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A generalization of modular forms

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We prove a transformation equation satisfied by a set of holomorphic functions with rational Fourier coefficients of cardinality $2^{\aleph_0}$ arising from modular forms. This generalizes the classical transformation property satisfied by modular forms with rational coefficients, which only applies to a set of cardinality $\aleph_0$ for a given weight.

Modular forms play a crucial role in number theory, complex analysis, and geometry. However, from a set-theoretic point of view, the $\mathbb{Q}$-vector space $M_r(\Gamma)$ of holomorphic modular forms of a given weight r on $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ is only a small subset of the meromorphic functions of $q = e^{2\pi iz}$ on the open unit disc D centered at the origin of the complex plane with rational power series coefficients. This is because the set of all modular forms of a given weight r with rational Fourier coefficients is countable (has cardinality $\aleph_0$), as can be seen from the fact that the algebra of all modular forms on Γ over $\mathbb{Q}$ is finitely generated by modular forms with rational coefficients [Ono 2004]. In contrast, since every meromorphic function of $q = e^{2\pi iz}$ on the unit disc D with a pole having at most finite order at $q = 0$ can be represented as a power series of the form

$$g(z) = \sum_{n=-m}^{\infty} a(n)e^{2\pi inz} \tag{1}$$

uniformly convergent on compact subsets of D and conversely, it is clear that the cardinality of the set of meromorphic or holomorphic functions of $q = e^{2\pi iz}$ with rational power series coefficients is $2^{\aleph_0}$. We discuss this in more detail in the proof of Corollary 4 and Proposition 5.

Since modular forms are only a small subset of the set of all meromorphic functions, it is interesting to ask whether or not it is possible to generalize the definition of modularity so as to encompass a set of functions with cardinality $2^{\aleph_0}$, while still

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preserving some of the remarkable transformation properties of modular forms. This can be done by allowing the level of the modular form to become infinite.

To be specific, we consider sequences of elements of $\mathrm{SL}(2, \mathbb{Z})$, that is, integer matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $|ad - bc| = 1$, where the entries depend on a positive integer k, which will be suppressed from the notation. We will assume that c is an increasing (and therefore unbounded) function of k, and that the quotient d/c approaches a finite limit as $k \to \infty$. Note that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to the modular group $\Gamma_0(c)$ — by definition, $\Gamma_0(N)$ consists of the matrices in $\mathrm{SL}(2, \mathbb{Z})$ whose lower left entry is a multiple of N. We let $\mathrm{SL}(2, \mathbb{Z})$ act on the upper half-plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ in the usual way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}.$$

Let r be a positive integer. Let g be a meromorphic function on the upper half-plane with a pole of at most finite order at $z = i\infty$. Suppose there is a sequence $c = c(k)$ with the property that, for *any* sequence $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of the form above *consistent with this choice of* c, the function g satisfies the transformation equation

$$\Big(z + \lim_{k\to\infty} \frac{d}{c}\Big)^r g(z) = \lim_{k\to\infty} c^{-r} g\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\Big). \tag{2}$$

In that case we say that g is a *generalized modular form of weight* r, or a *modular form of weight* r *and level infinity*.

To see that this notion is a generalization of traditional modular forms, consider a modular form g of weight r and level N, and take for c the sequence given by $c(k) = Nk$. Any element of any sequence $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ consistent with this choice of c is an element of $\Gamma_0(N)$; therefore, by the definition of a modular form, g satisfies

$$(cz + d)^r g(z) = g\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\Big)$$

for all k. Dividing both sides by c^r and taking the limit as $k \to \infty$ we see that (2) is satisfied.

We will now see how to create uncountably many generalized modular forms with rational coefficients. We recall the definition of Dirichlet multiplication for two sequences $\{h(n)\}$ and $\{C(n)\}$:

$$(h * C)_n = \sum_{d|n} h(d) C\Big(\frac{n}{d}\Big)$$

We will assume $C(1) \neq 0$ in order to guarantee the existence of the Dirichlet inverse $\{C^{-1}(n)\}$, the inverse of the sequence $\{C(n)\}$ under the operation of Dirichlet multiplication. For efficient notation, we use $\{A_n\}$ and $\{A(n)\}$ interchangeably for any sequence $\{A_n\}$. Here is our main result.

