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Zero forcing number, path cover number, and maximum nullity of cacti

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The zero forcing number of a graph is the minimum size of a zero forcing set. This parameter is useful in the minimum rank/maximum nullity problem, as it gives an upper bound to the maximum nullity. The path cover number of a graph is the minimum size of a path cover. Results for comparing the parameters are presented, with equality of zero forcing number and path cover number shown for all cacti and equality of zero forcing number and maximum nullity for a subset of cacti. (A cactus is a graph where each edge is in at most one cycle.)

1. Introduction

Throughout this paper, a *graph* $G = (V_G, E_G)$ will mean a simple (no loops, no multiple edges) undirected graph. We will assume a finite and non-empty vertex set V_G. The edge set E_G consists of two-element subsets of vertices. If $\{x, y\} \in E_G$, we say x and y are *neighbors* or x and y are *adjacent*, and write $x \sim y$.

The zero forcing number of a graph was introduced in [AIM 2008] and the related terminology was developed in [Barioli et al. 2009], [Barioli et al. 2010], and [Hogben 2010]. Referring to it as the graph infection number, physicists have used this parameter in studying quantum systems control [Burgarth and Giovannetti 2007; Burgarth and Maruyama 2009; Severini 2008]. Consider a black and white vertex coloring of a graph G. From the initial coloring, vertices change color according to the *color-change rule*: If v is the only white neighbor of a black vertex u, then change the color of v to black. Applying the color-change rule to u to change the color of v, we say u *forces* v and write $u \to v$. Given an initial coloring of G, the *derived set* is the set of vertices colored black after the color-change rule is applied until no more changes are possible. If the set Z of vertices initially colored black has derived set that is all the vertices of G, we say Z is a *zero forcing set* for G. A zero forcing set with the minimum number of vertices is called an *optimal zero forcing set*, and this minimum size of a zero forcing set for a graph G is the *zero forcing number* of the graph, denoted $\mathrm{Z}(G)$.

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The *path cover number* $\mathrm{P}(G)$ of a graph G is the smallest positive integer m such that there are m vertex-disjoint induced paths in G such that every vertex of G is a vertex of one of the paths.

An association between graphs and matrices is made in the following way. Denote by $S_n(\mathbb{R})$ the set of $n \times n$ real symmetric matrices. The *graph* of $A \in S_n(\mathbb{R})$, denoted $\mathscr{G}(A)$, is the graph with vertices $\{1, \ldots, n\}$ and edges $\{\{i, j\} : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Given a graph G, the *set of symmetric matrices described by* G is $\mathscr{S}(G) = \{A \in S_n(\mathbb{R}) : \mathscr{G}(A) = G\}$. The *minimum rank of* G is $\mathrm{mr}(G) = \min\{\operatorname{rank} A : A \in \mathscr{S}(G)\}$ and the *maximum nullity of* G is $\mathrm{M}(G) = \max\{\operatorname{null} A : A \in \mathscr{S}(G)\}$. Clearly $\mathrm{mr}(G) + \mathrm{M}(G) = |G|$, where the *order* $|G|$ is the number of vertices in G. Because of this relationship, finding the value of one of these two parameters for a graph is equivalent to finding the value for both.

Following are theorems relating the zero forcing number to path cover number and maximum nullity of a graph. These bounds will be used in later results.

Theorem 1.1 [Hogben 2010]. *For any graph G, $\mathrm{P}(G) \leq \mathrm{Z}(G)$.*

Theorem 1.2 [AIM 2008]. *For any graph G, $\mathrm{M}(G) \leq \mathrm{Z}(G)$.*

It is well known that if G is a tree then $\mathrm{P}(G) = \mathrm{Z}(G)$ [AIM 2008] and $\mathrm{P}(G) = \mathrm{M}(G)$ [Johnson and Duarte 1999], so the three parameters are equal.

In this paper, we compare the graph parameters $\mathrm{Z}(G)$, $\mathrm{P}(G)$, and $\mathrm{M}(G)$. In Section 2, we present the effect on the parameters after the deletion of a single vertex or the deletion of a single edge. These (mostly known) results will be utilized in later sections. Results of similar type for each of the graph parameters are presented in a unified format to emphasize the relationship to each other. The main result of Section 3 is equality of zero forcing number and path cover number for cacti, where a cactus is a graph where each edge is in at most one cycle. In Section 4, we prove zero forcing number is equal to maximum nullity for a restricted family of cacti. Section 5 summarizes our results and suggests further research.

Additional properties and some notation. Here we present additional terminology, notation, and theorems that will be used. For a given zero forcing set Z, a *chronological list of forces* is a listing of the forces used to construct the derived set in the order they are performed. A *forcing chain* for a chronological list of forces is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k-1$, $v_i \to v_{i+1}$, and a *maximal forcing chain* is a forcing chain that is not a proper subsequence of any other forcing chain. The collection of maximal forcing chains for a chronological list of forces is called the *chain set* of the chronological list of forces, and an *optimal chain set* is a chain set from a chronological list of forces of an optimal zero forcing set. When a chain set contains a chain consisting of a single vertex, we say that the chain set contains the vertex as a *singleton*. For a zero forcing set Z, a *reversal* of Z is the set of vertices which are last in the forcing chains in the chain set of some chronological list of forces [Barioli et al. 2010].

