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Elliptic curves, eta-quotients and hypergeometric functions

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The well-known fact that all elliptic curves are modular, proven by Wiles, Taylor, Breuil, Conrad and Diamond, leaves open the question whether there exists a nice representation of the modular form associated to each elliptic curve. Here we provide explicit representations of the modular forms associated to certain Legendre form elliptic curves ${}_2E_1(\lambda)$ as linear combinations of quotients of Dedekind's eta-function. We also give congruences for some of the modular forms' coefficients in terms of Gaussian hypergeometric functions.

1. Introduction and statement of results

Wiles and Taylor [1995] proved that all semistable elliptic curves over $\mathbb{Q}$ are modular. Their result was later extended by Breuil, Conrad, Diamond and Taylor [Breuil et al. 2001] to all elliptic curves over $\mathbb{Q}$.

This correspondence allows facts about elliptic curves to be proven using modular forms, and vice versa. (See [Koblitz 1993] for more background on the theory of elliptic curves and modular forms.)

Let E be an elliptic curve over $\mathbb{Q}$. If $q := e^{2\pi i z}$, $\mathrm{GF}(p)$ is the finite field with p elements, and $N(p)$ is the number of points on E over $\mathrm{GF}(p)$, then the modularity theorem implies that there exists a corresponding weight-2 newform $f(z) = \sum_{n=1}^{\infty} a(n)q^n$ such that if p is a prime of good reduction, then $a(p) = 1 + p - N(p)$.

For example, if $\eta(z)$ is Dedekind's eta-function,

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n),$$

then the elliptic curves $y^2 = x^3 + 1$ and $y^2 = x^3 - x$ have the corresponding modular forms $\eta(6z)^4$ and $\eta(4z)^2\eta(8z)^2$, respectively; see [Martin and Ono 1997].

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It is natural to ask which elliptic curves have corresponding modular forms that are quotients of eta-functions. Martin and Ono [1997] have answered this question by listing all such *eta-quotients*

$$f(z)=\prod_{\delta}\eta(\delta z)^{r_\delta}\quad(\delta, r_\delta\in\mathbb{Z})$$

which are weight-2 newforms, and they gave corresponding modular elliptic curves.

(For more on the theory of eta-quotients, see [Ono 2004, Section 1.4].)

We show, for certain values of $\lambda\in\mathbb{Q}\setminus\{0,1\}$, that the elliptic curves ${}_2E_1(\lambda)$ defined by

$${}_2E_1(\lambda): y^2=x(x-1)(x-\lambda) \tag{1-1}$$

correspond to modular forms which are linear combinations of eta-quotients.

Remark. The proof of Theorem 1.1 will make clear how one can generate many more such examples.

Let

$$f_\lambda(z):=\sum_{n=1}^{\infty}{}_2a_1(n;\lambda)q^n \tag{1-2}$$

be the weight-2 newform corresponding to the elliptic curve ${}_2E_1(\lambda)$. It will be convenient to express eta-quotients using the notation

$$\left[\prod_{\delta}\delta^{r_\delta}\right]:=\prod_{\delta}\eta(\delta z)^{r_\delta}. \tag{1-3}$$

For example, in place of $\dfrac{\eta(2z)^2\eta(4z)^2\eta(5z)\eta(40z)}{\eta(z)\eta(8z)}$ we write $[1^{-1}2^2 4^2 5^1 8^{-1} 40^1]$.

Theorem 1.1. *If* $\lambda\in\left\{\frac{27}{16}, 5, \frac{81}{49}, -\frac{7}{25}\right\}$, *then* ${}_2E_1(\lambda)$ *corresponds to the modular forms given here*:

λ	conductor N	eta-quotient $f_\lambda(z)$
$\frac{27}{16}$	33	$[1^2 11^2]+3\cdot[3^2 33^2]+3\cdot[1^1 3^1 11^1 33^1]$
5	40	$[1^{-1}2^2 4^2 5^1 8^{-1} 40^1]+[1^1 5^{-1} 8^1 10^2 20^2 40^{-1}]$
$\frac{81}{49}$	42	$2\cdot[1^{-1}2^2 3^1 7^2 14^{-1} 42^1]-3\cdot[3^1 6^1 21^1 42^1]$ $+[2^1 3^2 6^{-1} 7^1 21^{-1} 42^2]+[1^1 3^{-1} 6^2 14^1 21^2 42^{-1}]$
$-\frac{7}{25}$	70	$[1^{-1}2^2 5^2 7^{-1} 10^{-1} 14^2 35^2 70^{-1}]-[1^2 2^{-1} 5^{-1} 7^2 10^2 14^{-1} 35^{-1} 70^2]$

