathbb{C})$. Since $z \in J$, $z^2 = 0$, as elements in the radical are nilpotent. By the Jordan canonical theorem, there exists an invertible matrix $w \in I_2 \otimes M_2(\mathbb{C})$ such that

$$wzw^{-1} = I_2 \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Replacing R by wRw^{-1}, we may assume that R contains B and $I_2 \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Furthermore, we may assume that R is the algebra generated by $M_2(\mathbb{C}) \otimes I_2$ and $I_2 \otimes \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in M_2(\mathbb{C}) \right\}.$$

Simple computation shows that R does not have property P_1. This is a contradiction. Therefore $J = 0$ and $R = B$. □

Lemma 3.6. *Let B be a unital subalgebra of $M_{n^2}(\mathbb{C})$ such that $B \cong M_n(\mathbb{C})$. Then B is a maximal* P_1 *algebra.*

Proof. We may write $M_{n^2}(\mathbb{C})$ as $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ and assume $B = M_n(\mathbb{C}) \otimes I_n$. Note that with respect to the matrix units of $I_n \otimes M_n(\mathbb{C})$, each element of $B = M_n(\mathbb{C}) \otimes I_n$ has the form

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix}, \quad a \in M_n(\mathbb{C}).$$

By Corollary 2.6, B has property P_1. Assume $B \subsetneq R \subseteq M_{n^2}(\mathbb{C})$ and R is an algebra with property P_1. We can write $R = R_1 + J$, where $R_1 \supset B$ is the semisimple part and J is the radical of R. Since R has property P_1, R_1 has property P_1. By Lemma 3.2, $\dim R_1 \le n^2$. Since $\dim B = n^2$, we have $R_1 = B$.

Suppose $0 \ne x = (x_{ij})_{1\le i,j\le n} \in J$ with respect to the matrix units $I_n \otimes M_n(\mathbb{C})$. Without loss of generality, we may assume $x_{11} \ne 0$. By Lemma 3.3, there are finite sets of operators $b_1, \dots, b_k, c_1, \dots, c_k \in M_n(\mathbb{C})$, such that

$$\sum_{i=1}^{k} b_i x_{11} c_i = I_n. \tag{2}$$

Let $y = (y_{ij})_{1\le i,j\le n} = \sum_{i=1}^k (b_i \otimes I_n)x(c_i \otimes I_n) \in J$. By (2), we have $y_{11} = I_n$. Choose unitary matrices $u_1, \ldots, u_k$ as in Lemma 3.4. Let

$$z = (z_{ij}) = \sum_{i=1}^{k} (u_i \otimes I_n) y (u_i^* \otimes I_n) \in J.$$

Then, $z_{11} = I_n$ and $z_{ij} = \lambda_{ij} I_n$ for some $\lambda_{ij} \in \mathbb{C}$, $1 \le i,\ j \le n$. So, $z \in I_n \otimes M_n(\mathbb{C})$.

Since $z \in J$, $z^n = 0$, as elements in the radical are nilpotent. By the Jordan Canonical theorem, there exists an invertible matrix $w \in I_n \otimes M_n(\mathbb{C})$ such that $0 \ne wzw^{-1} = \bigoplus_{i=1}^k z_i \in I_n \otimes M_n(\mathbb{C})$ and each z_i is a Jordan block with diagonal 0. Replacing R by wRw^{-1}, we may assume R contains B and $wzw^{-1} \in I_n \otimes M_n(\mathbb{C})$.

Suppose $r = \max\{\text{rank} z_i : 1 \le i, \le k\}$. We may assume $\text{rank} z_1 = \cdots = \text{rank} z_s = r$ and rank $z_i < r$ for all $s < i \le k$. Then $z^{r-1} = I_n \otimes \left(\left(\bigoplus_{i=1}^s z^{r-1}\right) \oplus 0\right)$. Note that

$$z_i^{r-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We may assume R is the algebra generated by $M_n(\mathbb{C}) \otimes I_n$ and z^{r-1}.

Without loss of generality, we assume $r = 2$, and $s = n/2$. The general case can be proved similarly. Then

$$R = \left\{ \begin{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \end{pmatrix}_{s\times s} : a, b \in M_n(\mathbb{C}) \right\}.$$

Simple computations show that

$$R_\perp = \left\{ \begin{pmatrix} \begin{pmatrix} x_1 & * \\ y_1 & x_2 \end{pmatrix} & & * \\ & \ddots & \\ * & & \begin{pmatrix} x_{n-1} & * \\ y_s & x_n \end{pmatrix} \end{pmatrix}_{s\times s} : x_i, y_i \in M_n(\mathbb{C}),\ \sum_{i=1}^n x_i = \sum_{i=1}^s y_i = 0 \right\}.$$

Let

$$m = \begin{pmatrix} 0_n & 0_n \\ I_n & 0_n \end{pmatrix}.$$

Since R has property P_1, we can write $m^{(s)} = x + R_\perp$ such that the rank of x is at most 1. This implies that $I_n + y_1$, $I_n + y_2, \ldots, I_n + y_s$ are all rank-1 matrices for some $y_1, \ldots, y_s \in M_n(\mathbb{C})$ with $y_1 + \cdots + y_s = 0$. Therefore, the rank of $I_n + y_1 + I_n + y_2 + \cdots + I_n + y_s = sI_n$ is at most $s = \frac{n}{2} < n$. This is a contradiction. So $J = 0$ and $R = B$. $\square$

The following is a key lemma to prove Theorem 3.1, which has an independent interest.

Lemma 3.7. *Let* $\lambda \neq 0$ *be a complex number, and let* $y_1, y_2, \ldots, y_n \in M_n(\mathbb{C})$ *satisfy* $y_1 + y_2 + \cdots + y_n = 0$. *Suppose* $\eta_1, \eta_2, \ldots, \eta_n \in \mathbb{C}^n$ *are linearly dependent vectors, and*

$$t = \begin{pmatrix} \lambda & * & * & * \cdots & * \\ \eta_1 & I_n + y_1 & * & * \cdots & * \\ \eta_2 & * & I_n + y_2 & * \cdots & * \\ \vdots & \vdots & \vdots & \vdots \ddots & \vdots \\ \eta_n & * & * & * \cdots & I_n + y_n \end{pmatrix}.$$

Then $\operatorname{rank} t > 1$.

Proof. We may assume that $\eta_1, \ldots, \eta_{k-1}$, $k \leq n$, are linearly independent vectors, and each η_j, $k \leq j \leq n$, can be written as a linear combination of $\eta_1, \ldots, \eta_{k-1}$. Write

$$\eta_i = \begin{pmatrix} \sigma_{i1} \\ \vdots \\ \sigma_{in} \end{pmatrix}.$$

We may assume that the $(k-1) \times (k-1)$ matrix $(\sigma_{i,j})_{(k-1)\times(k-1)}$ is invertible. Using row reduction, we can transform t to a new matrix

$$\begin{pmatrix} \lambda & * & * & * \cdots & * \\ \eta'_1 & I_n + y'_1 & * & * \cdots & * \\ \eta'_2 & * & I_n + y'_2 & * \cdots & * \\ \vdots & \vdots & \vdots & \vdots \ddots & \vdots \\ \eta'_n & * & * & * \cdots & I_n + y'_n \end{pmatrix}$$

such that the k-th row of each η'_j is 0 for $1 \leq j \leq n$, and $y'_1 + \cdots + y'_n = 0$. So the $(jk+1, 1)$-th entry of t' is zero for all $1 \leq j \leq n$.