Theorem 1.3 [Barioli et al. 2010]. *If Z is a zero forcing set of G then so is any reversal of Z.*

Observation 1.4. If Z' is a reversal of Z, then $|Z'| = |Z|$. In particular, if Z is an optimal zero forcing set, then a reversal Z' of Z is also an optimal zero forcing set.

A vertex v is called *terminal* if it is the endpoint of a path in some minimum path cover. It is called *doubly terminal* if it is in a path by itself in some minimum path cover, and is called *simply terminal* if it is terminal but not doubly terminal.

For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$, the *induced subgraph* $G[W]$ is the graph with vertex set W and edge set $\{\{v, w\} \in E_G : v, w \in W\}$. The subgraph induced by $\overline{W} = V_G \setminus W$ will be denoted by $G - W$, or in the case W is a single vertex $\{v\}$, by $G - v$. For $e \in E_G$, the subgraph $(V_G, E_G \setminus \{e\})$ will be denoted by $G - e$.

A graph is called *connected* if any two vertices are linked by a path. If a graph is not connected, we say it is *disconnected*. The maximal connected subgraphs of a graph are called the *components* of the graph. If the graph $G - v$ has more connected components than G, then v is called a *cut-vertex* of G. Similarly, a *cut-edge* of a graph is one such that its deletion increases the number of connected components.

2. Edge spread and vertex spread

We present a number of (mostly known) results which will be used in later sections. They are grouped and formatted in such a way as to emphasize commonality between the types of results for the different parameters.

Edge spread. In this subsection, we consider the effects on zero forcing number, path cover number, and maximum nullity when deleting a single edge from a graph. For a graph G and an edge e of G, the *rank edge spread* of e in G is $\mathrm{r}_e(G) = \mathrm{mr}(G) - \mathrm{mr}(G - e)$, the *null edge spread* of e in G is $\mathrm{n}_e(G) = \mathrm{M}(G) - \mathrm{M}(G - e)$, and the *zero edge spread* of e in G is $\mathrm{z}_e(G) = \mathrm{Z}(G) - \mathrm{Z}(G - e)$ [Edholm et al. 2010]. Here we make an analogous definition concerning change in path cover number when deleting an edge.

Definition 2.1. The *path edge spread* of e in G is $\mathrm{p}_e(G) = \mathrm{P}(G) - \mathrm{P}(G - e)$.

First we present the bounds on the zero edge spread and path edge spread and attempt to characterize edges with a given edge spread value.

Theorem 2.2 [Edholm et al. 2010]. *For every graph G and every edge $e = \{v, w\}$ of G, $-1 \leq \mathrm{z}_e(G) \leq 1$. If $\mathrm{z}_e(G) = 1$, then there exists an optimal chain set such that e is not an edge in any chain.*

Theorem 2.3. *For every graph G and every edge $e = \{v, w\}$ of G, $-1 \leq \mathrm{p}_e(G) \leq 1$. If $\mathrm{p}_e(G) = 1$, then there exists a minimum path cover such that v and w are not in the same path.*

Proof. Let G be a graph and $e = \{v, w\}$ be an edge in G. Consider a minimum path cover of G. If v and w are not covered by the same path, then this path cover of G is also a path cover of $G - e$. If v and w are covered by the same path in the path cover of G, then splitting the path into two paths will create a path cover of $G - e$. Either way, $\mathrm{P}(G - e) \leq \mathrm{P}(G) + 1$ so $\mathrm{p}_e(G) \geq -1$.

Consider a minimum path cover of $G - e$. If v and w are not covered by the same path, then this path cover of $G - e$ is also a path cover of G (observe that this case cannot occur if $\mathrm{p}_e(G) = 1$). If v and w are covered by the same path in the path cover of $G - e$, there must be a vertex on the path between them. Let x be the vertex that is between v and w on the path and adjacent to v. Split the path between v and x. This is a path cover of G, but with one more than $\mathrm{P}(G - e)$ paths. In the case $\mathrm{p}_e(G) = 1$, this is a minimum path cover of G with v and w in different paths. Regardless of the path edge spread, $\mathrm{P}(G) \leq \mathrm{P}(G - e) + 1$ so $\mathrm{p}_e(G) \leq 1$. □

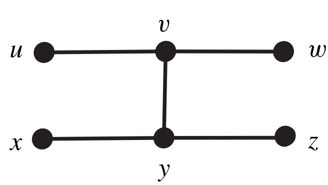
Theorem 2.4 [Edholm et al. 2010]. *Let $e = \{v, w\}$ be an edge of G. If $\mathrm{z}_e(G) = -1$, then for every optimal zero forcing chain set of G, e is an edge in a chain.*

Theorem 2.5. *Let $e = \{v, w\}$ be an edge of G. If $\mathrm{p}_e(G) = -1$, then for every minimum path cover of G, v and w are in the same path.*

Proof. The contrapositive will be proved. Let G be a graph and $e = \{v, w\}$ be an edge of G. Suppose there is a minimum path cover of G in which v and w are not in the same path. This path cover of G is also a path cover of $G - e$, so $\mathrm{P}(G - e) \leq \mathrm{P}(G)$. Hence $\mathrm{p}_e(G) \geq 0$. □

Theorem 2.5 can be viewed as a partial converse to the second statement in Theorem 2.3. Here we provide an example showing that the converse of the second statement in Theorem 2.3 is not true. This example also shows the converse of the second statement in Theorem 2.2 is false.