We show, for all $\lambda\in\mathbb{Q}\setminus\{0,1\}$, that the Fourier coefficients of all $f_\lambda(z)$ satisfy an interesting hypergeometric congruence. For a prime p and an integer n, define

$\operatorname{ord}_p(n)$ to be the power of p dividing n, and if $\alpha = \frac{a}{b} \in \mathbb{Q}$, then set $\operatorname{ord}_p(\alpha) = \operatorname{ord}_p(a) - \operatorname{ord}_p(b)$. We show that with this notation, the numbers ${}_2a_1(p; \lambda)$ satisfy the following congruences.

Theorem 1.2. *Let $\lambda \notin \{0, 1\}$ be rational and let $p = 2f + 1$ be an odd prime such that* $\operatorname{ord}_p(\lambda(\lambda - 1)) = 0$. *Then*

$${}_2a_1(p; \lambda) \equiv (-1)^{\frac{p+1}{2}}(p-1)\sum_{k=0}^{f}\binom{f+k}{k}\binom{f}{k}(-\lambda)^k \pmod{p}.$$

Remarks. In light of Theorem 1.1, this implies that the congruence in Theorem 1.2 holds for the coefficients of the linear combinations of eta-quotients given above.

- A well-known theorem of Hasse states that for every prime p,

$$|a(p)| < 2\sqrt{p}.$$

Theorem 1.2 therefore determines ${}_2a_1(p; \lambda)$ uniquely for primes $p > 16$.

Example. Consider $\lambda = \frac{27}{16}$. Then $\lambda(\lambda - 1) = \frac{3^3 \cdot 11}{2^8}$ and so for $p \notin \{2, 3, 11\}$ prime we observe the congruence by inspecting the coefficients of ${}_2E_1\left(\frac{27}{16}\right)$ for applicable primes $p < 30$, where $B(p; \lambda)$ is defined to be the right-hand side of the congruence in Theorem 1.2:

p	${}_2a_1\left(p; \frac{27}{16}\right)$	$B\left(p; \frac{27}{16}\right)$
5	$-2 \equiv 3 \pmod{5}$	3
7	$4 \equiv 4 \pmod{7}$	4
13	$-2 \equiv 11 \pmod{13}$	11
17	$-2 \equiv 15 \pmod{17}$	15
19	$0 \equiv 0 \pmod{19}$	0
23	$8 \equiv 8 \pmod{23}$	8
29	$-6 \equiv 23 \pmod{29}$	23

2. Elliptic curves and modular forms

In this section we prove Theorem 1.1. If E is an elliptic curve over $\mathbb{Q}$, then its conductor N is a product of the primes p of bad reduction for E, with exponents determined by the extent to which E is singular over $\mathrm{GF}(p)$. (An algorithm by Tate for computing conductors is given in [Cremona 1997].) Moreover, the modularity theorem implies that the modular form $f(z)$ corresponding to E is an element of $S_2(\Gamma_0(N))$. In particular, for an elliptic curve ${}_2E_1(\lambda)$, proving the correctness of any representation of $f_\lambda(z)$ in terms of eta-quotients amounts to checking that the given eta-quotients are elements of $S_2(\Gamma_0(N))$ and checking a finite number of coefficients of their Fourier expansions against those of f_λ.

We first provide a formula for the dimension of the space of cusp forms of weight 2 and level N, $S_2(\Gamma_0(N))$. We then show that the eta-quotients making up the linear combinations are elements of $S_2(\Gamma_0(N))$ and use the dimension formula to show that equality of two elements of $S_2(\Gamma_0(N))$ always depends only on some finite set of coefficients.

The linear combinations of eta-quotients in this paper were generated by the following algorithm:

(1) Given a rational number $\lambda \notin \{0, 1\}$, compute the conductor N of ${}_2E_1(\lambda)$. (The modular form corresponding to ${}_2E_1(\lambda)$ will be an element of $S_2(\Gamma_0(N))$.)
(2) Compute $\dim_{\mathbb{C}} S_2(\Gamma_0(N))$.
(3) Generate eta-quotients which are elements of $S_2(\Gamma_0(N))$.
(4) Attempt to construct a basis for $S_2(\Gamma_0(N))$ using these eta-quotients.

Of course, once one is armed with a basis of eta-quotients for $S_2(\Gamma_0(N))$, it is simple to express $f_\lambda(z)$ in terms of this basis.