Theorem 1. *Let*

$$\sum_{n=1}^{\infty} C(n)e^{2\pi inz}$$

be a cusp form of even weight $r > 0$ on Γ with $C(1) \neq 0$ and $\{|h(n)|\} \in \ell^1$ (i.e., $\sum_{n=1}^{\infty} h(n)$ is absolutely convergent). Then any holomorphic function on the upper half-plane of the form

$$g(z) = \sum_{n=1}^{\infty} (h * C)_n e^{2\pi inz} = \sum_{n=1}^{\infty} \sum_{d|n} h(d)C\Big(\frac{n}{d}\Big)e^{2\pi inz} \tag{3}$$

is a holomorphic generalized modular form of weight r and level infinity that satisfies the transformation equation

$$\Big(z + \lim_{c(k)\to\infty} \frac{d}{c}\Big)^r g(z) = \lim_{c(k)\to\infty} c^{-r} g\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\Big); \tag{4}$$

here $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(c)$ with $c(k) = \mathrm{lcm}(1, 2, 3, \dots, k)$. Thus $g(z)$ satisfies an approximate modular transformation equation, with its accuracy increasing as $c(k) \to \infty$. Here we define (4) *to be such an approximate modular transformation equation.*

This theorem generalizes the result

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^r f(z)$$

when $h(n)$ in (3) is the identity element of Dirichlet multiplication $I(n)$, since in this case $g(z)$ is a cusp form by definition:

$$g(z) = \sum_{n=1}^{\infty} (I * C)_n e^{2\pi inz} = \sum_{n=1}^{\infty} C(n)e^{2\pi inz},$$

$$\Big(z + \frac{d}{c}\Big)^r g(z) = c^{-r} g\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\Big).$$

Of course, in this case $\{|I(n)|\} \in \ell^1$ since $I(n) = 0$ for $n > 1$, and thus the hypotheses of Theorem 1 are satisfied. We also note that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)z$ approaches the real line as $c(k) \to \infty$, since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(c)$ implies

$$\frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)}, \qquad \lim_{c(k)\to\infty} \mathrm{Im}\, \frac{az+b}{cz+d} = 0.$$

Proof. We prove Theorem 1 using series of modular forms. In particular we use the cusp form of weight r on Γ given by

$$f(z) = \sum_{n=1}^{\infty} C(n)e^{2\pi inz},$$

where $\{C(n)\}$ is any cusp form coefficient sequence. It is well known that there exist functions that are analytic in the upper half-plane and satisfy the functional equation

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^r f(z),$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $ad - bc = 1$ (Γ being the modular group). From this property, it is easy to see that if n divides c, that is, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(n)$, then for positive integer n we have

$$f\Big(n\,\frac{az+b}{cz+d}\Big) = (cz+d)^r f(nz).$$

The Fourier expansion for $f(mz)$

$$f(mz) = \sum_{n=1}^{\infty} C(n) e^{2\pi imnz}$$

is absolutely convergent in the upper half-plane, since $C(n) = O(n^{r/2})$ by a standard argument of Hecke [Apostol 1990]. Assuming $A_m = O(m^p)$ for some natural number p we note that the double series

$$\sum_{m=1}^{\infty} A_m f(mz) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_m C_n e^{2\pi imnz}$$

is absolutely convergent, since both sequences A_m and C_n are bounded by polynomials, while of course $e^{2\pi imnz}$ decays exponentially in absolute value as m or n increases. Hence rearrangement is justified and we can write

$$\sum_{m=1}^{\infty} A_m f(mz) = \sum_{n=1}^{\infty}\sum_{d|n} A(d)C\Big(\frac{n}{d}\Big)e^{2\pi inz} = \sum_{n=1}^{\infty}(A * C)_n e^{2\pi inz}.$$