Suppose t is a rank 1 matrix. Then t' is also a rank 1 matrix. By the assumption, $\lambda \neq 0$. This implies that each entry of the $(jk+1)$-th row of t' is zero for all $1 \leq j \leq n$. In particular, the (k, k)-th entry of $I_n + y'_j$ is 0 for all $1 \leq j \leq n$. Therefore, the (k, k)-th of $I_n + y'_1 + I_n + y'_2 + \cdots + I_n + y'_n = nI_n$ is zero. This is a contradiction. So $\operatorname{rank} t > 1$. $\square$

The following lemma is a special case of Lemma 3.10. However, we include its proof to illustrate our idea.

Lemma 3.8. *Suppose* $\dim H = 5$ *and*

$$B = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : \lambda \in \mathbb{C}, a \in M_2(\mathbb{C}) \right\} \subset L(H) = M_5(\mathbb{C}).$$

Then, B *is a maximal* P_1 *algebra.*

Proof. Since B has a separating vector, B has property P_1 by Theorem 2.4. Suppose $B \subset R \subseteq M_5(\mathbb{C})$ and R has property P_1. We can write $R = R_1 + J$, where $R_1 \supset B$ is the semisimple part and J is the radical part. By Lemma 3.2, $B = R_1$.

Suppose $0 \neq x \in J$. Let

$$p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_2 & 0_2 \\ 0 & 0_2 & I_2 \end{pmatrix}.$$

Then $qBq \subseteq qRq \subset B(PH) = M_4(\mathbb{C})$. By Lemma 3.5, $qBq = qRq$. This implies that we may assume

$$0 \neq x = \begin{pmatrix} 0 & \xi^T & \eta^T \\ 0 & 0_2 & 0_2 \\ 0 & 0_2 & 0_2 \end{pmatrix}, \quad \text{where } \xi, \eta \in \mathbb{C}^2.$$

Case 1. ξ and η are linearly independent vectors. Note that

$$x \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & \xi^T a & \eta^T a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in R.$$

Since ξ and η are linearly independent, and $a \in M_2(\mathbb{C})$ is arbitrary, this implies that

$$R \supseteq \left\{ \begin{pmatrix} \lambda & \xi^T & \eta^T \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : \lambda \in \mathbb{C}, \xi, \eta \in \mathbb{C}^2, a \in M_2(\mathbb{C}) \right\}.$$

Simple computation shows that

$$R_\perp \subseteq \left\{ \begin{pmatrix} 0 & * & * \\ 0 & y_1 & * \\ 0 & * & y_2 \end{pmatrix} : y_1, y_2 \in M_2(\mathbb{C}), y_1 + y_2 = 0 \right\}.$$

Since R has property P_1, we can write $I_5 = x + R_\perp$ such that the rank of x is at most 1. This gives us a rank 1 matrix x of the form

$$R_\perp = \begin{pmatrix} 1 & * & * \\ 0 & y_1 + I_2 & * \\ 0 & * & y_2 + I_2 \end{pmatrix}, \quad \text{where } y_1 + y_2 = 0.$$

This contradicts Lemma 3.7.

Case 2. ξ and η are linearly dependent. Without loss of generality, assume $\eta = t\xi$. So

$$x = \begin{pmatrix} 0 & \xi^T & t\xi^T \\ 0 & 0_2 & 0_2 \\ 0 & 0_2 & 0_2 \end{pmatrix} \quad \text{and} \quad x\begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} = \begin{pmatrix} 0 & \xi^T a & t\xi^T a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\xi \neq 0$, and $a \in M_2(\mathbb{C})$ is arbitrary, this implies that

$$R \supset \left\{ \begin{pmatrix} \lambda & \xi^T & t\xi^T \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} : \lambda \in \mathbb{C}, \xi \in \mathbb{C}^2, a \in M_2(\mathbb{C}) \right\}.$$

Simple computation shows that

$$R_\perp \subset \left\{ \left(\begin{smallmatrix} 0 & * & * \\ \eta_1 & y_1 & * \\ \eta_2 & * & y_2 \end{smallmatrix} \right) y_1, y_2 \in M_2(\mathbb{C}) : y_1 + y_2 = 0,\ \eta_1, \eta_2 \in \mathbb{C}^2,\ \eta_1 + t\eta_2 = 0 \right\}. \quad (3)$$

Since R has property P_1, we can write $I_5 = x + R_\perp$ such that the rank of x is at most 1. This gives us a rank 1 matrix x of the form

$$R_\perp = \begin{pmatrix} 1 & * & * \\ \eta_1 & y_1 + I_2 & * \\ \eta_2 & * & y_2 + I_2 \end{pmatrix},$$

where $\eta_1 + t\eta_2 = 0$ and $y_1 + y_2 = 0$. This contradicts Lemma 3.7. $\square$

Lemma 3.9. *Suppose* $\{z_{ij}\}_{1 \le i \le s, 1 \le j \le r} \subseteq M_{sr}(\mathbb{C})$ *and* $\{c_{ji}\}_{1 \le i \le s, 1 \le j \le r} \subseteq M_{rs}(\mathbb{C})$ *such that*

$$\sum_{i=1}^{s} \sum_{j=1}^{r} z_{ij} a c_{ji} b = 0, \quad \textit{for all } a \in M_r(\mathbb{C}),\ \textit{for all } b \in M_s(\mathbb{C}).$$

If $c_{ji} \neq 0$ *for some* $1 \le i \le s$, $1 \le j \le r$, *then* z_{ij} *are linearly dependent.*

Proof. We may assume $c_{11} \neq 0$ and the $(1, 1)$ entry of c_{11} is 1. Replacing c_{ji} by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} c_{ji} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

we may assume

$$c_{ji} = \lambda_{ij} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \text{where } \lambda_{11} = 1.$$

Let z_{ij}^k be the k-th column of z_{ij}. Simple computation shows that

$$\sum_{i=1}^{s} \sum_{j=1}^{r} z_{ij} c_{ji} = 0$$

is equivalent to $\sum_{i=1}^{s} \sum_{j=1}^{r} \lambda_{ij} z_{ij}^1 = 0$. Let

$$a = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Simple computation shows that $\sum_{i=1}^{s} \sum_{j=1}^{r} z_{ij} a c_{ji} = 0$ is equivalent to

$$\sum_{i=1}^{s} \sum_{j=1}^{r} \lambda_{ij} z_{ij}^2 = 0.$$

Choosing a appropriately, we have $\sum_{i=1}^{s} \sum_{j=1}^{r} \lambda_{ij} z_{ij}^k = 0$ for all $1 \le k \le n$. This implies $\sum_{i=1}^{s} \sum_{j=1}^{r} \lambda_{ij} z_{ij} = 0$. □

Lemma 3.10. *Suppose* $\dim H = (r^2 + s^2)$ *and*

$$B = \{a^{(r)} \oplus b^{(s)} : a \in M_r(\mathbb{C}), b \in M_s(\mathbb{C})\} \subset L(H) = M_{(r^2+s^2)}(\mathbb{C}).$$

Then B is a maximal P_1 *algebra.*

Proof. Since B has a separating vector, B has property P_1 by Theorem 2.4. Suppose $B \subseteq R \subseteq M_{(r^2+s^2)}(\mathbb{C})$ and R has property P_1. We can write $R = R_1 + J$, where $R_1 \supset B$ is the semisimple part and J is the radical part. By Lemma 3.2, $B = R_1$.