Dimension of $S_2(\Gamma_0(N))$. It will be useful to know not only that $S_2(\Gamma_0(N))$ is finite-dimensional for every positive integer N, but also its exact dimension $d_N := \dim_{\mathbb{C}} S_2(\Gamma_0(N))$.

The following formula for d_N is a simplification of [Ono 2004, Theorem 1.34], which gives a formula for the quantity $\dim_{\mathbb{C}} S_k(\Gamma_0(N), \chi) - \dim_{\mathbb{C}} M_{2-k}(\Gamma_0(N), \chi)$, in the case where $k = 2$ and $\chi = \epsilon$ is the trivial character modulo N.

Proposition 2.1. *If N is a fixed positive integer and $r_p := \operatorname{ord}_p(N)$, define*

$$\lambda_p := \begin{cases} p^{\frac{r_p}{2}} + p^{\frac{r_p}{2}-1} & \text{if } r_p \equiv 0 \pmod 2, \\ 2p^{\frac{r_p-1}{2}} & \text{if } r_p \equiv 1 \pmod 2. \end{cases}$$

With this notation,

$$d_N = 1 + \frac{N}{12}\prod_{p|N}(1 + p^{-1}) - \frac{1}{2}\prod_{p|N}\lambda_p - \frac{1}{4}\sum_{\substack{x \pmod N \\ x^2+1\equiv 0 \pmod N}} 1 - \frac{1}{3}\sum_{\substack{x \pmod N \\ x^2+x+1\equiv 0 \pmod N}} 1.$$

Proof. This follows from [Ono 2004, Theorem 1.34], noting that the conductor of the trivial character is 1 and that $M_0(\Gamma_0(N), \epsilon)$ is the space of constant functions and hence has dimension 1. □

Proof of Theorem 1.1. Let N be the conductor of $E = {}_2E_1(\lambda)$ and let $d_N = \dim_{\mathbb{C}} S_2(\Gamma_0(N))$ as before. Conditions under which an eta-quotient is an element of $S_2(\Gamma_0(N)))$ are provided in [Ono 2004, Theorems 1.64 and 1.65]: If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient which vanishes at each cusp of $\Gamma_0(N)$, such that the pairs (δ, r_δ) satisfy $\sum_{\delta|N} r_\delta = 4$, $\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$, and $\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$, then $f(z) \in S_2(\Gamma_0(N))$. The order of vanishing of such an $f(z)$ at the cusp $\frac{c}{d}$ is given by [Ono 2004, Theorem 1.65] as

$$\frac{N}{24}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d})d\delta}. \tag{2-1}$$

It is straightforward to check that the formula above gives a positive order of vanishing for each eta-quotient at each cusp, that each eta-quotient satisfies the given congruence conditions, and that the r_δ of each eta-quotient sum to 4. These conditions guarantee that each eta-quotient appearing in the table above lies in $S_2(\Gamma_0(N))$.

The eta-quotients given for $\lambda = \frac{27}{16}$ form a basis for $S_2(\Gamma_0(33))$. Similarly, for $\lambda = 5$, the given eta-quotients along with $[2^2 10^2]$ form a basis; for $\lambda = \frac{81}{49}$ the given eta-quotients along with $[1^{-1}2^2 3^2 6^{-1} 7^{-1} 14^2 21^2 42^{-1}]$ form a basis; and for $\lambda = -\frac{7}{25}$ a complete basis is

$$\begin{aligned}\{&[5^2 7^2], [1^{-1}2^2 7^2 10^1 14^{-1} 35^1], [10^2 14^2],\\ &[1^2 2^{-1} 5^1 7^{-1} 14^2 70^1], [1^2 2^{-1} 5^{-1} 7^2 10^2 14^{-1} 35^{-1} 70^2],\\ &[1^1 5^1 7^1 35^1], [1^1 5^2 10^{-1} 14^1 35^{-1} 70^2], [5^1 10^1 35^1 70^1], [1^{-1} 2^2 5^1 7^1 35^{-1} 70^2]\}.\end{aligned}$$

To see this, let $g_{i,j}$ be the j-th Fourier coefficient of the i-th basis vector g_i and define $t_1 < \cdots < t_{d_N}$ to be the first ascending set of indices for which the vectors $\{(g_{i,t_j})_{j=1}^{d_N}\}_{i=1}^{d_N}$ are linearly independent. One can find such a sequence by direct computation of the Fourier coefficients and inspection of the matrices $[g_{i,t_j}]_{i,j=1}^{d_N}$ for various choices of small $t_1 < \cdots < t_{d_N}$.