We also need the identity

$$e^{2\pi iz} = \sum_{m=1}^{\infty} C^{-1}(m) f(mz) \tag{5}$$

where $C^{-1}(m)$ is the Dirichlet inverse of the cusp form coefficients. Assuming absolute convergence, identity (5) follows easily from the following rearrangement:

$$\begin{aligned}\sum_{m=1}^{\infty} C^{-1}(m) f(mz) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C^{-1}(m)C(n)e^{2\pi imnz} \\ &= \sum_{n=1}^{\infty}(C^{-1} * C)_n e^{2\pi inz} = e^{2\pi iz}.\end{aligned}$$

To prove absolute convergence it is sufficient to prove that $C^{-1}(m)$ is bounded by a polynomial in m. This follows from the fact that $C(n) = O(n^{r/2})$ [Apostol 1990], together with the following lemma:

Lemma 2. *If a sequence $\{C(n)\} \subseteq \mathbb{C}$ with $C(1) \neq 0$ is bounded by a polynomial in n, then its Dirichlet inverse $C^{-1}(m)$ is also bounded by a polynomial in n. In symbols, if $|C(n)| = O(n^{d_1})$ for some $d_1 \in \mathbb{R}$, there exists $d_2 \in \mathbb{R}$ such that $|C^{-1}(n)| = O(n^{d_2})$.*

Proof. We prove this by induction. If $|C(n)| = O(n^{d_1})$, then letting $|C(1)| = P$, we find that there exists $k \in \mathbb{R}$ such that $|C(n)| \leq Pn^k$ for any positive integer n. We use the standard recursive definition

$$C^{-1}(n) = -\frac{1}{C(1)} \sum_{\substack{d|n \\ d<n}} C\left(\frac{n}{d}\right) C^{-1}(d), \tag{6}$$

which is equivalent to $(C * C^{-1})_n = I(n)$, where $I(n)$ is the identity element of Dirichlet multiplication. We find that $|C(1)| = P$ implies $|C^{-1}(1)| = 1/P$. We make the inductive hypothesis

$$|C^{-1}(d)| \leq \frac{1}{P} d^{k+2} \quad \text{for all } d < n,\ d \in \mathbb{N}.$$

Using the recursive definition (6) we obtain

$$|C^{-1}(n)| \leq \left|\frac{1}{C(1)}\right| \sum_{\substack{d\,|n \\ d<n}} \left|C\left(\frac{n}{d}\right)\right| |C^{-1}(d)| \leq \left|\frac{1}{C(1)}\right| \sum_{\substack{d\,|n \\ d<n}} \left(\frac{n}{d}\right)^k d^{k+2} \leq \frac{1}{P} n^k \sum_{\substack{d\,|n \\ d<n}} d^2.$$

So,

$$|C^{-1}(n)| \leq \frac{1}{P} n^k \sum_{\substack{d\,|n \\ d<n}} d^2 = \frac{1}{P} n^{k+2} \sum_{\substack{d|n \\ d>1}} \frac{1}{d^2} \leq \frac{1}{P} n^{k+2} (\zeta(2) - 1) \leq \frac{1}{P} n^{k+2},$$

where $\zeta(s)$ is the Riemann zeta function. It follows that

$$|C^{-1}(n)| \leq \frac{1}{P} n^{k+2},$$

and this completes the induction. $\square$

Any complex analytic function $J(q)$ can be written as a power series for q in the open unit disc D centered at $q = 0$:

$$J(q) = \sum_{n=0}^{\infty} A_n q^n.$$

Making the substitution $q = e^{2\pi i z}$ with $J(e^{2\pi i z}) = g(z)$ and assuming $J(0) = 0$

for convenience, we find

$$g(z) = \sum_{n=1}^{\infty} A_n e^{2\pi i n z}.$$

Using the absolute convergence of

$$e^{2\pi i n z} = \sum_{m=1}^{\infty} C^{-1}(m) f(mnz)$$

in the upper half-plane, and assuming A_n is bounded by a polynomial in n, we use rearrangement of series to write

$$\begin{aligned}\sum_{n=1}^{\infty}(A * C^{-1})_n f(nz) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(A * C^{-1})_n C_m e^{2\pi i m n z} \\ &= \sum_{n=1}^{\infty}(A * C^{-1} * C)_n e^{2\pi i n z} = \sum_{n=1}^{\infty} A_n e^{2\pi i n z} = g(z).\end{aligned}$$