Suppose $0 \neq x \in J$. Let $p = I_r^{(r)} \oplus 0$ and $q = 0 \oplus I_s^{(s)}$. Then, $pBp \subseteq pRp \subseteq B(pH)$ and pRp has property P_1. By Lemma 3.6, $pRp = pBp$. Similarly, $qRq = qBq$. So we may assume

$$0 \neq x = \begin{pmatrix} 0_r^{(r)} & c \\ 0 & 0_s^{(s)} \end{pmatrix}.$$

Write $c = (c_{ij})_{1 \le i \le r, 1 \le j \le s}$. Note that $c \neq 0$.

Suppose

$$z = \begin{pmatrix} x_1 & * & \cdots & * & * & * & \cdots & * \\ * & x_2 & \cdots & * & * & * & \cdots & * \\ & & \ddots & & & & \ddots & \\ * & * & \cdots & x_r & * & * & \cdots & * \\ z_{11} & z_{12} & \cdots & z_{1r} & y_1 & * & \cdots & * \\ z_{21} & z_{22} & \cdots & z_{2r} & * & y_2 & \cdots & * \\ & & \ddots & & & & \ddots & \\ z_{s1} & z_{s2} & \cdots & z_{sr} & * & * & \cdots & y_s \end{pmatrix} \in R_\perp.$$

Since $R_\perp \subset B_\perp$, $x_1 + x_2 + \cdots + x_r = 0_r$ and $y_1 + y_2 + \cdots + y_s = 0_s$. Note that

$$x(a^{(r)} \oplus b^{(s)}) = \begin{pmatrix} 0_r^{(r)} & cb^{(s)} \\ 0 & 0_s^{(s)} \end{pmatrix}.$$

Since $x \in R_\perp$ and $x(a^{(r)} \oplus b^{(s)}) \in R$, we have

$$\operatorname{Tr}\left(\begin{pmatrix} z_{11} & \dots & z_{1r} \\ \vdots & & \\ z_{s1} & \dots & z_{sr} \end{pmatrix} \begin{pmatrix} c_{11} & \dots & c_{1s} \\ \vdots & & \\ c_{r1} & \dots & c_{rs} \end{pmatrix} \begin{pmatrix} b & & \\ & \ddots & \\ & & b \end{pmatrix} \right) = 0.$$

Simple computation shows that $\operatorname{Tr}(\sum_{i=1}^{s} \sum_{j=1}^{r} z_{ij} c_{ji} b) = 0$. Since $b \in M_s(\mathbb{C})$ is an arbitrary matrix, $\sum_{i=1}^{s} \sum_{j=1}^{r} z_{ij} c_{ji} = 0$.

Note that

$$(a^{(r)} \oplus 0)x(0 \oplus b^{(s)}) = \begin{pmatrix} 0_r^{(r)} & a^{(r)}cb^{(s)} \\ 0 & 0_s^{(s)} \end{pmatrix} = \begin{pmatrix} 0_r^{(r)} & (ac_{ij}b)_{1 \le i \le r, 1 \le j \le s} \\ 0 & 0_s^{(s)} \end{pmatrix}.$$

By similar arguments as above, we have $\sum_{i=1}^{s} \sum_{j=1}^{r} z_{ij} a c_{ji} b = 0$ for all $a \in M_r(\mathbb{C})$ and $b \in M_s(\mathbb{C})$. By Lemma 3.9, this implies that $\{z_{ij}\}_{1 \le i \le s, 1 \le j \le r}$ are linearly dependent matrices.

Since R has property P_1, $I_{r^2+s^2} = x + R_\perp$ for some x such that the rank of x is at most 1. So x is a matrix of the form

$$\begin{pmatrix} I_r + x_1 & * & \cdots & * & * & * & \cdots & * \\ * & I_r + x_2 & \cdots & * & * & * & \cdots & * \\ & & \ddots & & & & \ddots & \\ * & * & \cdots & I_r + x_r & * & * & \cdots & * \\ z_{11} & z_{12} & \cdots & z_{1r} & I_s + y_1 & * & \cdots & * \\ z_{21} & z_{22} & \cdots & z_{2r} & * & I_s + y_2 & \cdots & * \\ & & \ddots & & & & \ddots & \\ z_{s1} & z_{s2} & \cdots & z_{sr} & * & * & \cdots & I_s + y_s \end{pmatrix}.$$

Since x is a rank 1 matrix, $(z_{ij})_{1\le i\le s,1\le j\le r}$ are rank 1 matrices. So there are $\xi_1,\ldots,\xi_s\in\mathbb{C}^s$, $\eta_1,\ldots,\eta_r\in\mathbb{C}^r$ such that $z_{ij}=\xi_i\otimes\eta_j$ for $1\le i\le s$ and $1\le j\le r$. Since $\{z_{ij}\}_{1\le i\le s,1\le j\le r}$ are linearly dependent matrices, either $\{\xi_i\}_{i=1}^s$ are linearly dependent or $\{\eta_j\}_{j=1}^r$ are linearly dependent. Without loss of generality, assume $\{\xi_i\}_{i=1}^s$ are linearly dependent. Now, x is a matrix of the form

$$\begin{pmatrix} I_r+x_1 & * & \cdots & * & * & * & \cdots & * \\ * & I_r+x_2 & \cdots & * & * & * & \cdots & * \\ & & \ddots & & & & \ddots & \\ * & * & \cdots & I_r+x_r & * & * & \cdots & * \\ \xi_1\otimes\eta_1 & \xi_1\otimes\eta_2 & \cdots & \xi_1\otimes\eta_r & I_s+y_1 & * & \cdots & * \\ \xi_2\otimes\eta_1 & \xi_2\otimes\eta_2 & \cdots & \xi_2\otimes\eta_r & * & I_s+y_2 & \cdots & * \\ & & \ddots & & & & \ddots & \\ \xi_s\otimes\eta_1 & \xi_s\otimes\eta_1 & \cdots & \xi_s\otimes\eta_r & * & * & \cdots & I_s+y_s \end{pmatrix}.$$

Since $x_1+\cdots+x_r=0$, one entry of I_r+x_i is not zero for some $1\le i\le r$. We may assume the (1, 1) entry of I_r+x_1 is $\lambda\ne 0$. Let

$$\eta_1=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_r\end{pmatrix}.$$

Then the matrix

$$\begin{pmatrix} \lambda & * & \cdots & * \\ \alpha_1\xi_1 & I_s+y_1 & \cdots & * \\ \vdots & & \ddots & \\ \alpha_1\xi_s & * & \cdots & I_s+y_s \end{pmatrix}$$

has rank 1 since it is a submatrix of x. This contradicts Lemma 3.7. So $R=B$. $\square$

Proof of Theorem 3.1. By Lemma 3.2, if B has P_1, then $\dim B\le n$. Assume B has property P_1, and $\dim B=n$. We claim $B=\bigoplus_{i=1}^r M_{n_i}(\mathbb{C})^{(n_i)}$ and $n=\sum_{i=1}^r n_i^2$. We will proceed by induction on n. If $n=1$, this is clear. Assume our claim is true for $n\le k$. Let $B\subseteq M_{k+1}(\mathbb{C})$ be a semisimple P_1 algebra and $\dim B=k+1$. Suppose B has a nontrivial central projection p, $0<p<1$. Then, $B=pBp\oplus(1-p)B(1-p)$. By Lemma 2.1, $pBp\subseteq B(pH)$ and $(1-p)B(1-p)\subseteq B((1-p)H)$ are both semisimple algebras with property P_1. By Lemma 3.2, $\dim(pBp)=\dim(pH)$ and $\dim((1-p)B(1-p))=\dim((1-p)H)$. By induction, $pBp=\bigoplus_{i=1}^{r_1} M_{n_i}(\mathbb{C})^{(n_i)}$, $(1-p)B(1-p)=\bigoplus_{i=1}^{r_2} M_{m_i}(\mathbb{C})^{(m_i)}$, and $\sum_{i=1}^{r_1} n_i^2+\sum_{i=1}^{r_2} m_i^2=k+1$. Suppose B does not have a nontrivial central projection. Then $B=M_r(\mathbb{C})\subseteq M_{n+1}(\mathbb{C})$ and $\dim B=r^2=n+1$ by Lemma 2.5.