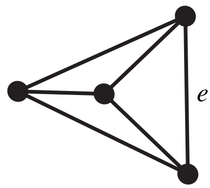
Example 2.6. Let G be this graph:



For $e = \{v, y\}$ we have $\mathrm{p}_e(G) = 0$, but v and y are not in the same path in the minimum path cover.

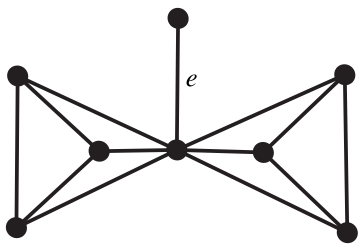
Although the bounds on $\mathrm{z}_e(G)$ and $\mathrm{p}_e(G)$ are the same, the parameters are not generally comparable, as can be seen in Examples 2.7 and 2.8 below. Null edge spread has the same bounds as well, and [Edholm et al. 2010] gives examples showing the incomparability of $\mathrm{z}_e(G)$ with $\mathrm{n}_e(G)$.

Example 2.7. Let G be this graph:



Here $\mathrm{Z}(G) = 3$ and $\mathrm{Z}(G-e) = \mathrm{P}(G) = \mathrm{P}(G-e) = 2$. Therefore, $\mathrm{z}_e(G) = 1 > 0 = \mathrm{p}_e(G)$.

Example 2.8. Let G be this graph:



Here $\mathrm{Z}(G) = 5$, $\mathrm{Z}(G-e) = 6$, and $\mathrm{P}(G) = \mathrm{P}(G-e) = 4$. Therefore, $\mathrm{z}_e(G) = -1 < 0 = \mathrm{p}_e(G)$.

Under the conditions of Observation 2.9 we can use one of parameters $\mathrm{z}_e(G)$ or $\mathrm{p}_e(G)$ to determine the other.

Observation 2.9. Let G be a graph such that $\mathrm{P}(G) = \mathrm{Z}(G)$ and let e be an edge of G. Then:

(1) $\mathrm{p}_e(G) \geq \mathrm{z}_e(G)$.

(2) If $\mathrm{z}_e(G) = 1$, then $\mathrm{p}_e(G) = 1$.

(3) If $\mathrm{p}_e(G) = -1$, then $\mathrm{z}_e(G) = -1$.

Next we consider edge spreads when the edge is a cut-edge.

Theorem 2.10 [Barioli et al. 2004]. *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{r}_e(G) = \begin{cases} 0 & \text{if } \max_{i=1,2}\{r_{v_i}(G_i)\} = 2, \\ 1 & \text{otherwise.} \end{cases}$$

Corollary 2.11. *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{n}_e(G) = \begin{cases} 0 & \text{if } \min_{i=1,2}\{n_{v_i}(G_i)\} = -1, \\ -1 & \text{otherwise.} \end{cases}$$

Proof. This follows from Theorem 2.10 and the fact that $\mathrm{r}_e(G) + \mathrm{n}_e(G) = 0$ for any graph G and any edge e of G. □

Theorem 2.12. *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{z}_e(G) = \begin{cases} -1 & \text{if } v_i \text{ is in an optimal zero forcing set in } G_i \text{ for } i = 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let Z_1 and Z_2 be optimal zero forcing sets for G_1 and G_2, respectively. Let $Z = Z_1 \cup Z_2$. Color the vertices of Z black and the remaining vertices white. Forces can be performed in G_1 until v_1 is black. Forces can be performed in G_2 until v_2 is black. Now the remaining forces can take place in G_1 and in G_2. Therefore Z is a zero forcing set for G and $\mathrm{Z}(G) \leq |Z| = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e)$. Hence $\mathrm{z}_e(G) \leq 0$.

Suppose v_1 is an optimal zero forcing set Z_1 for G_1 and v_2 is in an optimal zero forcing set Z_2 in G_2. Let Z_1' be a reversal of Z_1. Then by Observation 1.4, Z_1' is an optimal zero forcing set for G_1 and there is a chronological list of forces in which v_1 does not perform a force (i.e., v_1 is last in the maximal forcing chain which contains it). Let $Z = Z_1' \cup Z_2 \setminus \{v_2\}$. Color the vertices of Z black and the remaining vertices white. Forces can be performed in G_1 until all vertices of G_1 are black and v_1 has not performed a force. Now v_1 is black and v_2 is the only white neighbor of v_1, so $v_1 \to v_2$. Now all the vertices of Z_2 are black and none has performed a force, so all other vertices of G_2 can be forced black. Therefore Z is a zero forcing set for G and $\mathrm{Z}(G) \leq |Z| = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) - 1 = \mathrm{Z}(G - e) - 1$. Theorem 2.2 gives $\mathrm{z}_e(G) \geq -1$, so $\mathrm{z}_e(G) = -1$.

Suppose now that at least one of v_1 or v_2 is not in any optimal zero forcing set for the respective component. Without loss of generality, say v_1 is not in any optimal zero forcing set for G_1. Let Z be an optimal zero forcing set for G and consider the chronological list of forces. Examine the following cases.

