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The rank gradient and the lamplighter group

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We introduce the notion of the rank gradient function of a descending chain of subgroups of finite index and show that the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ has uncountably many 2-chains (that is, chains in which each subsequent group has index 2 in the previous group) with pairwise different rank gradient functions. In doing so, we obtain some information on subgroups of finite index in the lamplighter group.

1. Introduction

The lamplighter group, by which we mean the wreath product of the group of order 2 with the infinite cyclic group, denoted $\mathscr{L} = \mathbb{Z}_2 \wr \mathbb{Z}$, is a popular object in group theory and its applications. Just two illustrations of this are Chapter 6 in [Meier 2008] and some select sections in [Lubotzky and Segal 2003]. It is a 2-step solvable group (i.e., metabelian) of exponential growth, infinitely presented and scale invariant [Grigorchuk and Żuk 2001; Nekrashevych and Pete 2011], which is the cornerstone in all known results about the range of L^2-Betti numbers of groups on compact manifolds. In particular, Atiyah's problem about the existence of closed manifolds with noninteger and even irrational L^2-Betti numbers was completely solved on a base of considerations related to $\mathscr{L}$ [Grigorchuk and Żuk 2001; Grabowski 2010; Grigorchuk et al. 2000].

Lackenby [2005] introduced an interesting group-theoretical notion, the rank gradient, which happens to be useful in topology, the theory of countable equivalence relations, the study of amenable groups and other areas. Given a group G and a descending sequence $\{H_n\}_{n=1}^{\infty}$ of subgroups of finite index one can define

$$RG(G, \{H_n\}) = \lim_{n\to\infty} \frac{d(H_n) - 1}{[G : H_n]}$$

to be the rank gradient of the sequence $\{H_n\}$ with respect to G where $d(H)$ denotes the minimal number of generators of a group H.

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Amenable groups were introduced by J. von Neumann in 1929 and play an important role in many areas of mathematics [Nekrashevych and Pete 2011]. There are a number of results due to Lackenby, M. Abért, A. Jaikin-Zapirain and N. Nikolov showing that amenability of G or of certain normal subgroups of G usually implies vanishing of the rank gradient. For instance, finitely generated infinite amenable groups have $RG = 0$ with respect to any normal chain with trivial intersection; see [Abért et al. 2011, Theorem 5].

It is reasonable to study the rank gradient for sequences $\{H_n\}$ with trivial core (i.e., no nontrivial normal subgroups in the intersection $\bigcap_n H_n$). Indeed,

$$RG(G, \{H_n\}) = RG(G/N, \{H_n/N\})$$

if $N \lhd G$, $N < \bigcap_n H_n$. The most attention is given to the case when $\bigcap_n H_n = \{1\}$. One of the remaining open questions is this:

Question 1.1 [Abért et al. 2011]. *Let G be a finitely generated infinite amenable group. Is it true that $RG(G, \{H_n\}) = 0$ for any chain with trivial intersection*?

If $\bigcap_{n=1}^{\infty} H_n = H$ then H is a closed subgroup with respect to the profinite topology and $RG(G, \{H_n\})$ is a characteristic of the pair (G, H) which in some situations may characterize the pair (G, H) up to isomorphism. We say two pairs (G, H), (P, Q) are isomorphic if there is an isomorphism $\phi : G \to P$ such that $\phi(H) = Q$.

If $RG(G, \{H_n\}) = 0$ then one may be interested in the decay of the function of the natural argument $n \in \mathbb{N}$ given by

$$\mathrm{rg}(n) = \mathrm{rg}_{(G,\{H_n\})}(n) = \frac{d(H_n) - 1}{[G : H_n]}$$

which we call the *rank gradient function*. We may omit $(G, \{H_n\})$ if the group and chain in consideration are understood. Again, the rate of decay of $\mathrm{rg}(n)$ may be an invariant of the pair (G, H) and may characterize the way H lies in G as a subgroup. Note that the same subgroup can be obtained as the intersection of distinct chains: one can delete certain elements in H_n thereby allowing $\mathrm{rg}(n)$ to decay as fast as one would like and indeed this is not the only way to get different chains with the same intersection. Thus, we restrict our definition to the case when for some prime p, we have $[H_{n+1} : H_n] = p$ and in this case we say the chain is a *p-chain*. Our main result shows that $\mathrm{rg}(n)$ may be used to show that the lamplighter group contains 2-chains with distinct rates of decay of the rank gradient function.