Suppose $B \subsetneq R \subseteq M_k(\mathbb{C}) \in L(H)$ and R is an algebra with property P_1. Let $0 \neq x \in R \setminus B$. Note that $B = \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C})^{(n_i)}$. Let p_i be the projection of B that corresponds to the summand $M_{n_i}(\mathbb{C})^{(n_i)}$. Then, we have $p_i B p_i \subseteq p_i R p_i \subseteq L(p_i H)$ and $p_i R p_i$ has property P_1. By Lemma 3.6, $p_i R p_i = p_i B p_i$. So we may assume

$$0 \neq x = \begin{pmatrix} 0_{n_1}^{(n_1)} & x_{12} & x_{13} & \cdots & x_{1n_r} \\ & 0_{n_2}^{(n_2)} & x_{23} & \cdots & x_{2n_r} \\ & & \ddots & & \vdots \\ & & & 0_{n_{r-1}}^{(n_{r-1})} & x_{r-1r} \\ 0 & & & & 0_{n_r}^{(n_r)} \end{pmatrix}.$$

We may assume that $x_{12} \neq 0$. Then

$$(p_1 + p_2)x(p_1 + p_2) \in (p_1 + p_2)R(p_1 + p_2) \setminus (p_1 + p_2)B(p_1 + p_2).$$

By Lemma 2.1, $(p_1 + p_2)R(p_1 + p_2)$ has property P_1. By Lemma 3.10,

$$(p_1 + p_2)B(p_1 + p_2) = M_{n_1}(\mathbb{C})^{(n_1)} \oplus M_{n_2}(\mathbb{C})^{(n_2)}$$

is a maximal P_1 algebra. This is a contradiction. So B is a maximal P_1 algebra. □

4. Singly generated maximal P_1 algebras

In this section, we prove the following result.

Theorem 4.1. *Suppose B is a singly generated unital subalgebra of $M_n(\mathbb{C})$ and $\dim B = n$. Then B is a maximal* P_1 *algebra.*

To prove Theorem 4.1, we need several lemmas. Let J_n be the $n \times n$ Jordan block.

Lemma 4.2. *Let B be the unital subalgebra of $M_n(\mathbb{C})$ generated by the Jordan block J_n. If $N \supset B$ is a subalgebra of the upper-triangular algebra of $M_n(\mathbb{C})$ and N has property* P_1, *then $N = B$.*

Proof. Suppose $N \supsetneq B$ is a subalgebra of the upper-triangular algebra and N has property P_1. Note that

$$B = \left\{ \sum_{k=0}^{n-1} \lambda_k (J_n)^k : \lambda_0, \dots, \lambda_{n-1} \in \mathbb{C} \right\}.$$

A special case. Suppose N contains an operator x of the following form

$$x = \begin{pmatrix} 0 & \cdots & 0 & \lambda & 0 \\ & 0 & \cdots & 0 & \eta \\ & & 0 & \cdots & 0 \\ & & & \ddots & \vdots \\ & & & & 0 \end{pmatrix}, \tag{4}$$

where $\lambda \neq \eta$. Then N contains the algebra generated by B and x. Therefore,

$$N \supset \left\{ \begin{pmatrix} \lambda_1 & \cdots & \lambda_{n-2} & \alpha & \gamma \\ & \lambda_1 & \cdots & \lambda_{n-2} & \beta \\ & & \lambda_1 & \cdots & \lambda_{n-2} \\ & & & \ddots & \vdots \\ & & & & \lambda_1 \end{pmatrix} : \lambda_1, \ldots, \lambda_{n-2}, \alpha, \beta, \gamma \in \mathbb{C} \right\}.$$

Simple computation shows that

$$N_\perp \subset \left\{ \begin{pmatrix} * & \cdots & * & 0 & 0 \\ & * & \cdots & * & 0 \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{pmatrix} \right\}.$$

It is easy to see that the operator $(J_n)^{n-2}$ can not be written as a sum of a rank one operator and an operator in $N_\perp$. This contradicts the assumption that N has property P$_1$.

The general case. Suppose $z \in N \setminus B$. By the assumption of the lemma, $z = (z_{i,j})_{n\times n}$ is an upper-triangular matrix. Since $z \notin B$, we may assume that

$$z_{j,j+k-1} \neq z_{j+r,j+r+k-1}$$

for some positive integers j, k, r, and $z_{s,t} = 0$ for $t < s + k - 1$. Without loss of generality, we assume that $z_{1,k} \neq z_{2,1+k}$ and $1 \leq k \leq n-1$. If $k = n-1$, then this implies that N contains an x as in (4). If $k < n-2$, then $(J_n)^{k+1} z$ (or consider $z(J_n)^{k+1}$ if $z_{n-1,n-1} \neq z_{n,n}$) is a matrix in N. If we write

$$(J_n)^{k+1} z = (y_{ij})_{n\times n}.$$

Then $y_{1,k+1} \neq y_{2,k+2}$ and $y_{s,t} = 0$ for $t < s + k$. Repeating the above arguments, we can see that N contains an x as in (4). This completes the proof. $\square$

Lemma 4.3. *Let B be the unital subalgebra of $M_n(\mathbb{C})$ generated by the Jordan block J_n. Then B is a maximal* P$_1$ *algebra.*

Proof. Suppose $N \supset B$ is a subalgebra of $M_n(\mathbb{C})$ and N has property P_1. By Wedderburn's theorem,

$$N = M_{n_1}(\mathbb{C}) \oplus \cdots M_{n_s}(\mathbb{C}) \oplus J,$$

where J is the radical of N.

Case 1. $n_1 = \cdots = n_s = 1$. Then N is triangularizable, that is, there exists a unitary matrix $u \in M_n(\mathbb{C})$ such that uNu^* is contained in the algebra of upper-triangular matrices (see [Christensen 1999, Proposition 2.5]). Since $J_n \in B \subset N$, uJ_nu^* is a strictly upper-triangular matrix. Simple computation shows that u has to be a diagonal matrix. Therefore, $N = u^*(uNu^*)u$ is contained in the algebra of upper-triangular matrices. Since N has property P_1, $N = B$ by Lemma 4.2.