Case 1: Suppose $v_1 \to v_2$. Then v_1 cannot force any vertex of G_1. Since v_1 is not in any optimal zero forcing set for G_1, it is not at the end of a forcing chain for any optimal zero forcing set of G_1. Thus v_1 forcing v_2 requires $|Z \cap V_{G_1}| \geq \mathrm{Z}(G_1) + 1$. It must also be that $|Z \cap V_{G_2}| \geq \mathrm{Z}(G_2) - 1$. Then $\mathrm{Z}(G) = |Z| = |Z \cap V_{G_1}| + |Z \cap V_{G_2}| \geq \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e)$, so $\mathrm{z}_e(G) \geq 0$.

Case 2: Suppose $v_1 \not\to v_2$. Then $|Z \cap V_{G_2}| \geq \mathrm{Z}(G_2)$. Since v_1 is not in any optimal zero forcing set for G_1, it must be that $|Z \cap V_{G_1}| \geq \mathrm{Z}(G_1)$. Then $\mathrm{Z}(G) = |Z| = |Z \cap V_{G_1}| + |Z \cap V_{G_2}| \geq \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e)$, so $\mathrm{z}_e(G) \geq 0$. □

Theorem 2.13 [Barioli et al. 2004]. *Let $e = \{v_1, v_2\}$ be a cut-edge of a connected graph G. Let G_1 and G_2 be the connected components of $G - e$ with $v_1 \in G_1$ and $v_2 \in G_2$. Then*

$$\mathrm{p}_e(G) = \begin{cases} -1 & \textit{if } v_i \textit{ is terminal in } G_i \textit{ for } i = 1, 2, \\ 0 & \textit{otherwise.} \end{cases}$$

The converse of Theorem 2.4 is open from [Edholm et al. 2010], and the converse of Theorem 2.5 is left open in this paper. We will show that the converses of these theorems are true for a cut-edge.

Theorem 2.14. *Let $e = \{v, w\}$ be a cut-edge of G. If e is an edge in a chain for every optimal zero forcing chain set of G, then $\mathrm{z}_e(G) = -1$.*

Proof. The contrapositive will be proved. Suppose $\mathrm{z}_e(G) \neq -1$. By Theorem 2.12, $\mathrm{z}_e(G) = 0$. Let G_1 and G_2 be the connected components of $G - e$ with $v \in G_1$ and $w \in G_2$. Let Z_1 and Z_2 be optimal zero forcing sets for G_1 and G_2, respectively. Let $Z = Z_1 \cup Z_2$. Color the vertices of Z black and the remaining vertices white. Forces can be performed in G_1 until v is black. Forces can be performed in G_2 until w is black. Now the remaining forces can take place in G_1 and in G_2. Therefore Z is a zero forcing set for G and $e = \{v, w\}$ is not an edge in any chain. Also, $|Z| = \mathrm{Z}(G_1) + \mathrm{Z}(G_2) = \mathrm{Z}(G - e) = \mathrm{Z}(G) - \mathrm{z}_e(G) = \mathrm{Z}(G)$, so Z is an optimal zero forcing set for G. □

Theorem 2.15. *Let $e = \{v, w\}$ be a cut-edge of G. If v and w are in the same path for every minimum path cover of G, then $\mathrm{p}_e(G) = -1$.*

Proof. The contrapositive will be proved. Suppose $\mathrm{p}_e(G) \neq -1$. By Theorem 2.13, $\mathrm{p}_e(G) = 0$. Let G_1 and G_2 be the connected components of $G - e$ with $v \in G_1$ and $w \in G_2$. Consider a path cover of G consisting of minimum path covers of G_1 and G_2. Then v and w are not in the same path of this path cover of G. Also, since $\mathrm{p}_e(G) = 0$, this path cover of G is minimum. □

Vertex spread. In this section, we consider the effects on minimum rank, maximum nullity, zero forcing number, and path cover number when deleting a single vertex from a graph. For a graph G and a vertex v of G, the *rank spread* of v in G is $\mathrm{r}_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v)$ [Barioli et al. 2004], the *null spread* of v in G is $\mathrm{n}_v(G) = \mathrm{M}(G) - \mathrm{M}(G - v)$ [Edholm et al. 2010], the *zero spread* of v in G is $\mathrm{z}_v(G) = \mathrm{Z}(G) - \mathrm{Z}(G - v)$ [Edholm et al. 2010], and the *path spread* of v in G is $\mathrm{p}_v(G) = \mathrm{P}(G) - \mathrm{P}(G - v)$ [Barioli et al. 2005].

Theorem 2.16 [Edholm et al. 2010; Huang et al. 2010]. *For every graph G and vertex v of G, $-1 \le \mathrm{z}_v(G) \le 1$.*

Theorem 2.17 [Barioli et al. 2004; Barioli et al. 2005]. *For every graph G and vertex v of G, $-1 \le \mathrm{p}_v(G) \le 1$.*

Recall that v being contained as a singleton means it is in a forcing chain by itself in an optimal chain set, and v being doubly terminal means it is in a path by itself in a minimum path cover.

Theorem 2.18 [Edholm et al. 2010]. *Let v be a vertex of G. Then $\mathrm{z}_v(G) = 1$ if and only if there exists an optimal chain set of G that contains v as a singleton.*

Theorem 2.19 [Barioli et al. 2005]. *Let v be a vertex of G. Then $\mathrm{p}_v(G) = 1$ if and only if v is doubly terminal.*

Theorem 2.20 [Edholm et al. 2010]. *Let v be a vertex of G. If $\mathrm{z}_v(G) = -1$, then v is never in an optimal zero forcing set for G.*

Theorem 2.21 [Barioli et al. 2005]. *Let v be a vertex of G. If $\mathrm{p}_v(G) = -1$, then v is not terminal.*

The next theorems give the parameter spreads for a cut-vertex. Recall that v being simply terminal means that v is terminal but not doubly terminal. By Theorems 2.19 and 2.21, this is equivalent to the path spread being zero and v being an endpoint in some minimal path cover.

