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Darius Bayegan and Megumi Harada

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A Giambelli formula for the S^1-equivariant cohomology of type A Peterson varieties

Darius Bayegan and Megumi Harada

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We prove a *Giambelli formula* for the Peterson Schubert classes in the S^1-equivariant cohomology ring of a type A Peterson variety. The proof uses the Monk formula for the equivariant structure constants for the Peterson Schubert classes derived by Harada and Tymoczko. In addition, we give proofs of two facts observed by H. Naruse: firstly, that some constants that appear in the multiplicative structure of the S^1-equivariant cohomology of Peterson varieties are Stirling numbers of the second kind, and secondly, that the Peterson Schubert classes satisfy a stability property in a sense analogous to the stability of the classical equivariant Schubert classes in the T-equivariant cohomology of the flag variety.

1. Introduction

The main result of this note is a formula of Giambelli type in the S^1-equivariant cohomology[1] of type A Peterson varieties. Specifically, we give an explicit formula that expresses an arbitrary *Peterson Schubert class* in terms of the degree-2 Peterson Schubert classes. We call this a "Giambelli formula" by analogy with the standard Giambelli formula in classical Schubert calculus [Fulton 1997], which expresses an arbitrary Schubert class in terms of degree-2 Schubert classes.

We briefly recall the setting of our results. *Peterson varieties* in type A can be defined as the following subvariety Y of $\mathcal{F}\ell ags(\mathbb{C}^n)$:

$$Y := \{V_\bullet = (0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n) \mid NV_i \subseteq V_{i+1} \text{ for all } i = 1, \dots, n-1\}, \quad \text{(1-1)}$$

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[1] All our cohomology rings are with coefficients in $\mathbb{C}$.

where $N : \mathbb{C}^n \to \mathbb{C}^n$ denotes the principal nilpotent operator. These varieties have been much studied due to their relation to the quantum cohomology of the flag variety [Kostant 1996; Rietsch 2003]. Thus it is natural to study their topology, including the structure of their (equivariant) cohomology rings. We do so through Schubert calculus techniques. Our results extend techniques initiated and developed in [Harada and Tymoczko 2010; 2011], to which we refer the reader for further details and motivation.

There is a natural circle subgroup of $U(n, \mathbb{C})$, recalled in Section 2, that acts on Y. The inclusion of Y into $\mathcal{F}\ell ags(\mathbb{C}^n)$ induces a natural ring homomorphism

$$H_T^*(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H_{S^1}^*(Y) \tag{1-2}$$

where T is the subgroup of diagonal matrices of $U(n, \mathbb{C})$ acting in the usual way on $\mathcal{F}\ell ags(\mathbb{C}^n)$. One of the main results of [Harada and Tymoczko 2011] is that a certain subset of the equivariant Schubert classes $\{\sigma_w\}_{w \in S_n}$ in $H_T^*(\mathcal{F}\ell ags(\mathbb{C}^n))$ maps under the projection (1-2) to a computationally convenient module basis of $H_{S^1}^*(Y)$. We refer to the images via (1-2) of $\{\sigma_w\}_{w \in S_n}$ in $H_{S^1}^*(Y)$ as *Peterson Schubert classes*. Theorem 6.12 of the same reference gives a manifestly positive *Monk formula* for the product of a degree-2 Peterson Schubert class with an arbitrary Peterson Schubert class, expressed as a $H_{S^1}^*(\text{pt})$-linear combination of Peterson Schubert classes. This is an example of equivariant Schubert calculus in the realm of Hessenberg varieties (of which Peterson varieties are a special case), and we view the Giambelli formula (Theorem 3.2) as a further development of this theory. The Giambelli formula for Peterson varieties was also independently observed by H. Naruse.

Our Giambelli formula also allows us to simplify the presentation of the ring $H_{S^1}^*(Y)$ given in [Harada and Tymoczko 2011, Section 6]. This is because the previous presentation used as its generators all of the elements in the module basis given by Peterson Schubert classes, although the ring $H_{S^1}^*(Y)$ is multiplicatively generated by only the degree-2 Peterson Schubert classes. Details are explained starting on page 123 below, where we also give a concrete example in $n = 4$ to illustrate our results. We also formulate a conjecture (cf. Remark 3.12), suggested to us by the referee of this manuscript, that the ideal of defining relations is in fact generated by the quadratic relations only. If true, this would be a significant further simplification of the presentation of this ring and would lead to interesting further questions (both combinatorial and geometric).

In Sections 4 and 5, we present proofs of two facts concerning Peterson Schubert classes, which we learned from H. Naruse. The results are due to Naruse but the proofs given here are our own. We chose to include these results because they do not appear elsewhere in the literature. The first fact is that *Stirling numbers of the second kind* (see Section 4 for the definition) appear naturally in the product structure of $H^*_{S^1}(Y)$. The second is that the Peterson Schubert classes satisfy a *stability condition* with respect to the natural inclusions of Peterson varieties induced from the inclusions $\mathscr{F}\ell ags(\mathbb{C}^n) \hookrightarrow \mathscr{F}\ell ags(\mathbb{C}^{n+1})$.

2. Peterson varieties and S^1-fixed points

In this section we briefly recall the objects under study. For details we refer the reader to [Harada and Tymoczko 2011]. Since we work exclusively in Lie type A we henceforth omit it from our terminology.

