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Soap film realization of isoperimetric surfaces with boundary

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We examine surfaces of the type proved to be minimizing under a connectivity condition by Dorff et al. We determine which of these surfaces are stable soap films. The connectivity condition is shown to be very restrictive; few of these surfaces are stable (locally minimizing) without it.

1. Introduction

Surface area minimization in soap bubbles and soap films is one of the more fascinating subjects in mathematics today. Metacalibration techniques — a generalization of the calibrations popularized by Harvey and Lawson [1982] (see also [Morgan 1988, Chapter 6]) — were developed to investigate the problems that arise in surface minimization. In particular metacalibration techniques prove very useful in solving a new class of problems with both fixed volume and fixed boundary constraints. We call these problems *equitent problems* after Lawlor et al. Equitent stands for equal content (volume condition) and equal extent (boundary condition) [Dorff et al. 2008].

In this paper we consider a certain class of equitent problems addressed in [Dorff et al. 2011]. It was shown there that certain equitent surfaces are globally minimizing under a connectivity condition that restricts the surfaces' homotopy class. This connectivity condition is not however true for general minimizing surfaces. We examine which of these surfaces are locally minimizing without the connectivity condition. This is equivalent to showing these surfaces are realizable as a soap film. We demonstrate this for those surfaces that are proved to be locally minimal.

2. The surfaces of Dorff et al.

Equitent surfaces are constructed via the union of sections of spheres and planes. Starting with a cone over a wire-frame polyhedron, the center of the cone is then

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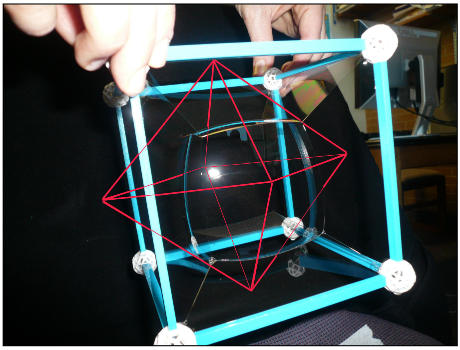


Figure 1. Equitent surface constructed on a cube wireframe.

replaced by a volume (bubble) that is enclosed by spherical caps in the same polyhedral arrangement. See the example soap film in Figure 1. Dorff et al. categorize these figures by the dual figure to the wire frame polyhedron. This dual figure, called the *connectivity graph*, is used to define the planes and spheres used in the construction of these surfaces and describes the adjacency conditions on the resulting surface. The specifics of the construction are not requisite to our results.

In their paper Dorff et al. also define a connectivity condition, which is that exterior regions share boundary only if the corresponding vertices in the connectivity graph are adjacent. They prove that the constructed surfaces are globally area minimizing among all surfaces that enclose the same fixed volume, have the same wire frame polyhedral boundary, and satisfy the connectivity condition.

3. Soap film stability

Theorem. *Among all the minimal surfaces of Dorff et al. in* $\mathbb{R}^3$, *there are only six that are stable as a soap film*: *those whose connectivity graphs are a single point*, *edge*, *equilateral triangle*, *regular tetrahedron*, *regular octahedron*, *or regular icosahedron.*

Proof. We relax the connectivity condition and look at which surfaces are locally area minimizing among surfaces that enclose the same fixed volume and have the same wire frame polyhedral boundary. We reduce conditions for local minimality to conditions on the connectivity graph.

First, in the constuction of the surfaces Dorff et al. require that the connectivity graph to be a uniform polyhedron (polytope) of unit edge length. A uniform polyhedron is one with regular polygon faces and congruent vertices. This guarantees the existence of particular vector fields needed in the minimization proof. They also require the circumradius of the connectivity graph to be strictly less than 1. A circumradius greater than or equal to 1 would create a central bubble of volume zero.

Uniform polyhedra that meet this condition are limited to the tetrahedron, cube, octahedron, icosahedron, triangular prism, pentagonal prism, square antiprism, and pentagonal antiprism.

Minimality conditions come from the work of Jean Taylor [1976]. She proved that Plateau's rules for soap films must hold for locally minimizing surfaces in $\mathbb{R}^3$. These are:

(1) Soap films are made of smooth surfaces of constant mean curvature.

(2) Soap films always meet in threes along a smooth curve, meeting at equal angles of 120^o.

(3) These curves meet in fours at a point, meeting at equal angles of $\cos^{-1}(-\frac{1}{3})$ (approximately 109^o).

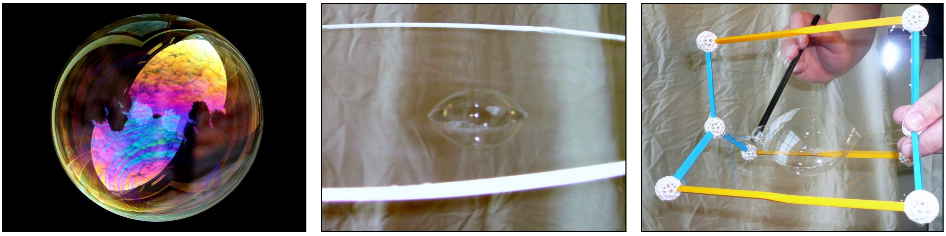
The first and third rules always hold as a result of the surface's construction. The second rule, however, further limits the number of connectivity graphs that can be formed. In the construction, each face of the connectivity graph corresponds to one of these curves (from a vertex of the wire-frame polyhedron) and each edge corresponds to a smooth surface connecting to this curve (from an edge of the wire-frame polyhedron). Thus the second rule implies that connectivity graphs must be constrained to have only triangular faces.