Theorem 1.2. *The group $\mathcal{L}$ has uncountably many 2-chains with pairwise distinct rank gradient functions.*

This result is obtained by explicitly describing subgroups of index 2 in the "higher rank" lamplighter groups $\mathcal{L}_n = \mathbb{Z}_2^n \wr \mathbb{Z}$.

Theorem 1.3. *For any* 2*-chain* $\{H_n\}$ *in* $\mathcal{L}$ *each member* H_n *is isomorphic to* $\mathcal{L}_i = \mathbb{Z}_2^i \wr \mathbb{Z}$ *for some* $i \le n$.

This is a corollary of Theorem 2.1 below.

2. Subgroups of index 2 in $\mathcal{L}_n$

Let $\mathcal{L}_n = \mathbb{Z}_2^n \wr \mathbb{Z} = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2^n \rtimes \mathbb{Z}$ (by $\mathbb{Z}_2$ we mean the group of order 2 and the generator of $\mathbb{Z}$ acts by shifting in the direct sum) and let $\mathcal{A}_n = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2^n$ be the base group of $\mathcal{L}_n$. Observe that $\mathcal{L}_n$ is generated by the elements a_i, $i = 1, 2, \dots, n$ and t where t is a generator of the infinite multiplicative cyclic group which we nevertheless denote in the additive way $\mathbb{Z}$, and $a_i \in \mathcal{A}_n$, $i = 1, 2, \dots, n$ are elements given by an $n \times \infty$ matrix with all entries zero except one located in the i-th row and column at position 0 (we assume that the columns are enumerated by the elements of $\mathbb{Z}$). So $\mathcal{L}_n = \langle a_1, \dots, a_n, t\rangle$. We will use similar notation for generation in the remainder of the paper. Observe that if we identify elements of the base group $\mathcal{A}_n$ with two sided infinite (bi-infinite) sequences of columns of dimension n over $\mathbb{Z}_2$ then conjugation by t acts on them as a shift τ in the set of sequences. We will use this fact later.

Theorem 2.1. *Let* $H < \mathcal{L}_n$ *be a subgroup of index* 2. *Then either* $H \simeq \mathcal{L}_n$ *or* $H \simeq \mathcal{L}_{2n}$. *There are* $2^{n+1} - 2$ *subgroups of the first type and* 1 *subgroup of the second type.*

In the proof, we use the following well known result.

Lemma 2.2. *Let* $M = \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p \oplus \cdots$ *be a finite or infinite direct sum of cyclic groups* $\mathbb{Z}_p$ *with* p *a prime. Then every subgroup* $P < M$ *is a direct summand*: $M = P \oplus Q$ *for some* Q. (See [Kargapolov and Merzljakov 1977, Chapter 10].)

We will often interpret $\mathbb{Z}_p^n$ as a vector space of dimension n over the prime field $\mathbb{F}_p \simeq \mathbb{Z}_p$. Before we present a proof of Theorem 2.1, we will need the following lemma.

Lemma 2.3. *Let* $M = \mathbb{Z}_p^n$. *Every subgroup* $P < M$ *of index* p *has a unique "orthogonal" complement* $Q < M$ *such that* $M = P \oplus Q$. *The group* Q *is generated by the element* $\bar{a} = (a_1, \dots, a_n)$ *which is determined by* P. *Then* P *consists of elements* $\bar{x} = (x_1, \dots, x_n)$ *whose coordinates satisfy the "orthogonality" condition*

$$a_1 x_1 + \cdots + a_n x_n \equiv 0 \pmod{p}.$$

Proof. Let $[M : P] = p$. Consider the subgroup P as a subspace of the vector space $M = \mathbb{Z}_p^n$. Choose a basis of P consisting of elements $\bar{b}_1, \dots, \bar{b}_{n-1}$

$$\bar{b}_1 = (b_{1,1}, \dots, b_{1,n}), \quad \dots, \quad \bar{b}_{n-1} = (b_{n-1,1}, \dots, b_{n-1,n}),$$

with $b_{i,j} \in \mathbb{Z}_p$. Now define the $(n-1) \times n$ matrix $B = (b_{ij})$, which has rank $n-1$, and consider the system of equations

$$\begin{aligned} b_{1,1}x_1 + \cdots + b_{1,n}x_n &= 0 \\ &\vdots \\ b_{n-1,1}x_1 + \cdots + b_{n-1,n}x_n &= 0. \end{aligned}$$