This is justified by the discussion above and Lemma 2, which imply that all the series above are absolutely convergent. Now consider the partial sums of the series

$$g_k(z) = \sum_{n=1}^{k}(A * C^{-1})_n f(nz).$$

From this definition, assuming $z = x + iy$, we have

$$|g(z) - g_k(z)| = O(e^{-2\pi k y}). \tag{7}$$

This is because the cusp forms $f(nz)$ decay exponentially as n increases [Shimura 2007], so there exists $M \in \mathbb{R}^+$ such that $|f(nz)| < Me^{-2\pi n y}$ for all n. Hence, as $k \to \infty$ we have by the triangle inequality:

$$\begin{aligned}|g(z) - g_k(z)| &= \left|\sum_{n=k+1}^{\infty}(A * C^{-1})_n f(nz)\right| \\ &< Me^{-2\pi k y}\sum_{n=1}^{\infty}(A * C^{-1})_n e^{-2\pi n y} = O(e^{-2\pi k y}).\end{aligned}$$

From the functional equation $f\left(n\,\dfrac{az+b}{cz+d}\right) = (cz+d)^r f(nz)$, valid if $n|c$ and $ad - bc = 1$, we obtain

$$\left(z + \frac{d}{c}\right)^r g_k(z) = c^{-r}\sum_{n=1}^{k}(A * C^{-1})_n f\left(n\,\frac{az+b}{cz+d}\right), \quad ad - bc = 1,$$

by choosing $c(k) = \mathrm{lcm}[1, 2, 3, \ldots, k]$.

Given this c, we can always choose a, b, d such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(n)$ for all $n \leq k$ and with d/c approaching a finite limit as $k \to \infty$. For example, one can take $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$ or, more generally,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c\nu + 1 & -c\nu^2 \\ c & -c\nu + 1 \end{pmatrix}$$

for some integer ν. Hence, we can write

$$\left(z + \frac{d}{c}\right)^r g_k(z) = c^{-r} \sum_{n=1}^{k} (A * C^{-1})_n f\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right). \tag{8}$$

This approach, however, does not work for arbitrary holomorphic functions $f(z)$ since the error term

$$c^{-r} \sum_{n=k+1}^{\infty} (A * C^{-1})_n f\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)$$

diverges as k and c approach ∞. One way to circumvent this difficulty is to choose an sequence of real numbers $h(n)$ with $\{|h(n)|\} \in \ell^1$, and set

$$A_n = (h * C)_n, \quad \text{or, equivalently,} \quad (A * C^{-1})_n = h(n), \tag{9}$$

so that

$$g(z) = \sum_{n=1}^{\infty} (h * C)_n e^{2\pi i n z}. \tag{10}$$

In this case, A_n is bounded by a polynomial in n and the error term is

$$c^{-r} \sum_{n=k+1}^{\infty} h(n) f\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right).$$

Lemma 3. *Let* $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ *be a sequence as on page 16. As* $c \to \infty$, *we have*

$$\left| f\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) \right| < M n^{-r/2} \frac{|cz + d|^r}{(\operatorname{Im} z)^{r/2}},$$

where the constant M *does not depend on* n, a, b, d.