Case 2. Suppose $n_i \geq 2$ for some i, $1 \leq i \leq s$. Choose a nonzero partial isometry $v \in M_{n_i}(\mathbb{C})$ such that $v^2 = 0$. Then either $v \notin B$ or $v^* \notin B$ since B does not contain any nontrivial projections. We may assume that $v \notin B$. Consider the subalgebra $\tilde{N}$ generated by v and B. An element of $\tilde{N}$ can be written as $b_1 v b_2 v \cdots v b_n$, where $b_i \in J$ for $2 \leq i \leq n-1$, $b_1 = 1$ or $b_1 \in J$, $b_n = 1$ or $b_n \in J$. By Lemma 2.1 of [Christensen 1999], $\tilde{N} = \mathbb{C}1 \oplus \tilde{J}$, where $\tilde{J}$ is the radical part of $\tilde{N}$ such that $v \in \tilde{J}$. Note that $\tilde{N}$ also has property P_1. By Case 1, $\tilde{N} = B$. So $v \in B$. This is a contradiction. □

Lemma 4.4. *Let $B_i \subset M_{n_i}(\mathbb{C})$ be the unital subalgebra generated by the Jordan block J_{n_i} for $i = 1, 2$. Then $B = B_1 \oplus B_2$ is a maximal* P_1 *subalgebra of $M_{n_1+n_2}(\mathbb{C})$.*

Proof. Suppose $B \subsetneq N \subset M_{n_1+n_2}(\mathbb{C})$ and N has property P_1. Let p_i be the central projections of B corresponding to B_i. Then $B_1 \subset p_1 N p_1 \subset M_{n_1}(\mathbb{C})$ and $p_1 N p_1$ has property P_1. By Lemma 4.3, $p_1 N p_1 = B_1$. Similarly, $p_2 N p_2 = B_2$. Suppose $x \in N \setminus B$. Then we may assume that $0 \neq x = p_1 x p_2$. With respect to matrix units of $M_{n_1}(\mathbb{C})$ and $M_{n_2}(\mathbb{C})$, we can write x as

$$x = \begin{pmatrix} 0 & (x_{ij})_{n_1 \times n_2} \\ 0 & 0 \end{pmatrix},$$

where $(x_{ij})_{n_1 \times n_2}$ is a nonzero matrix. Multiplying on the left by a suitable matrix of B, we may assume that $x_{ij} = 0$ for all $i \geq 2$ (which can be easily seen for the case $n_2 = 1$, other cases are similar). Multiplying on the right by another suitable matrix of B, we may further assume that $x_{1,n_2} = 1$ and $x_{1,j} = 0$ for $1 \leq j \leq n_2 - 1$.

So we may assume that

$$x = \begin{pmatrix} 0_{n_1\times n_1} & \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}_{n_1\times n_2} \\ 0 & 0_{n_2\times n_2} \end{pmatrix}.$$

Let $\tilde{N}$ be the algebra generated by B and x above. Then

$$\tilde{N} = \left\{ \begin{pmatrix} \begin{pmatrix} \lambda_1 & \cdots & \lambda_{n_1} \\ & \ddots & \vdots \\ 0 & & \lambda_1 \end{pmatrix} & \begin{pmatrix} 0 & \cdots & \alpha \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix}_{n_1\times n_2} \\ 0 & \begin{pmatrix} \eta_1 & \cdots & \eta_{n_2} \\ & \ddots & \vdots \\ 0 & & \eta_1 \end{pmatrix} \end{pmatrix} : \lambda_i, \eta_j, \alpha \in \mathbb{C} \right\}.$$

Simple computation shows that

$$\tilde{N}_\perp \subset \left\{ \begin{pmatrix} \begin{pmatrix} * & \cdots & 0 \\ & \ddots & \vdots \\ * & & * \end{pmatrix} & \begin{pmatrix} * & \cdots & 0 \\ * & \cdots & * \\ \cdots & \cdots & \cdots \\ * & \cdots & * \end{pmatrix} \\ * & \begin{pmatrix} * & \cdots & 0 \\ & \ddots & \vdots \\ * & & * \end{pmatrix} \end{pmatrix} \right\}.$$

Let

$$y = \begin{pmatrix} \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} & 0_{n_1\times n_2} \\ 0 & \begin{pmatrix} 0 & \cdots & 1 \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & 0 \end{pmatrix} \end{pmatrix}.$$

It is easy to see that the operator y cannot be written as a sum of a rank one operator and an operator in $\tilde{N}_\perp$. This contradicts the fact that $\tilde{N}$ has property P_1. $\square$

Proof of Theorem 4.1. Suppose B is generated by a matrix T. By the Jordan canonical form theorem, we may assume that $T = \bigoplus_{i=1}^r (\lambda_i + J_{n_i})$ and $\sum_{i=1}^r n_i = n$. Note that $\dim(B) = n$ if and only if $\lambda_i \neq \lambda_j$ for $i \neq j$, and if and only if

$B = \bigoplus_{i=1}^{r} B_i$, where each B_i is the subalgebra of $M_{n_i}(\mathbb{C})$ generated by the Jordan block J_{n_i}.

Suppose $B \subsetneq N \subset M_n(\mathbb{C})$ and N has property P_1. Let p_i be the central projection of B corresponding to B_i. Then $B_i \subset p_i N p_i \subset M_{n_i}(\mathbb{C})$ and $p_i N p_i$ has property P_1. By Lemma 4.3, $B_i = p_i N p_i$. Since $B \neq N$, there is an element $0 \neq x \in N$ such that $x = p_i x p_j$ for some $i \neq j$. Without loss of generality, we may assume that $0 \neq x = p_1 x p_2$. Now we have $B_1 \oplus B_2 \subsetneq (p_1 + p_2)N(p_1 + p_2) \subseteq M_{n_1+n_2}(\mathbb{C})$ and $(p_1 + p_2)N(p_1 + p_2)$ also has property P_1. On the other hand, by Lemma 4.4, $B_1 \oplus B_2 = (p_1 + p_2)N(p_1 + p_2)$. This is a contradiction. □

5. P_1 algebras in $M_n(\mathbb{C})$, $n \leq 4$

Let B be a subalgebra of $M_n(\mathbb{C})$. Then $B = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C}) \oplus J$, where J is the radical part of B. If $n_1, \ldots, n_s = 1$, then B is upper-triangularizable, that is, there exists a unitary matrix u such that uBu^* is a subalgebra of the upper-triangular algebra of $M_n(\mathbb{C})$ (see [Christensen 1999, Proposition 2.5] or [Humphreys 1972, Corollary A, page 17]). The following lemma will be useful.

Lemma 5.1. [Azoff] *Let S be a subspace of $L(H)$ and consider the subalgebras of $L(H^{(2)})$ defined by*

$$B = \left\{ \begin{pmatrix} \lambda e & a \\ 0 & \lambda e \end{pmatrix} : \lambda \in \mathbb{C}, a \in S \right\}, \quad C = \left\{ \begin{pmatrix} \lambda e & a \\ 0 & \mu e \end{pmatrix} : \lambda, \mu \in \mathbb{C}, a \in S \right\}.$$

(1) *B has property* P_1 *if and only if S has property* P_1.

(2) *C has property* P_1 *if and only if S has property* P_1 *and is intransitive.*

Proposition 5.2. *Let B be a unital subalgebra of $M_2(\mathbb{C})$ with property* P_1. *Then B is unitarily equivalent to one of the following three subalgebras*:

$$\left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \eta \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

Proof. It is easy to verify that the above algebras have property P_1. Suppose B has property P_1. Then the semisimple part of B must be abelian. Conjugating by a unitary matrix, we may assume that B is a subalgebra of the algebra of upper-triangluar matrices. Note that the algebra of upper-triangular matrices does not have property P_1. So B must be one of the algebras listed in the lemma. □

Proposition 5.3. *Let B be a unital subalgebra of $M_3(\mathbb{C})$ with property* P_1. *Then either B or B^* has a separating vector. Therefore*, $\dim B \leq 3$. *Furthermore*, *if*

$\dim B = 3$, *then B is similarly conjugate to one of the following algebras*

$$A_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, \; A_2 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

$$A_3 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_3 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, \; A_4 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

$$A_5 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}, \; A_6 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_1 & \lambda_3 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

Proof. Suppose B has property P_1. Then the semisimple part of B must be abelian. Conjugating by a unitary matrix, we may assume that B is a subalgebra of the algebra of upper-triangluar matrices. We consider the following cases.