Theorem 2.22 [Barioli et al. 2004]. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \dots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Then*

$$\mathrm{r}_v(G) = \min\left\{\sum_{i=1}^{k} \mathrm{r}_v(G_i), 2\right\}$$

Corollary 2.23. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \dots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote $\min_{1 \le j \le k}\{\mathrm{n}_v(G_j)\}$, and t denote the number of the G_i's in which $\mathrm{n}_v(G_i) = 0$. Then*

$$\mathrm{n}_v(G) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = 0 \text{ and } t = 1, \\ -1 & \text{if } m = 0 \text{ and } t \ge 2, \text{ or if } m = -1. \end{cases}$$

Proof. This follows from Theorem 2.22 and the fact that $\mathrm{r}_v(G) + \mathrm{n}_v(G) = 1$ for any graph G and any vertex v of G. □

Theorem 2.24 [Row 2011]. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \dots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote* $\min_{1 \le j \le k}\{\mathrm{z}_v(G_j)\}$, *and t denote the number of the G_i's in which $\mathrm{z}_v(G_i) = 0$ and v is in an optimal zero forcing set. Then*

$$\mathrm{z}_v(G) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = 0 \text{ and } t \le 1, \\ -1 & \text{if } m = 0 \text{ and } t \ge 2, \text{ or if } m = -1. \end{cases}$$

Theorem 2.25 [Barioli et al. 2005]. *Let $G = (V_G, E_G)$ be a graph with cut-vertex $v \in V_G$. Let $W_1, \dots, W_k$ be the vertex sets for the connected components of $G - v$, and for $1 \le i \le k$, let $G_i = G[W_i \cup \{v\}]$. Let m denote* $\min_{1 \le j \le k}\{\mathrm{p}_v(G_j)\}$, *and t denote the number of the G_i's in which v is simply terminal. Then*

$$\mathrm{p}_v(G) = \begin{cases} 1 & \text{if } m = 1, \\ 0 & \text{if } m = 0 \text{ and } t \le 1, \\ -1 & \text{if } m = 0 \text{ and } t \ge 2, \text{ or if } m = -1. \end{cases}$$

3. Comparing Z(G) and P(G) for cacti

A block of a graph is a maximal connected subgraph without a cut-vertex. A *cactus* is a graph in which each block is either a cycle or an edge. In other words, a cactus is a graph in which any two cycles share at most one vertex. An example of a cactus is shown in Figure 1. In this section, we prove $\mathrm{Z}(G) = \mathrm{P}(G)$ for any cactus G. We begin with a few preliminaries.

Theorem 3.1 [Row 2011]. *Let G be a unicyclic graph. Then* $\mathrm{Z}(G) = \mathrm{P}(G)$.

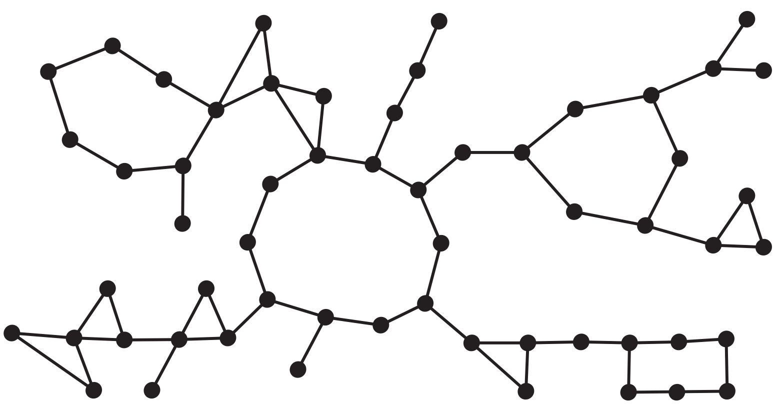


Figure 1. A cactus. No edge is in more than one cycle.

Lemma 3.2. *Let G be a graph, v a vertex in G, and H the graph constructed by appending a leaf w to v in G. Suppose $\mathrm{Z}(G) = \mathrm{P}(G)$ and $\mathrm{Z}(H) = \mathrm{P}(H)$. The vertex v is in an optimal zero forcing set for G if and only if v is terminal in G.*

Proof. Suppose v is in an optimal zero forcing set for G. An optimal chain set from this optimal zero forcing set determines a path cover of G with $\mathrm{Z}(G) = \mathrm{P}(G)$ paths and v as an endpoint of a path. Hence v is terminal.