Proof. Since f is a cusp form of weight r, we have

$$|f(z)| (\operatorname{Im} z)^{r/2} < M \tag{11}$$

in the upper half-plane, for some bound $M > 0$. We sketch the proof; see [Apostol 1990] for details. Let $\varphi(z) = |f(z)| (\operatorname{Im} z)^{r/2}$ First, $\varphi(z) \to 0$ as $\operatorname{Im} z \to +\infty$, since f decays exponentially with $\operatorname{Im} z$, and therefore faster than any polynomial. By compactness, then, $\varphi(z)$ must be bounded in the fundamental region

$$\left\{ z : \operatorname{Im} z > 0,\ |z| \geq 1,\ \operatorname{Re} z \leq \tfrac{1}{2} \right\}$$

for the action of the modular group Γ on the upper half-plane. But φ is invariant under Γ (basically because $\operatorname{Im} z$ acts like the absolute value of a modular form of weight -2, so the weights cancel out). Thus the value of φ at any point z equals its value at some point in the fundamental domain, and is therefore bounded.

From (11) we can write

$$\left| f\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) \right| \left(\operatorname{Im}\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)\right)^{r/2} < M.$$

Since

$$\left(\operatorname{Im}\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)\right)^{r/2} = |cz+d|^{-r} n^{r/2} (\operatorname{Im} z)^{r/2},$$

we obtain the desired inequality. □

We recall that $r > 0$ for holomorphic cusp forms [Apostol 1990]. Thus, if $\{|h(n)|\} \in \ell^1$, the error term

$$c^{-r} \sum_{n=k+1}^{\infty} h(n) f\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)$$

is clearly absolutely convergent and approaches 0 as $k \to \infty$. Hence, from (7), (8), and (9), we have successively

$$\left(z + \frac{d}{c}\right)^r g_k(z) = c^{-r} \sum_{n=1}^{\infty} h(n) f\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) + O(\epsilon(k)), , \tag{12}$$

for some function $\epsilon(k)$ satisfying $\lim_{k\to\infty} \epsilon(k) = 0$. This leads to

$$\left(z + \frac{d}{c}\right)^r g(z) = c^{-r} \sum_{n=1}^{\infty} h(n) f\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) + O(\epsilon(k)) + O(e^{-2\pi k y}),$$

$$\left(z + \frac{d}{c}\right)^r g(z) = c^{-r} \sum_{n=1}^{\infty} h(n) f\left(n \begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) + O(\epsilon(k)).$$

From (12) we obtain, using the Fourier expansion $f(nz) = \sum_{m=1}^{\infty} C(m) e^{2\pi i m n z}$ and absolute convergence to justify rearrangements,

$$\begin{aligned} \left(z + \frac{d}{c}\right)^r g(z) &= c^{-r} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} h(n) C(m) e^{2\pi i m n \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z} + O(\epsilon(k)) \\ &= c^{-r} \sum_{n=1}^{\infty} (h * C)_n e^{2\pi i n \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z} + O(\epsilon(k)). \end{aligned}$$

From (10) we have

$$\left(z + \lim_{c(k)\to\infty} \frac{d}{c}\right)^r g(z) = \lim_{c(k)\to\infty} c^{-r} g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right),$$

with $c(k) = \operatorname{lcm}(1, 2, 3, \ldots, k)$, which completes the proof of Theorem 1. We note that $g(z)$ is holomorphic in the upper half-plane since $|h(n)|\} \in \ell^1$ and $C(n) = O(n^{r/2})$ result in uniform convergence of the series (10) on compact subsets. $\square$

Corollary 4. *If there exists a cusp form of even weight r over $\mathbb{Q}$ with $C(1) \neq 0$, then the set G of generalized modular forms of weight r and level infinity with rational coefficients has cardinality $2^{\aleph_0}$:*

$$|G| = 2^{\aleph_0}.$$

Proof. This follows from Theorem 1, which implies that, for all $\{h(n)\}$ such that $\{|h(n)|\} \in \ell^1$,

$$g(z) = \sum_{n=1}^{\infty} (h * C)_n e^{2\pi i n z}$$

is a generalized modular form of weight r over $\mathbb{Q}$, assuming that $\{C(n)\}$ is the rational Fourier coefficient sequence of a weight r cusp form with $C(1) \neq 0$.