The uniform polyhedra that meet the conditions on the construction and satisfy this second rule are limited to the tetrahedron, octahedron, and icosahedron. For connectivity graphs in lower dimensions that also satisfy these conditions, we have a single point (0 dimensions), a line segment (1 dimension), and an equilateral triangle (2 dimensions). □

These conditions are very restrictive; out of the 18 convex uniform polyhedrons and infinite sets of prisms, antiprisms, and lower dimensional figures, only six equitent surfaces can be created in $\mathbb{R}^3$. In the next section we demonstrate each of these surfaces as a soap film.

4. Realization of the bubbles

Equitent surfaces can be realized as a soap film by dipping a wire-frame in a soap solution and blowing a soap bubble onto the surface. (It may however take several tries to get a surface of a particular homotopy class, and have it last long enough to take a picture!) Each of the six connectivity graphs identified in the last section do generate a stable minimal surface when realized as a soap film this way. Note that the wire-frame polyhedron in each case is the dual figure to the connectivity graph. Also note that the number of vertices in the connectivity graph corresponds to the number of exterior regions separated by the equitent surface.

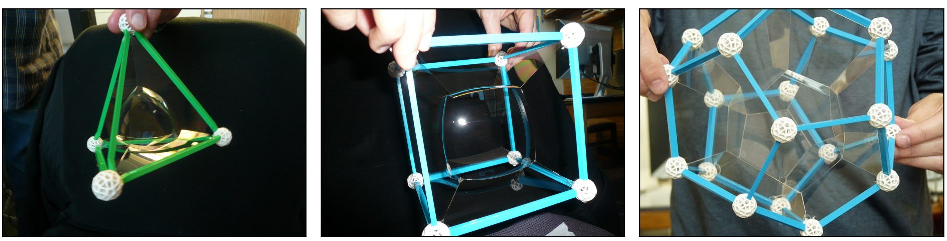


Dimension 0 Dimension 1 Dimension 2

Figure 2. Equitent surfaces with lower dimensional connectivity graph.

For lower dimensional connectivity graphs we see that the surface realized from a single point is a spherical bubble with no wire frame (Figure 2, left). A single edge as a connectivity graph yields a lens shaped bubble on a planar surface. Here we represent the wire-frame as a circle (any polygon in two dimensions will do); see Figure 2, middle. From an equilateral triangle we have a "football" shaped bubble connected to three planar surfaces (Figure 2, right).

For the three dimensional connectivity graphs, a polyhedral shaped bubble with spherical caps will be formed. These figures will also have planar surfaces connecting to each edge of the bubble. For tetrahedral, octahedral, or icosahedral connectivity graphs we get a tetrahedron-, cube-, or dodecahedron-shaped bubble, respectively. See Figure 3.



Tetrahedral graph Octahedral graph Icosahedral graph

Figure 3. Equitent surfaces with dimension-3 connectivity graph.

5. Conclusion

As noted earlier, we have seen that the connectivity condition of Dorff et al. is a very restrictive condition. Each of the locally minimizing surfaces were known prior to their work, though perhaps not yet proven to be minimal. The real impact of their paper comes from the pioneering new method of metacalibration and how we can use it to tackle equitent problems. Their paper gives the first new results

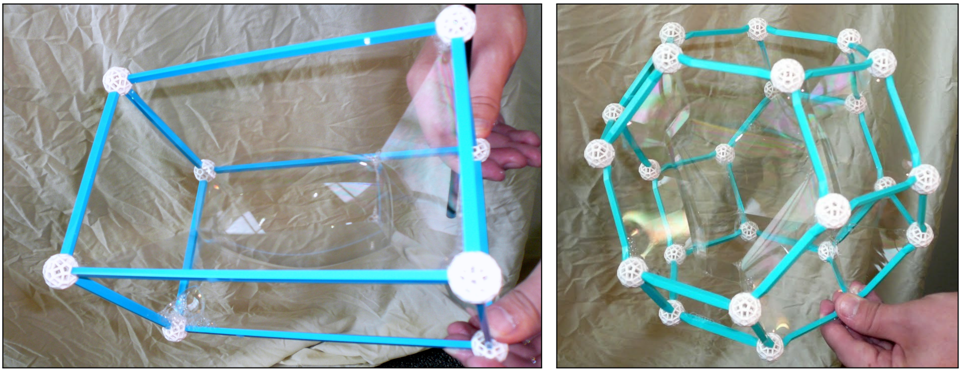


Figure 4. Other examples of equitent surfaces: rectangular prism wire-frame (left) and negative-pressure soap bubbles (right).

proven using this method, though it has also been used to provide new proofs of some multiple bubble problems [Dilts et al. $\geq$ 2011].

We hope to be able to generalize the metacalibration approach to handle further equitent problems. This includes finding an alternate construction of equitent surfaces that relaxes the uniformity condition on the connectivity graphs. This would allow us to investigate surfaces such as those generated on a rectangular prism wire-frame, not just a cube (Figure 4, left).

Another problem to consider are equitent surfaces that would be generated by connectivity graphs of circumradius greater than or equal to 1. Such surfaces are stable in $\mathbb{R}^2$ and $\mathbb{R}^3$, though the central bubble has negative pressure and the faces bow inwards (Figure 4, right).

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