Now let $v_i = (g_{i,t_1}, \dots, g_{i,t_{d_N}})$ and let $b_1, \dots, b_{d_N}$ be a basis for $S_2(\Gamma_0(N))$. If we have $h_1, h_2 \in S_2(\Gamma_0(N))$ with equal t_i-th coefficients, then these coefficients are zero in the difference $h_1 - h_2$. But $h_1 - h_2$ can be written as a linear combination $\sum c_i b_i$ of basis elements, for constants c_i. Hence $\sum c_i v_i = 0$ in $\mathbb{R}^{d_N}$, so by linear independence all $c_i = 0$, and thus $h_1 - h_2 = 0$. It therefore suffices to check that the coefficients of f_λ on $q^{t_1}, \dots, q^{t_{d_N}}$ match the coefficients that result from the linear combination of eta-quotients. □

Remark. In practice, these computations can be done using a computer algebra system such as SAGE.

Example. We show that the modular form corresponding to ${}_2E_1\big(\frac{27}{16}\big)$ is

$$g(z) := [1^2 11^2] + 3\cdot[3^2 33^2] + 3\cdot[1^1 3^1 11^1 33^1].$$

For convenience, let $G = \{[1^2 11^2], [3^2 33^2], [1^1 3^1 11^1 33^1]\}$ be the set of eta-quotients making up the linear combination $g(z)$. The conductor of ${}_2E_1\big(\frac{27}{16}\big)$ is 33 and so the corresponding modular form $f_{\frac{27}{16}}(z)$ is an element of $S_2(\Gamma_0(33))$.

To show that $g(z)$ is also an element of $S_2(\Gamma_0(33))$, it suffices to show that $G \subset S_2(\Gamma_0(33))$. Take $g_i(z) \in G$. By [Ono 2004, Theorem 1.64], $g_i(z)$ is a modular form of weight 2 for $\Gamma_0(33)$. By [Ono 2004, Theorem 1.65], $g_i(z)$ vanishes at all cusps of $\Gamma_0(33)$, and thus $g_i(z) \in S_2(\Gamma_0(33))$.

Since $\mathrm{ord}_3(33) = \mathrm{ord}_{11}(33) = 1$, we have $\lambda_3 = \lambda_{11} = 2$ and evaluation of the dimension formula in Proposition 2.1 gives

$$\begin{aligned}
&\dim_{\mathbb{C}} S_2(\Gamma_0(33)) \\
&\quad = 1 + \frac{33}{12}\prod_{p\mid 33}(1+p^{-1}) - \frac{1}{2}\prod_{p\mid 33}\lambda_p - \frac{1}{4}\sum_{\substack{x \pmod{33} \\ x^2+1\equiv 0 \pmod{33}}} 1 - \frac{1}{3}\sum_{\substack{x \pmod{33} \\ x^2+x+1\equiv 0 \pmod{33}}} 1 \\
&\quad = 1 + \tfrac{33}{12}\left(1+\tfrac{1}{3}\right)\left(1+\tfrac{1}{11}\right) - \tfrac{1}{2}(\lambda_3)(\lambda_{11}) - \tfrac{1}{4}(0) - \tfrac{1}{3}(0) \\
&\quad = 3.
\end{aligned}$$

It remains to show that G is a basis for $S_2(\Gamma_0(33))$. Any dependence relation satisfied by the elements of G would imply a dependence relation among their coefficients. It thus suffices to find a set of indices $t_1 < t_2 < t_3$ such that the 3×3 matrix formed by the t_i-th coefficients of these eta-quotients is nonsingular. For this particular λ, the first three coefficients suffice.

This implies that any two elements of $S_2(\Gamma_0(33))$ which agree on the first three coefficients are equal. In fact, we observe that the first three coefficients of the modular form corresponding to ${}_2E_1\left(\frac{27}{16}\right)$ are the same as the first three coefficients of $g(z)$. That is, the coefficients of $g(z) = q + q^2 - q^3 - q^4 + \cdots$ agree with the coefficients of $f_{\frac{27}{16}}(z)$.

3. Gaussian hypergeometric functions and proof of Theorem 1.2

We recall some facts about Gaussian hypergeometric functions over finite fields of prime order and use the Gaussian hypergeometric function ${}_2F_1\left(\begin{smallmatrix}\phi, & \phi \\ & \epsilon\end{smallmatrix} \mid \lambda\right)$ to prove Theorem 1.2.