This system has the nontrivial solution $\bar{a} = (a_1, \ldots, a_n)$ and every other solution is some constant multiple of $\bar{a}$. It is then easy to see that $M = P \oplus \langle \bar{a} \rangle$. It is also clear that given some $\bar{a} \in M$ with $\bar{a} \neq 0$, the set of solutions of $a_1x_1 + \cdots + a_nx_n \equiv 0 \pmod{p}$ yields a subgroup P of index p in M. $\square$

Although we do not this, observe that by using tools from linear algebra, the notion of orthogonal complement can be defined in a similar way as we did for a subgroup of index p in an elementary p-group of finite rank. We will use the notation $H^{\perp}$ to denote the orthogonal complement of a subgroup $H < M$ in M.

Corollary 2.4. *There is a bijection between subgroups of index p in $M = \mathbb{Z}_p^n$ and subgroups of order p given by*

$$H \to H^{\perp}.$$

We now restrict our attention to the case when $p = 2$.

Proof of Theorem 2.1. Observe that the abelianization $A := (\mathscr{L}_n)_{ab}$ is isomorphic to $\mathbb{Z}_2^n \times \mathbb{Z}$. Define $A^2 < A$ to be the subgroup generated by the squares of elements in A. Then, $A/A^2 \simeq \mathbb{Z}_2^{n+1} = \langle \bar{a}_1, \ldots, \bar{a}_n, \bar{t} \rangle$ where as before $\mathbb{Z} = \langle t \rangle$ denotes the multiplicative infinite cyclic group generated by t, and a bar over some generator, $\bar{a}_i$ or $\bar{t}$ for example, denotes that we are considering the element corresponding to a_i or t of $\mathscr{L}_n$ as an element of the quotient group $\mathscr{L}_n/[\mathscr{L}_n, \mathscr{L}_n]\mathscr{L}_n^2 \simeq \mathbb{Z}_2^{n+1}$. If we consider a_i as an $n \times \infty$ matrix, then it is of the form

$$\begin{pmatrix} \cdots & 0 & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & 0 & 1 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & \cdots \end{pmatrix},$$

where the 1 is in the i-th row and the 0-th column. Recall that each a_i is the i-th generator of $\mathscr{A}_n^0$, where we define

$$\mathscr{A}_n = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2^n = \bigoplus_{j \in \mathbb{Z}} \mathscr{A}_n^j.$$

The number of subgroups of index 2 in $\mathscr{L}_n$ is equal to the number of epimorphisms $\mathscr{L}_n \to \mathbb{Z}_2$ which is equal to the number of subgroups of index 2 in $\mathbb{Z}_2^{n+1}$ which is equal to $2^{n+1}-1$ since the kernel of any such epimorphism is an orthogonal complement to a subgroup of order 2 generated by some nonidentity element. We have a short exact sequence

$$1 \to \mathscr{A}_n \to \mathscr{L}_n \xrightarrow{\phi} \langle t \rangle \to 1$$

where ϕ is the natural projection onto $\mathbb{Z} = \langle t \rangle$. Let $H < \mathscr{L}_n$ be of index 2. Then H is normal in $\mathscr{L}_n$ and therefore shift invariant.

There are two cases: either $\phi[H] = \langle t^2 \rangle$ or $\phi[H] = \langle t \rangle$.

Case 1. Assume $\phi[H] = \langle t^2 \rangle$. In this case $H \cap \mathscr{A}_n = \mathscr{A}_n$, since otherwise we would have $[\mathscr{L}_n : H] \geq 4$ and there is only one subgroup H of index 2 in $\mathscr{L}_n$ with this property. Furthermore, $t^2 \in H$ and $H = \mathscr{A}_n \rtimes \langle t^2 \rangle$.