Case 1. Suppose the semisimple part of B is $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Then $B = A_1$ by Theorem 3.1.

Case 2. Suppose the semisimple part of B is $\mathbb{C} \oplus \mathbb{C}$. We may assume that the semisimple part of B consists of matrices

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}.$$

We consider two subcases.

Subcase 2.1. Suppose B is contained in the following algebra

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_1 & \lambda_4 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Simple computation shows that B_1 does not have property P_1 (the identity matrix can not be written as $x + (B_1)_\perp$ such that the rank of x is at most 1). So B is a proper subalgebra of B_1. This implies that there exist α, β such that

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_3\alpha \\ 0 & \lambda_1 & \lambda_3\beta \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

If $\alpha \neq 0$, let

$$s = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Simple computation shows that $sA_2s^{-1} = B$, that is, $s^{-1}Bs = A_2$. If $\alpha = 0$, $\beta \neq 0$, let

$$s = \begin{pmatrix} 0 & 1 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $sA_2s^{-1} = B$, that is, $s^{-1}Bs = A_2$. If $\alpha = \beta = 0$, then clearly B has a separating vector.

Subcase 2.2. Suppose B is not contained in B_1. Since B is an algebra, B contains A_3. It is easy to see that A_3 is the algebra generated by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\dim A_3 = 3$. So $B = A_3$ by Theorem 4.1.

Case 3. Suppose the semisimple part of B is $\mathbb{C}$. Then B is contained in the following algebra

$$B_3 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & \lambda_4 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that B_3 does not have property P_1. So B is a proper subalgebra of B_3. We consider the following subcases.

Subcase 3.1. Suppose B contains an element

$$b = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix},$$

such that $\alpha \neq 0$ and $\beta \neq 0$. Conjugating by an invertible upper-triangular matrix, we may assume that $b = J_3$ is the Jordan block. So B contains A_4. By Theorem 4.1, $B = A_4$.

Subcase 3.2. Suppose B does not contain an element b as in subcase 3.2. Then $B \subseteq A_5$ or $B \subseteq A_6$. Note that A_5^* has a separating vector and A_6 has a separating vector. So both A_5 and A_6 have property P_1. $\square$

Lemma 5.4. *Let*

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\} \subset M_4(\mathbb{C}).$$

Then B is a maximal P_1 *algebra.*

Proof. Note that B^* has a separating vector. So B has property P_1. Suppose $A \supsetneq B$ is a P_1 algebra. Suppose A contains a matrix

$$a_1 = \begin{pmatrix} 0 & \alpha & * & * \\ 0 & 0 & \beta & * \\ 0 & 0 & \lambda_1 & \gamma \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix},$$

such that $\gamma \neq 0$. Since $B \subset A$, we may assume that $\alpha \neq 0$ and $\beta \neq 0$. Conjugating by an upper-triangular invertible matrix, we may assume that A contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So A is the algebra generated by the Jordan block by Theorem 4.1 and $\dim A = 4$. However, $\dim B = 4$ and $B \subsetneq A$. This is a contradiction.

Therefore, A is contained in

$$\left\{ \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_1 & * & * \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1 \in \mathbb{C} \right\}.$$

Since A is an algebra containing B and $A \neq B$, we may assume that A contains a matrix of the following form

$$a_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix},$$

where either $s \neq 0$ or $t \neq 0$. Furthermore, we can assume that $s = 1$ and $t \neq 0$. Let A_1 be the algebra generated by B and a_2. Then

$$A_1 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 + \lambda_5 & t\lambda_5 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \ldots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that the predual space of A_1 is

$$\left\{\begin{pmatrix} \eta_1 & * & * & * \\ t\eta_5 & \eta_2 & * & * \\ 0 & -t\eta_5 & \eta_3 & 0 \\ 0 & \eta_5 & 0 & \eta_4 \end{pmatrix} : \eta_1, \ldots, \eta_4 \in \mathbb{C}, \eta_1 + \eta_2 + \eta_3 + \eta_4 = 0\right\}.$$

It is easy to show that the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -t & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

cannot be written as $x + (A_1)_\perp$ such that the rank of x is at most 1. This is a contradiction. So B is a maximal P_1 algebra. $\square$

Proposition 5.5. *Let B be a unital subalgebra of $M_4(\mathbb{C})$ with property* P_1. *Then B satisfies one of the following conditions*:

(i) *B has a separating vector.*

(ii) *B^* has a separating vector.*

(iii) *B is similarly conjugate to an algebra of the form*

$$\left\{\begin{pmatrix} \lambda I_2 & s \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C}, s \in S\right\},$$

where S is a subspace of $M_2(\mathbb{C})$ with dimension 2.

In particular, $\dim B \leq 4$.

Proof. Suppose B has property P_1. Then the semisimple part of B must be $M_2(\mathbb{C})$ or abelian. If the semisimple part of B is $M_2(\mathbb{C})$, then $B = M_2(\mathbb{C})^{(2)}$ by Theorem 3.1. So B has a separating vector. Suppose the semisimple part of B is abelian. Conjugating by a unitary matrix, we may assume that B is a subalgebra of the algebra of upper triangluar matrices. We consider the following cases.

Case 1. Suppose the semisimple part of B is $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$. Then

$$B = \left\{\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C}\right\}$$

by Theorem 3.1. So B has a separating vector.

Case 2. Suppose the semisimple part of B is $\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$. We may assume that the semisimple part of B consists of matrices

$$\begin{pmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \lambda_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 0\\ 0 & 0 & 0 & \lambda_3\end{pmatrix}.$$

Let

$$e_1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad e_2=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix},\quad e_3=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}.$$

By Lemma 2.1, $(e_2+e_3)B(e_2+e_3)\subset M_2(\mathbb{C})$ has property P_1. By Theorem 3.1 and the assumption of Case 2,

$$(e_2+e_3)B(e_2+e_3)=\left\{\begin{pmatrix}\lambda_2 & 0\\ 0 & \lambda_3\end{pmatrix} : \lambda_2,\lambda_3\in\mathbb{C}\right\}.$$

We consider two subcases.