Gaussian hypergeometric functions. Greene [1987] defined *Gaussian hypergeometric functions* over arbitrary finite fields and showed that they have properties analogous to those of classical hypergeometric functions. We recall some definitions and notation from [Ono 1998] in the case of fields of prime order.

Definition 3.1. If p is an odd prime, $\mathrm{GF}(p)$ is the field with p elements, and A and B are characters of $\mathrm{GF}(p)$, define

$$\binom{A}{B} := \frac{B(-1)}{p} J(A, \bar{B}) = \frac{B(-1)}{p} \sum_{x \in \mathrm{GF}(p)} A(x)\bar{B}(1-x).$$

Furthermore, if $A_0, \dots, A_n$ and $B_1, \dots, B_n$ are characters of $\mathrm{GF}(p)$, define the Gaussian hypergeometric series ${}_{n+1}F_n\left(\begin{smallmatrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{smallmatrix} \mid x\right)$ by the following sum over all characters χ of $\mathrm{GF}(p)$:

$${}_{n+1}F_n\left(\begin{smallmatrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{smallmatrix} \mid x\right) := \frac{p}{p-1} \sum_{\chi} \binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi} \cdots \binom{A_n\chi}{B_n\chi}\chi(x)$$

In particular, we are concerned with the Gaussian hypergeometric series ${}_2F_1(\lambda)$ defined by

$${}_2F_1(\lambda) := {}_2F_1\left(\begin{smallmatrix} \phi, & \phi \\ & \epsilon \end{smallmatrix} \mid \lambda\right) = \frac{p}{p-1} \sum_{\chi} \binom{\phi\chi}{\chi}^2 \chi(\lambda)$$

where ϕ is the quadratic character of $\mathrm{GF}(p)$. It is shown in [Ono 1998] that if $\lambda \in \mathbb{Q} \setminus \{0, 1\}$, then

$${}_2F_1(\lambda) = -\frac{\phi(-1)\,{}_2a_1(p; \lambda)}{p} \tag{3-1}$$

for every odd prime p such that $\mathrm{ord}_p(\lambda(\lambda - 1)) = 0$.

In addition, define the generalized Apéry number $D(n; m, l, r)$ for every $r \in \mathbb{Q}$ and every pair of nonnegative integers m and l by

$$D(n; m, l, r) := \sum_{k=0}^{n} \binom{n+k}{k}^m \binom{n}{k}^l r^{lk}.$$

Ono also shows (ibid.) that if $p = 2f + 1$ is an odd prime and $w = l + m$, then

$$D(f; m, l, r) \equiv \left(\frac{p}{p-1}\right)^{w-1} {}_wF_{w-1}\left(\begin{smallmatrix} \phi, & \phi, & \dots, & \phi \\ & \epsilon, & \dots, & \epsilon \end{smallmatrix} \mid (-r)^l\right) \pmod{p}. \tag{3-2}$$

Proof of Theorem 1.2. By (3-1) and the fact that $\phi(-1) = (-1)^{\frac{p-1}{2}}$, we have that

$$\frac{p}{p-1}\,{}_2F_1(\lambda) = \frac{(-1)^{\frac{p+1}{2}}\,{}_2a_1(p; \lambda)}{p-1}.$$

By (3-2), letting $l = m = 1$ (and thus $w = 2$) and $r = -\lambda$, we have

$$\frac{p}{p-1}\,{}_2F_1(\lambda) \equiv D\,(f; 1, 1, -\lambda) \pmod{p}.$$

Combining these two equations and rearranging, we get

$${}_2a_1(p; \lambda) \equiv (-1)^{\frac{p+1}{2}}(p-1)D\,(f; 1, 1, -\lambda) \pmod{p}.$$

Since

$$D\,(f; 1, 1, -\lambda) = \sum_{k=0}^{n} \binom{f+k}{k}\binom{f}{k}(-\lambda)^k,$$

we have

$$ {}_2a_1(p;\lambda) \equiv (-1)^{\frac{p+1}{2}}(p-1)\sum_{k=0}^{f}\binom{f+k}{k}\binom{f}{k}(-\lambda)^k \pmod{p}. \qquad \square $$

Remark. The binomial product $\binom{f+k}{k}\binom{f}{k}$ can be combined into the multinomial coefficient $\binom{f+k}{k,\ k,\ f-k}$ and so the congruence in Theorem 1.2 can also be written as

$$ {}_2a_1(p;\lambda) \equiv (-1)^{\frac{p+1}{2}}(p-1)\sum_{k=0}^{f}\binom{f+k}{k,k,f-k}(-\lambda)^k \pmod{p}. $$

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