Suppose v is terminal in G. Then $e = \{v, w\}$ is a cut edge and the graph $H' = (\{w\}, \varnothing)$ is a single isolated vertex. Therefore, w is terminal in H'. By Theorem 2.13, $\mathrm{p}_e(H) = -1$. By Observation 2.9, $\mathrm{z}_e(H) = -1$. By Theorem 2.12, v is in an optimal zero forcing set for G. □

Theorem 3.3. *Let G be a cactus. Then $\mathrm{Z}(G) = \mathrm{P}(G)$.*

Proof. The theorem will be proved by induction on the number of cycles in the cactus. If there is one cycle, G is a unicyclic graph and by Theorem 3.1, $\mathrm{Z}(G) = \mathrm{P}(G)$. Suppose now that for some $m \geq 2$ any cactus G with less than m cycles satisfies $\mathrm{Z}(G) = \mathrm{P}(G)$. Let G be a cactus with m cycles. Since the cycles are edge disjoint, there is a cut-vertex v such that $G - v$ has connected components with vertex sets $W_1, \dots, W_k$ and each graph $G_i = G[W_i \cup \{v\}], \forall i = 1, \dots k$ is a cactus with fewer than m cycles. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{P}(G_i), \forall i = 1, \dots, k$ and $\mathrm{Z}(G_i - v) = \mathrm{P}(G_i - v), \forall i = 1, \dots, k$, so $\mathrm{z}_v(G_i) = \mathrm{p}_v(G_i), \forall i = 1, \dots, k$. Therefore, $\min_{1 \leq j \leq k}\{\mathrm{z}_v(G_j)\} = \min_{1 \leq j \leq k}\{\mathrm{p}_v(G_j)\}$. For all $i = 1, \dots k$, consider the graphs H_i constructed by appending a leaf w_i to v in G_i. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{P}(G_i), \forall i = 1, \dots k$ and $\mathrm{Z}(H_i) = \mathrm{P}(H_i), \forall i = 1, \dots k$. By Lemma 3.2, v is in an optimal zero forcing set for G_j if and only if v is terminal in G_j. Then $\mathrm{z}_v(G_j) = 0$ and v is in an optimal zero forcing set for G_j if and only if $\mathrm{p}_v(G_j) = 0$ and v is terminal in G_j if and only if v is simply terminal in G_j by the contrapositive of Theorem 2.19. Then by Theorems 2.24 and 2.25, $\mathrm{z}_v(G) = \mathrm{p}_v(G)$. Hence $\mathrm{Z}(G) = \sum_{i=1}^{k} \mathrm{Z}(G_i - v) + \mathrm{z}_v(G) = \sum_{i=1}^{k} \mathrm{P}(G_i - v) + \mathrm{p}_v(G) = \mathrm{P}(G)$. □

4. Comparing $\mathrm{Z}(G)$ and $\mathrm{M}(G)$ for cacti

In Section 3 we showed equality of $\mathrm{Z}(G)$ and $\mathrm{P}(G)$ for all cacti G by utilizing Theorem 3.1 for the base case in the induction proof. Since it is not true that $\mathrm{Z}(G) = \mathrm{M}(G)$ for all unicyclic graphs, in this section we focus on a subset of cacti and prove $\mathrm{Z}(G) = \mathrm{M}(G)$ for each graph in this subset.

Let C_n be an n-cycle and let $U \subseteq V_{C_n}$. The graph H obtained from C_n by appending a leaf to each vertex in U is called a *partial n-sun*. If $U = V_{C_n}$, then H is called an n-sun. It was shown in [Barioli et al. 2005] that $\mathrm{M}(H) = \mathrm{P}(H)$ for partial n-suns except for n-suns with $n > 3$ odd.

If there are at least two components of the graph $G - v$ which are paths, each joined to v in G at only one endpoint, then vertex v is called *appropriate*. A vertex v is called a *peripheral leaf* if v is adjacent to only one other vertex u, and u is adjacent to no more than two vertices. The *trimmed form* of a graph G is an induced subgraph obtained by a sequence of deletions of appropriate vertices, isolated paths, and peripheral leaves until no more such deletions are possible.

Theorem 4.1 [Row 2011]. *If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then $\mathrm{Z}(G) = \mathrm{Z}(\breve{G}) + n_2 - n_1$.*

Theorem 4.2 [Barioli et al. 2005]. *If the trimmed form of G, $\breve{G}$, can be obtained by performing n_1 deletions of appropriate vertices, n_2 deletions of isolated paths, and n_3 deletions of peripheral leaves, then $\mathrm{M}(G) = \mathrm{M}(\breve{G}) + n_2 - n_1$.*

Theorem 4.3 [Barioli et al. 2005]. *The trimmed form of a unicyclic graph G is either the empty graph or a partial n-sun.*

Observation 4.4. The trimmed form of a unicyclic graph G in which at least one of the cycle vertices has only two neighbors is not an n-sun.

The following theorem and lemma will be used in the proof of Theorem 4.7, the main result of this section.

Theorem 4.5. *Let G be a unicyclic graph in which the cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. Then $\mathrm{Z}(G) = \mathrm{M}(G)$.*

Proof. Let $\breve{G}$ be the trimmed form of G. By Theorem 4.3 and Observation 4.4, $\breve{G}$ is either the empty graph or a partial n-sun, but not an n-sun with n odd and greater than three. The formulas from [Barioli et al. 2005] give $\mathrm{M}(\breve{G}) = \mathrm{P}(\breve{G})$. Theorem 3.1 gives $\mathrm{Z}(\breve{G}) = \mathrm{P}(\breve{G})$, so $\mathrm{Z}(\breve{G}) = \mathrm{M}(\breve{G})$. Then $\mathrm{Z}(G) = \mathrm{M}(G)$ by Theorems 4.1 and 4.2. □