Let $D_0 < \mathscr{A}_n$, $D_0 \simeq \mathbb{Z}_2^{2n}$ be a subgroup of $n \times \infty$ matrices where the only nonzero entries belong to columns with position 0 and 1. Define $D_j = t^{-2j} D_0 t^{2j}$. Then notice $D_i \cap D_j = 0$ for $i \neq j$ and $\mathscr{A}_n = \bigoplus_{j \in \mathbb{Z}} D_j$. The element t^2 acts by conjugation on $\bigoplus_{j \in \mathbb{Z}} D_j$ as a one-step shift. This implies $H \simeq \mathscr{L}_{2n}$.

Case 2. Now we assume $\phi[H] = \langle t \rangle$. We have $2^{n+1}-2$ such subgroups H. In this case, $H \cap \mathscr{A}_n = P$ is a shift invariant subgroup of index 2 in $\mathscr{A}_n$. Because P is shift invariant, there must be some $x \in \mathscr{A}_n$ whose matrix representation has only one nonzero column, namely the column with position 0, such that $x \notin P$. Let $q \in \mathbb{Z}_2^n$ be the vector with coordinates the same as x. That is, we consider x as an $n \times 1$ vector and relabel it q for clarity. Then let Q^0 be the orthogonal complement to $\langle q \rangle$:

$$\mathscr{A}_n^0 = \langle q \rangle \oplus Q^0,$$

where as before we have $\mathscr{A}_n = \bigoplus_{i \in \mathbb{Z}} \mathscr{A}_n^i$. Note that we are considering Q^0 and $\langle q \rangle$ as subgroups of $\mathscr{A}_n^0$ and so Q^0 is a subgroup of H since otherwise we would have $[\mathscr{L} : H] \geq 4$. Define

$$Q = \bigoplus_{i \in \mathbb{Z}} Q^i, \quad \text{where } Q^i = t^{-i} Q^0 t^i.$$

Let $\mathscr{R} = \mathbb{Z}_2[t, t^{-1}]$ be the ring of Laurent polynomials in $\mathbb{Z}_2$. It is isomorphic to the group ring $\mathbb{Z}_2[\mathbb{Z}]$ where as before $\mathbb{Z}$ is the additive notation for the multiplicative infinite cyclic group generated by t. The group $\mathscr{A}_n$ can be converted into an $\mathscr{R}$-module M_n by agreeing that the generator t acts on $\mathscr{A}_n$ as the previously defined right-shifting element τ (remember that elements of $\mathscr{A}_n$ can be considered as bi-infinite sequences of columns representing the elements of $\mathbb{Z}_2^n$). Moreover, $\mathscr{A}_n$ is the additive group of this module, M_n is a free $\mathscr{R}$-module of rank n and is isomorphic to $\mathscr{R}^n$.

Observe that Q is a shift invariant subgroup of H. Because of Lemma 2.2 there is a subgroup $S < P$ such that the decomposition $P = Q \oplus S$ holds. Note that

S is also a shift invariant subgroup of P and therefore can be interpreted as an $\mathcal{R}$-module. Therefore P, Q and S can be considered as submodules of M_n and the decomposition of modules $P = Q \oplus S$ holds (we will not change the notation for P, Q, S when considering them as modules or vise versa since it will be clear by the context if we are considering these objects as abelian groups or as $\mathcal{R}$-modules).

We will need the following lemma. Any graduate level textbook in Algebra will contain the fact that a ring of polynomials with coefficients in some field is a principal ideal domain. The ring $\mathcal{R}$ is the localization of the polynomial ring in the multiplicative set consisting of the nonnegative powers of t [Reid 1988]. Many properties of the Laurent polynomial ring follow from the general properties of localization as well as the next one which is a well known fact. However, we were unable to find a suitable reference for this so we add a proof of it below.

