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Congruence properties of S-partition functions

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We study the function $p(S; n)$ that counts the number of partitions of n with elements in S, where S is a set of integers. Generalizing previous work of Kronholm, we find that given a positive integer m, the coefficients of the generating function of $p(S; n)$ are periodic modulo m, and we use this periodicity to obtain families of S-partition congruences. In particular, we obtain families of congruences between partition functions $p(S_1; n)$ and $p(S_2; n)$.

1. Introduction and statement of results

The *partition function* $p(n)$ is the number of nonincreasing sequences of positive integers that sum to n. Ramanujan proved the following congruences for $p(n)$:

$$p(5n+4) \equiv 0 \pmod{5},$$
$$p(7n+5) \equiv 0 \pmod{7},$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Let S be a finite set of positive integers. An *S-partition* of an integer n is any nonincreasing sequence of integers in S that sums to n. The S-partition function $p(S; n)$ counts the number of S-partitions of n. The generating function for $p(S; n)$ is

$$G(S; q) := \sum_{n=0}^{\infty} p(S; n) q^n = \frac{1}{\prod_{s \in S}(1-q^s)} \in \mathbb{Z}[[q]]. \tag{1-1}$$

Kronholm [2005; 2007] found elegant "Ramanujan-type" congruences for the partition function

$$p(n, m) = p(\{1, \dots, m\}; n-m).$$

In this paper we reinterpret his idea of periodicity and we generalize it in the context of sets of positive integers. We first show that the coefficients of the generating function above are periodic modulo m.

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Theorem 1.1. *For a finite set of positive integers S and a positive integer m, there exists a positive integer $\gamma_m(S)$ such that for every integer n and all nonnegative k, we have*

$$p(S; n) \equiv p(S; n + k\gamma_m(S)) \pmod{m}. \tag{1-2}$$

Example. This theorem immediately implies many Ramanujan-type congruences. For example, if $S = \{1, 2, 3, 5\}$, one easily verifies that $\gamma_7(S) = 210$. Therefore, the fact that $p(S; 20) = 91 \equiv 0 \pmod{7}$ gives the Ramanujan congruence

$$p(S; 210n + 20) \equiv 0 \pmod{7}.$$

Example. This theorem is analogous to Theorem 2 of [Kronholm 2007]. As Kronholm states, let d be a multiple of $\operatorname{lcm}\{1, \dots, t\}$ and for the odd prime m, let m^α be a primary factor of d. Kronholm shows that if we let $\gamma_m(\{1, \dots, t\}) := d$, the congruences (1-2) hold. In particular, he proves that if

$$\sum_{\delta \geq 0} m^\delta \left(\left\lfloor \frac{t}{m^\delta} \right\rfloor - \left\lfloor \frac{t}{m^{\delta+1}} \right\rfloor \right) \leq m^\alpha, \tag{1-3}$$

then for $n \geq d - \sum_{j=2}^{t-1} j$, we have

$$p(\{1, \dots, t\}; n - t) \equiv p(\{1, \dots, t\}; n - t - d) \pmod{m}. \tag{1-4}$$

Example. Theorem 1.1 extends Theorem 2 of [Kronholm 2007] in that m does not have to be an odd prime. Let $S := \{2, 3, 11\}$ and let $m := 12$. Given this choice of S, it is clear that $p(S; 1) = 0$. We find that $\gamma_{12}(S) = 792$. By our theorem, for all positive k, $p(S; 1 + 792k) \equiv 0 \pmod{12}$.

For convenience, for sets S we let

$$\Phi_S(q) := \prod_{s \in S} (1 - q^s). \tag{1-5}$$

Corollary 1.2. *Let S_1 and S_2 be finite sets of positive integers and let m be a positive integer. If $\Phi_{S_1}(q)$ divides $\Phi_{S_2}(q)$ in $(\mathbb{Z}/m\mathbb{Z})[q]$, then for any nonnegative integer d, let*

$$X(q) := q^{d\gamma_m(S_1)} \frac{\Phi_{S_2}(q)}{\Phi_{S_1}(q)},$$

where

$$q^{d\gamma_m(S_1)} \frac{\Phi_{S_2}(q)}{\Phi_{S_1}(q)} :\equiv \sum_{i=0}^{c} a_i q^i \pmod{m}.$$

For $n \geq c$ and for any nonnegative k_1 and k_2, we have

$$p(S_1; n + k_1\gamma_m(S_1)) \equiv \sum_{i=0}^{c} a_i\, p(S_2; n - i + k_2\gamma_m(S_2)) \pmod{m}. \tag{1-6}$$

Note that Corollary 1.2 applies for all m when $S_1 \subseteq S_2$.

Example. Let $S_1 := \{1, 17\}$ and $S_2 := \{17, 289\}$ and let $m := 17$. Let $X(q) := \Phi_{S_2}(q)/\Phi_{S_1}(q)$. Then clearly $X(q) = \sum_{i=0}^{288} q^i$. For all nonnegative k_1 and k_2 and for all $n \geq 288$,

$$p(S_1; n + 289k_1) \equiv \sum_{i=0}^{288} p(S_2; n - i + 4913k_2) \pmod{17}.$$

2. Proof of Theorem 1.1

For convenience, we let $S := \{s_1, s_2, \ldots, s_t\}$ and let $d_S = \sum_{i=1}^{t} s_i$. Let

$$\Phi_S(q) = \prod_{s \in S}(1 - q^s) := \sum_{n=0}^{d_s} b(S; n)q^n.$$

From the identity

$$1 = \Phi_S(q)G(S; q) = \left(\sum_{n=0}^{d_s} b(S; n)q^n\right)\left(\sum_{n=0}^{\infty} p(S; n)q^n\right),$$

we have

$$1 = \sum_{i \geq 0} b(S; 0)p(S; i)q^i + \sum_{i \geq 0} b(S; 1)p(S; i)q^{i+1} + \cdots + \sum_{i \geq 0} b(S; d_S)p(S; i)q^{i+d_S}.$$