Subcase 2.1. Suppose B is contained in the following algebra

$$\left\{\begin{pmatrix}\lambda_1 & 0 & \lambda_4 & \lambda_6\\ 0 & \lambda_1 & \lambda_5 & \lambda_7\\ 0 & 0 & \lambda_2 & 0\\ 0 & 0 & 0 & \lambda_3\end{pmatrix} : \lambda_1,\dots,\lambda_7\in\mathbb{C}\right\}.$$

By Lemma 2.1, $(e_1+e_2)B(e_1+e_2)\subset M_3(\mathbb{C})$ has property P_1. Note that

$$(e_1+e_2)B(e_1+e_2)\subseteq\left\{\begin{pmatrix}\lambda_1 & 0 & \lambda_4\\ 0 & \lambda_1 & \lambda_5\\ 0 & 0 & \lambda_2\end{pmatrix} : \lambda_1,\dots,\lambda_5\in\mathbb{C}\right\}.$$

By the proof of Subcase 2.1 of Proposition 5.3, there exists an invertible matrix

$$s=\begin{pmatrix}* & * & *\\ * & * & *\\ 0 & 0 & 1\end{pmatrix},$$

such that

$$s^{-1}[(e_1+e_2)B(e_1+e_2)]s\subseteq\left\{\begin{pmatrix}\lambda_1 & 0 & \lambda_3\\ 0 & \lambda_1 & 0\\ 0 & 0 & \lambda_2\end{pmatrix} : \lambda_1,\lambda_2,\lambda_3\in\mathbb{C}\right\}.$$

Conjugating by $(s \oplus 1)^{-1} \in M_4(\mathbb{C})$, we may assume that B is contained in the algebra

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & \lambda_5 \\ 0 & \lambda_1 & 0 & \lambda_6 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_6 \in \mathbb{C} \right\}.$$

It is easy to see that B_1 is similarly conjugate to the algebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_5 & 0 \\ 0 & \lambda_1 & \lambda_6 & \lambda_4 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_6 \in \mathbb{C} \right\}.$$

So we may assume that

$$B_1 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & \lambda_5 & \lambda_6 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_6 \in \mathbb{C} \right\}.$$

Repeating the above arguments, we may assume that B is contained in the algebra

$$B_2 = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & \lambda_5 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that B_2 does not have property P_1 (the identity matrix can not be written as $x + (B_2)_\perp$ such that the rank of x is at most 1). So B is a proper subalgebra of B_2. Therefore, there exist α, β such that

$$B = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4\alpha & 0 \\ 0 & \lambda_1 & 0 & \lambda_4\beta \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

If $\alpha = \beta = 0$, then clearly B has a separating vector.

If $\alpha \neq 0$ and $\beta \neq 0$, let

$$t = \begin{pmatrix} \alpha^{-1} & 0 & 0 & 0 \\ 0 & \beta^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Simple computation shows that

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & \lambda_4 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

So B has a separating vector.

If $\alpha \neq 0$, $\beta = 0$ or $\alpha = 0$, $\beta \neq 0$, then B is similarly conjugate to the algebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_4 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

So B has a separating vector.

Subcase 2.2. Suppose B is not contained in B_1. Since B is an algebra, B contains the algebra

$$B_3 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_4 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that B_3 is the algebra generated by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and $\dim B_3 = 4$. So $B = B_3$ by Theorem 4.1 and B has a separating vector.

Case 3. Suppose the semisimple part of B is $\mathbb{C} \oplus \mathbb{C}$.

Subcase 3.1. Suppose B contains the following subalgebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

Let

$$f_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 2.1, $f_iBf_i \subset M_2(\mathbb{C})$ has property P_1. By Proposition 5.2,

$$f_iBf_i = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\} \quad \text{or} \quad f_iBf_i = \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

We consider the following subsubcases.

Subsubcase 3.1.1. Suppose

$$f_1Bf_1 = f_2Bf_2 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\}.$$

This implies that

$$B \subset \left\{ \begin{pmatrix} \lambda I_2 & * \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

By Lemma 5.1,

$$B = \left\{ \begin{pmatrix} \lambda I_2 & S \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\},$$

where S has property P_1 and is intransitive. By [Azoff 1973, Table 5A, page 34], S is equivalent to one of the following spaces: zero space, or

$$\left\{ \begin{pmatrix} \zeta & 0 \\ 0 & 0 \end{pmatrix} : \zeta \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix} : \zeta \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & \xi \\ 0 & 0 \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\},$$

$$\left\{ \begin{pmatrix} \zeta & 0 \\ \xi & 0 \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & 0 \\ 0 & \xi \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}, \quad \left\{ \begin{pmatrix} \zeta & \xi \\ 0 & \zeta \end{pmatrix} : \zeta, \xi \in \mathbb{C} \right\}.$$

Note that in the last four cases, neither B nor B^* has a separating vector.

Subsubcase 3.1.2. Suppose

$$f_1Bf_1 = f_2Bf_2 = \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

This implies that B contains the following subalgebra

$$B_4 = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_3 & \lambda_4 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that B_4 is the algebra generated by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\dim B_4 = 4$. So $B = B_4$ by Theorem 4.1, and B has a separating vector.

Subsubcase 3.1.3. Suppose

$$f_1 B f_1 = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \right\} \quad \text{and} \quad f_2 B f_2 = \left\{ \begin{pmatrix} \lambda & \eta \\ 0 & \lambda \end{pmatrix} : \lambda, \eta \in \mathbb{C} \right\}.$$

If $\dim B > 3$, then B contains a nonzero matrix

$$b = \begin{pmatrix} 0_2 & a \\ 0_2 & 0_2 \end{pmatrix}.$$

Let B_5 be the subalgebra generated by $f_1 B f_1$, $f_2 B f_2$ and b. Then $\dim B_5 = 4$ and B_5 is the algebra generated by the matrix

$$\begin{pmatrix} 0_2 & a \\ 0_2 & \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}.$$

So $B = B_5$ by Theorem 4.1 and

$$B = \left\{ \begin{pmatrix} \lambda_1 I_2 & \lambda_4 a \\ 0_2 & \begin{pmatrix} \lambda_2 & \lambda_3 \\ 0 & \lambda_2 \end{pmatrix} \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\},$$

where a is a 2×2 matrix. Let

$$t = \begin{pmatrix} b & 0 \\ 0_2 & I_2 \end{pmatrix}.$$

Then

$$t B t^{-1} = \left\{ \begin{pmatrix} \lambda_1 I_2 & \lambda_4 ba \\ 0_2 & \begin{pmatrix} \lambda_2 & \lambda_3 \\ 0 & \lambda_2 \end{pmatrix} \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

So we can choose b appropriately such that $ba = 0_2$, or $ba = I_2$, or

$$ba = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad ba = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad ba = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{or} \quad ba = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

In each case, B has a separating vector.

Subcase 3.2. Suppose B contains the following subalgebra

$$\left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

Let

$$p = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By Lemma 2.1, $pBp \subset M_3(\mathbb{C})$ has property P_1. By Proposition 5.2,

$$pBp = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & \lambda_2 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}$$

or

$$pBp = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

We consider the following subsubcases.

Subsubcase 3.2.1. Suppose

$$pBp = \left\{ \begin{pmatrix} \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_2 & \lambda_3 \\ 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C} \right\}.$$

Then B contains the following subalgebra

$$B_6 = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & 0 & \lambda_2 & \lambda_3 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that B_6 is the algebra generated by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\dim B_6 = 4$. So $B = B_6$ by Theorem 4.1, and B has a separating vector.

Subsubcase 3.2.2. Suppose

$$pBp = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{C} \right\}.$$

If $\dim B > 3$, then B contains a nonzero matrix

$$b = \begin{pmatrix} 0 & a \\ 0 & 0_3 \end{pmatrix}.$$

Let B_7 be the subalgebra generated by $(1-p)B(1-p)$, pBp and b. Then $\dim B_7 = 4$ and B_7 is the algebra generated the matrix

$$\begin{pmatrix} 0 & a \\ 0 & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix}.$$

So $B = B_7$ by Theorem 4.1 and

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_4 a \\ 0 & \begin{pmatrix} \lambda_2 & 0 & \lambda_3 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

Conjugating by an appropriate invertible matrix

$$t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & * & * \\ 0 & 0 & \eta & * \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

we have

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\},$$

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\},$$

or

$$tBt^{-1} = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & \lambda_2 \\ 0 & \lambda_2 & 0 & \lambda_3 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

In each case, B^* has a separating vector.