Lemma 2.5. *The ring $\mathcal{R}$ is a principal ideal domain.*

Proof. Let I be an ideal in $\mathcal{R}$. Then $I \cap \mathbb{Z}_2[t]$ is an ideal in $\mathbb{Z}_2[t]$ and since the ring of polynomials over a field is a principal ideal domain, $I \cap \mathbb{Z}_2[t] = (f)$ for some $f \in \mathbb{Z}_2[t]$. Then $\mathcal{R}f \subset I$. For $h \in I$, $h = t^{-k}g$ for some $k \in \mathbb{N}$ and $g \in \mathbb{Z}_2[t]$. Thus $g \in I \cap \mathbb{Z}_2[t] = (f)$, and so $h = t^{-k}fa \in \mathcal{R}f$ for some a in $\mathcal{R}$. Therefore $\mathcal{R}f = I$. □

Since they are submodules of a finitely generated free module $M_n \simeq \mathcal{R}^n$ over a principal ideal domain $\mathcal{R}$, the modules P, Q and S are also free. As P is a subgroup of index 2 in $\mathcal{A}_n$, the module P is free of rank n, Q is free of rank $n-1$ and S is free of rank 1. Thus the $\mathcal{R}^n$-module P, when considered as a group generated by the additive group P and the element t which acts by conjugation on P as the shift element τ, becomes isomorphic to $\mathcal{R}^n \rtimes \mathbb{Z} \simeq \mathcal{L}_n$.

We have $2^{n+1}-2$ subgroups H which can be obtained in the second case. Indeed, there are $2^n - 1$ choices for the vector q and therefore the subgroup Q. And to each choice of Q we have two choices to construct H: either to assume that $t \in H$ or that $t \notin H$. In this way, we get $2(2^n - 1) = 2^{n+1} - 2$ subgroups corresponding to *Case 2*. This finishes the proof of the first theorem. □

3. Construction of chains

Since $\mathbb{Z}_2[x, x^{-1}]$ is a principal ideal domain by Lemma 2.5, a shift invariant subgroup T of $\mathcal{A}_1 = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$ corresponds to a principal ideal $\mathcal{I}$ such that

$$\mathbb{Z}_2[x, x^{-1}]/\mathcal{I} \simeq \mathbb{Z}_2$$

which is a field. This implies that $\mathcal{I}$ is a maximal ideal generated by some irreducible polynomial of degree 1. Thus, $\mathcal{I} = \langle f \rangle$ with $\deg(f) = 1$ so $f = x + 1$. The corresponding element of T is then $\xi = (\ldots, 0, 1, 1, 0, \ldots)$ where the 1's are in the 0 and 1 place respectively. Additionally, ξ is a generator of T as an $\mathcal{R}$-module. One then concludes that T consists of sequences

$$(\ldots, a_{-1}, a_0, a_1, \ldots),$$

where

$$\sum_n a_n \equiv 0 \pmod{2}. \tag{1}$$

This observation gives an effective way to construct a subgroup H of index 2 in $\mathcal{L}_n$ with $H \simeq \mathcal{L}_n$. Choose a basis E of $\mathbb{Z}_2^n$ and write elements of $\mathcal{A}_n$ as $n \times \infty$ matrices (where the columns are indexed by $\mathbb{Z}$ as usual) with respect to this basis at position $i \in \mathbb{Z}$. Then take a subgroup of $\mathcal{A}_n$ consisting of elements which satisfy the relation (1) in the first row. After this, choose $t \in H$ or $t \notin H$.