Lemma 4.6. *Let G be a graph, v a vertex in G, and H the graph constructed from G by appending a leaf w to v, then appending a leaf x to w. Suppose $\mathrm{Z}(G) = \mathrm{M}(G)$ and $\mathrm{Z}(H) = \mathrm{M}(H)$. The vertex v is in an optimal zero forcing set for G if and only if $\mathrm{n}_v(G) = 0$.*

Proof. By construction, $e = \{v, w\}$ is a cut edge and the graph

$$H' = \{\{w, x\}, \{\{w, x\}\}\}$$

is a path on two vertices. Since $\mathrm{Z}(H') = \mathrm{M}(H')$, $\mathrm{z}_e(H) = \mathrm{n}_e(H)$. Also, w is in an optimal zero forcing set for H' and $n_w(H') = 0$. Then $\mathrm{n}_v(G) = 0 \Leftrightarrow \mathrm{n}_e(H) = -1 \Leftrightarrow \mathrm{z}_e(H) = -1 \Leftrightarrow v$ is in an optimal zero forcing set for G by Corollary 2.11 and Theorem 2.12. □

Here we present the main result of the section.

Theorem 4.7. *Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. Then* $\mathrm{Z}(G) = \mathrm{M}(G)$.

Proof. Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors. The theorem will be proved by induction on the number of cycles in the cactus. If there is one cycle, G is a unicyclic graph in which the cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors, and by Theorem 4.5, $\mathrm{Z}(G) = \mathrm{M}(G)$. Suppose now that for some $m \geq 2$ any cactus G in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors with less than m cycles satisfies $\mathrm{Z}(G) = \mathrm{M}(G)$. Let G be a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors with m cycles. Since the cycles are edge disjoint, there is a cut-vertex v such that $G - v$ has connected components with vertex sets $W_1, \ldots, W_k$ and each graph $G_i = G[W_i \cup \{v\}], \forall i = 1, \ldots k$ is a cactus in which each cycle has three vertices, an even number of vertices, or a vertex which has only two neighbors with fewer than m cycles. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{M}(G_i), \forall i = 1, \ldots, k$ and $\mathrm{Z}(G_i - v) = \mathrm{M}(G_i - v), \forall i = 1, \ldots, k$, so $\mathrm{z}_v(G_i) = \mathrm{n}_v(G_i), \forall i = 1, \ldots, k$. Therefore, $\min_{1 \leq j \leq k}\{\mathrm{z}_v(G_j)\} = \min_{1 \leq j \leq k}\{\mathrm{n}_v(G_j)\}$. For all $i = 1, \ldots k$, consider the graphs H_i constructed by appending a leaf w_i to v in G_i then appending a leaf x_i to w_i. By the inductive hypothesis, $\mathrm{Z}(G_i) = \mathrm{M}(G_i), \forall i = 1, \ldots k$ and $\mathrm{Z}(H_i) = \mathrm{M}(H_i), \forall i = 1, \ldots k$. By Lemma 4.6, v is in an optimal zero forcing set for G_j if and only if $\mathrm{n}_v(G_j) = 0$. Then $\mathrm{z}_v(G_j) = 0$ and v is in an optimal zero forcing set for G_j if and only if $\mathrm{n}_v(G_j) = 0$. Then by Theorem 2.24 and Corollary 2.23, $\mathrm{z}_v(G) = \mathrm{n}_v(G)$. Hence

$$\mathrm{Z}(G) = \sum_{i=1}^{k} \mathrm{Z}(G_i - v) + \mathrm{z}_v(G) = \sum_{i=1}^{k} \mathrm{M}(G_i - v) + \mathrm{n}_v(G) = \mathrm{M}(G). \qquad \square$$

The restrictions imposed on the cacti in this section are sufficient for $\mathrm{Z}(G) = \mathrm{M}(G)$, but are not necessary, as can be seen in the following example.

Example 4.8. The graph G shown in Figure 2 does not satisfy the property that each odd cycle of size five or more has at least one vertex with only two neighbors, but does satisfy $\mathrm{Z}(G) = \mathrm{M}(G)$.

5. Conclusions and open questions

We utilized cut-vertex and cut-edge results for zero forcing number, path cover number, and maximum nullity to build graphs having equality of parameters from smaller graphs having equality of the same parameters. Specifically, from knowing $\mathrm{Z}(G) = \mathrm{P}(G)$ for unicyclic graphs we showed $\mathrm{Z}(G) = \mathrm{P}(G)$ for cacti, and from

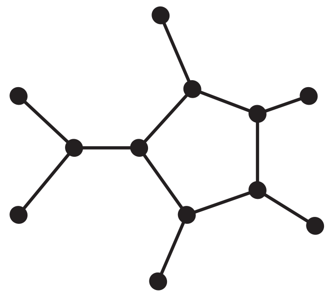


Figure 2. A cactus G that is not in the restricted family but which satisfies $\mathrm{Z}(G) = \mathrm{M}(G)$.

$\mathrm{Z}(G) = \mathrm{M}(G)$ for a restricted family of unicyclic graphs we showed $\mathrm{Z}(G) = \mathrm{M}(G)$ for a restricted family of cacti.

Question 5.1. *What other graphs with equality of some parameters have additional properties that would allow cut-vertex and cut-edge results to be utilized to "build" larger graphs having equality of the parameters?*

Question 5.2. *What are necessary conditions for a cactus to satisfy* $\mathrm{Z}(G) = \mathrm{M}(G)$?