Case 4. Suppose the semisimple part of B is $\mathbb{C}$. Consider matrices in B with the form

$$b = \begin{pmatrix} 0 & \alpha & * & * \\ 0 & 0 & \beta & * \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Subcase 4.1. B contains a matrix b with $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$. Conjugating by an upper-triangular invertible matrix, we may assume that B contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So B is the algebra generated by the Jordan block by Theorem 4.1. Note that B has a separating vector.

Subcase 4.2. B does not contain a matrix b as in Subcase 4.1 and B contains a matrix b with two elements of α, β, γ nonzero. We may assume that $\alpha \neq 0$ and $\beta \neq 0$. Conjugating by an upper-triangular invertible matrix, we may assume that B contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and therefore} \quad B \supseteq \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

By the assumption of Subcase 4.2, we have

$$B \subset \left\{ \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_1 & * & * \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1 \in \mathbb{C} \right\}. \tag{5}$$

Subsubcase 4.2.1. Suppose the $(2, 4)$-entry of every matrix in B is zero. Then B is contained in the algebra

$$B_8 \subset \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_4 & \lambda_5 \\ 0 & \lambda_1 & \lambda_3 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that B_8 does not have property P_1. So B is a proper subalgebra of B_8. By (5), there exist α, β such that

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4\alpha \\ 0 & \lambda_1 & \lambda_2 + \lambda_4\beta & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

If $\alpha = 0$ and $\beta \neq 0$, then B does not have property P_1. So we may assume that $\alpha \neq 0$. It is easy to see that B^* has a separating vector.

Subsubcase 4.2.2. Suppose the $(2,4)$-entry of a matrix in B is not zero. By (5), B contains an element

$$b = \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \beta & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\gamma \neq 0$. Since B is an algebra, B contains

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} b = \begin{pmatrix} 0 & 0 & \beta & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By (5), B contains

$$\begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since B is an algebra, B contains the subalgebra

$$B_9 \subseteq \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \ldots, \lambda_4 \in \mathbb{C} \right\}.$$

By Lemma 5.4, B_9 is a maximal P_1 algebra. Hence, $B = B_9$ and B^* has a separating vector.

Subcase 4.3. B does not contain a matrix b as in subcase 4.1, subcase 4.2, and B contains a matrix b with one element of α, β, γ nonzero. We may assume that $\alpha \neq 0$. Conjugating by an upper-triangular invertible matrix, we may assume that

B contains the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

By the assumption of subcase 4.3, B is contained in the algebra

$$B_{10} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & 0 & \lambda_5 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_5 \in \mathbb{C} \right\}.$$

Simple computation shows that B_{10} does not have property P_1. So B is a proper subalgebra of B_{10}. We consider the following subsubcases.

Subsubcase 4.3.1. . If the $(1,3)$ entry of each element of B is zero, then B is contained in the algebra

$$B_{11} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_3 \\ 0 & \lambda_1 & 0 & \lambda_4 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Simple computation shows that B_{11} does not have property P_1. So there exist α, β such that

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & 0 & \lambda_3\alpha \\ 0 & \lambda_1 & 0 & \lambda_3\beta \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

If $\beta = 0$, then B^* has a separating vector. If $\beta \neq 0$, then B has a separating vector.

Subsubcase 4.3.2. If the $(2,4)$ entry of each element of B is zero, then B is contained in the algebra

$$B_{12} = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Note that B_{12}^* has a separating vector and hence B^* has a separating vector.

Subsubcase 4.3.3. Suppose B contains an element

$$b = \begin{pmatrix} 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha \neq 0$ and $\gamma \neq 0$. Let

$$t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha^{-1} & -\frac{\beta}{\alpha\gamma} \\ 0 & 0 & 0 & \gamma^{-1} \end{pmatrix}.$$

Then

$$t^{-1}bt = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Conjugating by t^{-1} if necessary, we may assume that $\alpha = \gamma = 1$ and $\beta = 0$. Since B is a proper subalgebra of B_{10}, B is the algebra,

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ 0 & \lambda_1 & 0 & \lambda_3 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

It is easy to see that B^* has a separating vector.

Subcase 4.4. B does not contain a matrix B as in subcase 4.1, subcase 4.2, and subcase 4.3. Then

$$B \subset \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & 0 & \lambda_4 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \dots, \lambda_4 \in \mathbb{C} \right\}.$$

Combining Lemma 5.1 [Azoff 1973, Table 5A, page 34], and similar arguments as in Subsubcase 3.1.1,

$$B = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

or

$$B = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & \lambda_2 \\ 0 & \lambda_1 & 0 & \lambda_3 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

or

$$B = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & \lambda_3 \\ 0 & \lambda_1 & 0 & \lambda_2 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\},$$

or

$$B = \left\{ \begin{pmatrix} \lambda_1 & 0 & \lambda_2 & 0 \\ 0 & \lambda_1 & 0 & \lambda_3 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \right\}.$$

It is easy to show that in each case either B or B^* has a separating vector. $\square$

6. 2-reflexivity and property P_1

Let H be a Hilbert space. The usual notation $\mathrm{Lat}(B)$ will denote the lattice of invariant subspaces (or projections) for a subset $B \subseteq L(H)$, and $\mathrm{Alg}(L)$ will denote the algebra of bounded linear operators leaving invariant every member of a family L of subspaces (or projections). An algebra B is called reflexive if $B = \mathrm{AlgLat}(B)$. An algebra B is called n-reflexive if the n-fold inflation $B^{(n)} = \{b^{(n)} : b \in B\}$, acting on $\mathcal{H}^{(n)}$, is reflexive [Azoff 1986]. In [Larson 1982], the third author proved the following result: An algebra B is n-reflexive if and only if $B_\perp$, the preannihilator of B, is the trace class norm closed linear span of operators of rank$\leq n$. In [Larson 1982], the third author also showed the following connection between n-reflexivity and the P_1 property: If an algebra B has property P_1, then B is 3-fold reflexive. (This result also holds for linear subspaces with the same proof). He raised the following problem: Suppose $\dim H = n \in \mathbb{N}$ and $B \subset L(H) \equiv M_n(\mathbb{C})$ is a unital operator algebra with property P_1. Is B 2-reflexive? Note that this question also makes sense for linear subspaces. Azoff [1986] showed that the answer to the above question is affirmative for $n = 3$ (for all linear subspaces of $M_3(\mathbb{C})$ with property P_1). In this section, we prove the following result.

Proposition 6.1. *If* $\dim H = 4$ *and* $B \subset L(H) \equiv M_4(\mathbb{C})$ *is a unital operator algebra with property* P_1, *then* B *is* 2*-reflexive.*

Proof. By Proposition 5.5, either B or B^* has a separating vector or B is similarly conjugate to an algebra of the form

$$\left\{ \begin{pmatrix} \lambda I_2 & s \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C}, s \in S \right\},$$

where S is a subspace of $M_2(\mathbb{C})$ with dimension two. If B has a separating vector or B^* has a separating vector, then the fact that B is 2-reflexive follows from the proofs of Corollary 7 of [Larson 1982] and Proposition 1.2 of [Herrero et al. 1991]. If B is similarly conjugate to an algebra of the form

$$\left\{ \begin{pmatrix} \lambda I_2 & s \\ 0 & \eta I_2 \end{pmatrix} : \lambda, \eta \in \mathbb{C}, s \in S \right\},$$

where S is a subspace of $M_2(\mathbb{C})$ with dimension two, then the fact that B is 2-reflexive follows from Proposition 1 of [Kraus and Larson 1985]. □

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