Now let

$$A = (\mathbb{Q}[0, 1])^{\mathbb{N}} = \{(a : \mathbb{N} \to \mathbb{Q}[0, 1])\}$$

be the set of sequences $\{a(n)\}$ with $a(n) \in \mathbb{Q}[0, 1]$ for $n \in \mathbb{N}$. We recall from set theory that $|\mathbb{Q}[0, 1]| = \aleph_0$ and $|(\mathbb{Q}[0, 1])^{\mathbb{N}}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$ [Jech 1997]. Further, let

$$B = \{\{h(n)\} \in A : \{|h(n)|\} \in \ell^1\}$$

be the subset of A consisting of sequences whose sum converges absolutely. We know that $\{a(n)\} \in A$ implies $\{a(n)/n^2\} \in B$, since $|a(n)| \leq 1$ and by the comparison test for series and the absolute convergence of $\sum_{n=1}^{\infty} 1/n^2$. Thus the mapping $\{a(n)\} \to \{a(n)/n^2\}$ defines an injection $\beta : A \to B$.

Next, Theorem 1 implies that there exists an injection $\gamma : B \to G$, which sends a sequence $\{h(n)\} \in B$ to

$$g(z) = \sum_{n=1}^{\infty} (h * C)_n e^{2\pi i n z},$$

with $g(z) \in G$. The composite map $\gamma\beta : A \to G$ thus defines an injection from A to G, as long as $\sum_{n=1}^{\infty} C(n) e^{2\pi i n z}$ is a cusp form of weight r with $C(1) \neq 0$. Hence $|G| \geq 2^{\aleph_0} = |A|$.

At the same time, there is an injection from G into the set S of all formal power series of $q = e^{2\pi i z}$ over $\mathbb{Q}$. This set has the same cardinality as the set $\mathbb{Q}^{\mathbb{N}}$ of maps $\mathbb{N} \to \mathbb{Q}$. Hence $|S| = |\mathbb{Q}^{\mathbb{N}}| = \aleph_0^{\aleph_0} = 2^{\aleph_0}$. We conclude that $|G| \leq 2^{\aleph_0}$. Hence $|G| = 2^{\aleph_0}$. $\square$

We note that Corollary 4 holds for $r = 12$ and all even $r \geq 16$. This is because the standard $\Delta(z)$ function is a cusp form of weight 12 with $C(1) \neq 0$, and for even $r \geq 16$ an example of such a cusp form is $\Delta(z)E_{r-12}(z)$ with $E_{r-12}(z)$ an Eisenstein series.

Proposition 5. *$M_r(\Gamma)$ has cardinality $\aleph_0$ as a vector space over $\mathbb{Q}$.*

Proof. This follows from the result that every entire modular form $f \in M_r(\Gamma)$ is a polynomial of the form [Ono 2004]

$$f = \sum_{4a+6b=r} c_{a,b} G_4^a G_6^b,$$

where G_4 and G_6 are Eisenstein series with integer coefficients, $c_{a,b} \in \mathbb{C}$, and $a, b \in \mathbb{Z}^+$. If f has rational coefficients, then we conclude $c_{a,b} \in \mathbb{Q}$ since G_4 and G_6 have integer coefficients. Algebraically, this implies the following vector space isomorphism over $\mathbb{Q}$:

$$M_r(\Gamma) \cong \mathbb{Q}^{\dim M_r(\Gamma)}.$$

It is a well-known theorem in set theory that $\mathbb{Q}$ is countable, and in general the Cartesian products of any finite number of countable sets is countable [Jech 1997]. Thus, we conclude $M_r(\Gamma)$ over $\mathbb{Q}$ has cardinality $\aleph_0$. These results allow us to gauge the strength of Theorem 1, which generalizes the notion of modularity to encompass a much larger set of holomorphic functions than the classical entire modular forms. □

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