We know that $\mathcal{L}$ contains 3 subgroups of index 2, where 2 are isomorphic to $\mathcal{L}$ and the other is isomorphic to $\mathcal{L}_2$. Furthermore, $\mathcal{L}_2$ has 7 subgroups of index 2, where 1 is isomorphic to $\mathcal{L}_4$ and 6 are isomorphic to $\mathcal{L}_2$, etc. If we take a subgroup $H < \mathcal{L}$ of index 2^k obtained from $\mathcal{L}$ by taking a descending chain of subgroups of index 2 in the previous member of the chain then we have $H \simeq \mathcal{L}_{2^i}$ for some $i \leq k$. We can then take a subgroup of index 2 isomorphic to $\mathcal{L}_{2^i}$ (call this choice type 0) or to $\mathcal{L}_{2^{i+1}}$ (call this choice type 1). It is clear that $d(\mathcal{L}_n) = n + 1$ (obviously $\mathcal{L}_n$ is generated by $n + 1$ elements and the abelianization of $\mathcal{L}_n$ is $\mathbb{Z}_2^n \times \mathbb{Z}$ and is $(n + 1)$-generated). Now let $\omega \in \{0, 1\}^{\mathbb{N}}$ be a sequence. Then these two types of choices for subgroups of index 2 allow us to construct a chain $\{H_n^\omega\}$ such that the subgroup H_n^ω is obtained from the previous subgroup H_{n-1}^ω by looking at the n-th term in our sequence ω. That is, a 0 dictates we make a choice of type 0 and a 1 dictates we make a choice of type 1. It is clear that in such a way we obtain uncountably many different chains $\{H_n^\omega\}$ such that each of the functions $\mathrm{rg}^\omega(n)$ are distinct. This provides the proof of Theorem 1.2.

Remark. If $r^\omega = \lim_{n\to\infty} \mathrm{rg}^\omega(n) > 0$ then $r^\omega = 2^{-k}$ for some k and the rank gradient of the chain $\{H_n^\omega\}$ is positive where the number of 0's in the sequence ω is k. In this case, $H^\omega = \bigcap_n H_n^\omega$ contains a nontrivial normal subgroup. In all other cases the rank gradient of the 2-chain is 0.

4. Conclusion

It is clear that the same method used to construct uncountably many rank gradient functions of 2-chains in $\mathcal{L}$ allows one to construct uncountably many 2-chains with pairwise distinct types of decay of the rank gradient function. For instance, one can consider a family of functions $\delta_\alpha(n) = 2^{-n^\alpha}$ with $0 < \alpha < 1$ where to each such function we have a corresponding sequence ω with the property that the rank gradient function $\mathrm{rg}^\omega(n)$ is the best approximation of the function 2^{-n^α}. By "best approximation", we mean the following. Starting with any subgroup $H_1 \simeq \mathscr{L}_2$ of index 2 in $\mathscr{L}$ (which corresponds to the value $\omega_1 = 1$ of the sequence ω and the value $\mathrm{rg}^{\omega}(1) = 1 > \frac{1}{2} = \delta_\alpha(1)$), one can make a choice of type 0 until the rank gradient function becomes less than the value of the function $\delta_\alpha(n)$ for the corresponding value of the argument n. Then make the choice of type 1 until the rank gradient function becomes greater or equal to $\delta_\alpha(n)$ for the corresponding value of n. Then again make the choice of type 0, etc. By continuing this process, we construct a 2-chain that best approximates $\delta_\alpha(n)$. Since the rates of decay of the functions $\delta_\alpha(n)$ are clearly different for different values of α, the (rates of decay of the) corresponding rank gradient functions are also distinct.

Our study is the first step in understanding what types of decay of the rank gradient function may arise in the case of finitely generated residually finite amenable groups.

If $\{H_n\}_{n=1}^{\infty}$ is a descending chain of subgroups of finite index in a residually finite group G, then the intersection $H_* = \bigcap_{n=1}^{\infty} H_n$ is a subgroup of G closed with respect to the profinite topology and indeed any closed subgroup can be obtained in this way. The rank gradient function of the chain $\{H_n\}_{n=1}^{\infty}$ introduced by us may serve as a certain characteristic of the subgroup H_*. Right now it is unclear how $\mathrm{rg}(n)$ depends on the chain $\{H_n\}_{n=1}^{\infty}$ with fixed intersection H_*. Even in the case when $H_* = \{1\}$, it may be that different p-chains with trivial intersection have different rates of decay of $\mathrm{rg}(n)$ but we do not have any examples of this. Of course, it is reasonable to only consider chains with the property that H_{n+1} is a maximal subgroup in $\{H_n\}$. While we have considered the case of the lamplighter group, it will also be interesting to study the decay of the rank gradient function with respect to other amenable groups such as with respect to the 3-generated infinite torsion 2-group of intermediate growth constructed in [Grigorchuk 1980; 1984].

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