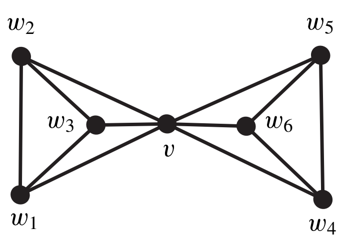
The converse of Theorem 2.4 is open from [Edholm et al. 2010]. We proved the converse holds if e is a cut-edge. We also proved the converse of Theorem 2.5 holds for a cut-edge.

Question 5.3. *Is the converse of Theorem 2.5 true? That is, if v and w are in the same path for every minimum path cover of G, does $\mathrm{p}_e(G) = -1$ where $e = \{v, w\}$?*

In general, v being in an optimal zero forcing set does not imply it being terminal, nor does v being terminal imply it being in an optimal zero forcing set, as evidenced by Examples 5.5 and 5.6 below. With the hypothesis that $\mathrm{Z}(G) = \mathrm{P}(G)$, we do get v in an optimal zero forcing set implying v terminal, as can be seen in the first part of the proof for Lemma 3.2 where the graph H is not used. The hypothesis about H is needed in Lemma 4.6 (see Example 5.7).

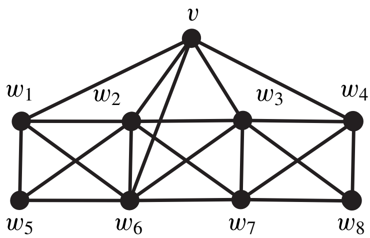
Question 5.4. *Is the graph H from the hypothesis of Lemma 3.2 necessary for the conclusion? For a graph G with $\mathrm{Z}(G) = \mathrm{P}(G)$, does vertex v being terminal imply v is in an optimal zero forcing set?*

Example 5.5. The vertex v is a cut-vertex for this graph G:



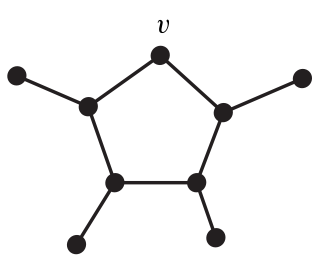
Now both $G[\{v, w_1, w_2, w_3\}]$ and $G[\{v, w_4, w_5, w_6\}]$ are K_4, so we can write $\mathrm{z}_v(G[\{v, w_1, w_2, w_3\}]) = \mathrm{z}_v(G[\{v, w_4, w_5, w_6\}]) = 1$ and v is simply terminal in $G[\{v, w_1, w_2, w_3\}]$ and $G[\{v, w_4, w_5, w_6\}]$. Hence $\mathrm{z}_v(G) = 1$ and $\mathrm{p}_v(G) = -1$ by Theorems 2.24 and 2.25. Therefore, v is in an optimal zero forcing set but not terminal by Theorems 2.18 and 2.21.

Example 5.6. Let G be this graph:



Then $\mathrm{Z}(G - v) = 5$ by [AIM 2008]. By Theorem 2.16, $\mathrm{Z}(G) \geq 4$ and moreover $\{w_2, w_3, w_5, w_6\}$ is a zero forcing set, so $\mathrm{Z}(G) = 4$. The graph $G - v$ is not a path, so $\mathrm{P}(G - v) \geq 2$ and $\{(w_1, w_2, w_3, w_4, w_5), (w_6, w_7, w_8, w_9, w_{10})\}$ is a path cover for $G - v$. Therefore, $\mathrm{P}(G - v) = 2$. By Theorem 2.17, and considering G is not a path, $2 \leq \mathrm{P}(G) \leq 3$. To show $\mathrm{P}(G) \neq 2$, attempt to cover G with two induced paths and consider w_5. If w_5 was in a path by itself, the other eight vertices cannot be covered with a single induced path, so w_5 has to be in a path with other vertices. Since the three neighbors of w_5 are all neighbors of each other, w_5 has to be an endpoint of an induced path. Consider which neighbor is in the path with w_5. If w_1 is with w_5, then w_2 and w_6 have to be in the other path, then v, w_3, and w_7 have to be with w_5 and w_1, then w_4 and w_8 have to be with w_2 and w_6, but $G[\{w_2, w_4, w_6, w_8\}]$ is not a path. If w_2 is with w_5, then w_1 and w_6 have to be in the other path, then v has to be with w_5 and w_2, then w_3 has to be with w_1 and w_6, then w_7 has to be with w_5, w_2, and v, but $G[\{v, w_2, w_5, w_7\}]$ is not a path. If w_6 is with w_5, then w_1 and w_2 have to be in the other path, then v has to be with w_5 and w_6, then w_3 has to be with w_1 and w_2, then w_7 has to be with w_5, w_6, and v, but $G[\{v, w_5, w_6, w_7\}]$ is not a path. So $\mathrm{P}(G) \geq 3$. Hence $\mathrm{z}_v(G) = -1$ and $\mathrm{p}_v(G) = 1$. Hence, v is terminal but never in an optimal zero forcing set by Theorems 2.19 and 2.20.

Example 5.7. Let G be this graph:



Then $\mathrm{Z}(G) = \mathrm{M}(G)$ and $\mathrm{n}_v(G) = 0$, but v is not in an optimal zero forcing set for G.

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