Looking at coefficients of q^N for $N \geq 1$, we observe that

$$\sum_{n=0}^{d_S} b(S; n)p(S; N - n) = 0. \tag{2-1}$$

This defines a linear recurrence relation. Noting that $b(S; 0) = 1$, we have

$$p(S; N) = -\sum_{n=1}^{d_S} b(S; n)p(S; N - n).$$

We consider consecutive d_S-tuples of consecutive partition values. Arranging these tuples in order, we have

$$\big(p(S; 0), p(S; 1), \ldots, p(S; d_S - 1)\big),$$
$$\big(p(S; d_S), p(S; d_S + 1), \ldots, p(S; 2d_S - 1)\big),$$

and so on. By reducing modulo m, we will find a first pair of d_S-tuples that agrees modulo m. Indeed, the maximal possible number of different tuples is m^{d_S}. Suppose that the first tuple of this pair starts at $p(S; n_0)$, and the second tuple starts at $p(S; n_1)$. Since by the linear recurrence relation, each tuple determines the next tuple, we have inductively that for all nonnegative i,

$$p(S; n_0 + i) \equiv p(S; n_1 + i) \pmod{m}.$$

We will first show that the residue classes of each tuple after the first determines the preceding tuple's residue classes. For any $a_0 = vd_S$, with $v \geq 1$, we consider the tuple

$$\big(p(S; a_0), \dots, p(S; a_0 + d_S - 1)\big).$$

By (2-1), and noting that $b(S; d_S) = (-1)^t$, we have

$$(-1)^{t+1} p(S; N - d_S) = \sum_{n=0}^{d_S-1} b(S; n) p(S; N - n).$$

It follows immediately that

$$\begin{aligned}
(-1)^{t+1} p(S; a_0 - 1) &\equiv \sum_{i=0}^{d_S-1} b(S; i) p(S; a_0 + d_S - 1 - i) \pmod{m},\\
(-1)^{t+1} p(S; a_0 - 2) &\equiv \left(\sum_{i=0}^{d_S-2} b(S; i) p(S; a_0 + d_S - 2 - i) \right)\\
&\qquad\qquad + b(S; d_S - 1) p(S; a_0 - 1) \pmod{m},\\
&\ \ \vdots\\
(-1)^{t+1} p(S; a_0 - d_S) &\equiv b(S; 0) p(S; a_0) + \sum_{i=1}^{d_S-1} b(S; i) p(S; a_0 - i) \pmod{m}.
\end{aligned}$$

Therefore, the residue classes of $\big(p(S; a_0), \dots, p(S; a_0 + d_S - 1)\big)$ reduced modulo m uniquely determine the residue classes of $\big(p(S; a_0 - d_S), \dots, p(S; a_0 - 1)\big)$ reduced modulo m.

To complete the proof, we must show that $n_0 = 0$. By hypothesis,

$$\big(p(S; n_0), \dots, p(S; d_S - 1)\big) \equiv \big(p(S; n_1), \dots, p(S; n_1 + d_S - 1)\big) \pmod{m}.$$

Suppose $n_0 = vd_S$ where $v \geq 1$, (i.e., $n_0 \neq 0$). Then, by the argument above, we have

$$\big(p(S; n_0 - d_S), \dots, p(S; n_0 - 1)\big) \equiv \big(p(S; n_1 - d_S), \dots, p(S; n_1 - 1)\big) \pmod{m}.$$

This result contradicts our hypothesis that the first-repeated tuple started for the first time at $p(S; n_0)$. Therefore, we can conclude that $n_0 = 0$, and so we let $\gamma_m(S) := n_1$. In particular, for any nonnegative k, we have (1-2).

3. Proof of Corollary 1.2

By Theorem 1.1, for any nonnegative k_1, we have

$$p(S_1; n + k_1\gamma_m(S_1)) \equiv p(S_1; n) \pmod{m}.$$

Clearly, for any nonnegative k_2, we have

$$\sum_{i=0}^{c} a_i\, p(S_2; n - i + k_2\gamma_m(S_2)) \equiv \sum_{i=0}^{c} a_i\, p(S_2; n - i) \pmod{m}.$$

Thus, subtracting two congruences, we have (1-6):

$$p(S_1; n + k_1\gamma_m(S_1)) - \sum_{i=0}^{c} a_i\, p(S_2; n - i + k_2\gamma_m(S_2)) \equiv p(S_1; n) - \sum_{i=0}^{c} a_i\, p(S_2; n - i) \pmod{m}. \quad (3\text{-}1)$$

Since $\dfrac{\Phi_{S_2}(q)}{\Phi_{S_1}(q)} G(S_2; q) = G(S_1; q)$, we know

$$G(S_1; q) - X(q)G(S_2; q) \equiv G(S_1; q) - q^{d\gamma_m(S_1)} G(S_1; q) \pmod{m}. \quad (3\text{-}2)$$

By comparing coefficients in (3-2), we have, for $n \geq d\gamma_m(S_1)$,

$$p(S_1; n) - \sum_{i=0}^{c} a_i\, p(S_2; n - i) \equiv 0 \pmod{m}. \quad (3\text{-}3)$$

Thus by (3-1), if $n \geq c$, then for any nonnegative k_1 and k_2, since $c \geq d\gamma_m(S_1)$, we have

$$p(S_1; n + k_1\gamma_m(S_1)) \equiv \sum_{i=0}^{c} a_i\, p(S_2; n - i + k_2\gamma_m(S_2)) \pmod{m}.$$

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