By the *flag variety* we mean the homogeneous space $\mathrm{GL}(n,\mathbb{C})/B$, where B is the standard Borel subgroup of upper-triangular invertible matrices. The flag variety can also be identified with the space of nested subspaces in $\mathbb{C}^n$, that is,

$$\begin{aligned}\mathscr{F}\ell ags(\mathbb{C}^n) &:= \{V_\bullet = (\{0\} \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V_n = \mathbb{C}^n) \mid \dim_{\mathbb{C}}(V_i) = i\}\\ &\cong \mathrm{GL}(n,\mathbb{C})/B.\end{aligned}$$

Let N be the $n \times n$ principal nilpotent operator given with respect to the standard basis of $\mathbb{C}^n$ as the matrix with one $n \times n$ Jordan block of eigenvalue 0, that is,

$$N = \begin{bmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ & & & & 0 & 0 \end{bmatrix}. \tag{2-1}$$

Fix n a positive integer. The main geometric object under study, the *Peterson variety* Y, is the subvariety of $\mathscr{F}\ell ags(\mathbb{C}^n)$ defined in (1-1) where N is the standard principal nilpotent in (2-1). The variety Y is a (singular) projective variety of complex dimension $n-1$.

We recall some facts from [Harada and Tymoczko 2011]. The following circle subgroup of $U(n,\mathbb{C})$ preserves Y:

$$S^1 = \left\{ \begin{bmatrix} t^n & 0 & \cdots & 0 \\ 0 & t^{n-1} & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & t \end{bmatrix} \;\middle|\; t \in \mathbb{C},\ \|t\| = 1 \right\} \subseteq T^n \subseteq U(n,\mathbb{C}). \tag{2-2}$$

Here T^n is the standard maximal torus of $U(n,\mathbb{C})$ consisting of diagonal unitary matrices. The S^1-fixed points of Y are isolated and are a subset of the T^n-fixed points of $\mathscr{F}\ell ags(\mathbb{C}^n)$. As is standard, we identify the T^n-fixed points in $\mathscr{F}\ell ags(\mathbb{C}^n)$ with the permutations S_n. In particular since Y^{S^1} is a subset of $\mathscr{F}\ell ags(\mathbb{C}^n)^{T^n}$, we think of the Peterson fixed points as permutations in S_n. There is a natural bijective correspondence between the Peterson fixed points Y^{S^1} and subsets of $\{1, 2, \ldots, n-1\}$ which we now briefly recall. It is explained in [Harada and Tymoczko 2011, Section 2.3] that a permutation $w \in S_n$ is in Y^{S^1} precisely when the one-line notation of w^{-1} is of the form

$$w^{-1} = \underbrace{j_1\ j_1 - 1\ \cdots\ 1}_{j_1 \text{ entries}}\ \underbrace{j_2\ j_2 - 1\ \cdots\ j_1 + 1}_{j_2 - j_1 \text{ entries}}\ \cdots\ \underbrace{n\ n - 1\ \cdots\ j_m + 1}_{n - j_m \text{ entries}} \tag{2-3}$$

where $1 \leq j_1 < j_2 < \cdots < j_m < n$ is any sequence of strictly increasing integers. For example, for $n = 9$, $m = 2$ and $j_1 = 3$, $j_2 = 7$, then the permutation w^{-1} in (2-3) has one-line notation 321765498. Thus for each permutation $w \in S_n$ satisfying (2-3) we define

$$\mathcal{A} := \{i : w^{-1}(i) = w^{-1}(i+1) + 1 \text{ for } 1 \leq i \leq n-1\} \subseteq \{1, 2, \ldots, n-1\}.$$

This gives a one-to-one correspondence between the power set of $\{1, 2, \ldots, n-1\}$ and Y^{S^1}. We denote the Peterson fixed point corresponding to a subset $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$ by $w_{\mathcal{A}}$.

Example 2.1. Let $n = 5$ and suppose $\mathcal{A} = \{1, 2, 4\}$. Then the associated permutation is $w_{\mathcal{A}} = 32154$.

Indeed, for a fixed n, we can also easily enumerate all the Peterson fixed points by using this correspondence.

Example 2.2. Let $n = 4$. Then Y^{S^1} consists of $2^3 = 8$ elements in correspondence with the subsets of $\{1, 2, 3\}$, namely: $w_{\varnothing} = 1234$, $w_{\{1\}} = 2134$, $w_{\{2\}} = 1324$, $w_{\{3\}} = 1243$, $w_{\{1,2\}} = 3214$, $w_{\{2,3\}} = 1432$, $w_{\{1,3\}} = 2143$, $w_{\{1,2,3\}} = 4321$.

Given a choice of subset $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$, there is a natural decomposition of $\mathcal{A}$ as follows. We say that a set of consecutive integers

$$\{a, a+1, \ldots, a+k\} \subseteq \mathcal{A}$$

is a *maximal consecutive* (*sub*)*string* of $\mathcal{A}$ if a and k are such that neither $a-1$ nor $a+k+1$ is in $\mathcal{A}$. For $a_1 := a$ and $a_2 := a_1 + k$, we denote the corresponding maximal consecutive substring by $[a_1, a_2]$. It is straightforward to see that any $\mathcal{A}$ uniquely decomposes into a disjoint union of maximal consecutive substrings

$$\mathcal{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m].$$

For instance, if $\mathcal{A} = \{1, 2, 3, 5, 6, 8\}$, then its decomposition into maximal consecutive substrings is $\{1, 2, 3\} \cup \{5, 6\} \cup \{8\} = [1, 3] \cup [5, 6] \cup [8, 8]$.

Suppose $\mathcal{A} = \{j_1 < j_2 < \cdots < j_m\}$. Finally we recall that we can associate to each $w_{\mathcal{A}}$ a permutation $v_{\mathcal{A}}$ by the recipe

$$w_{\mathcal{A}} \mapsto v_{\mathcal{A}} := s_{j_1} s_{j_2} \cdots s_{j_m} \tag{2-4}$$

where an s_i denotes the simple transposition $(i, i+1)$ in S_n.

3. The Giambelli formula for Peterson varieties

The Giambelli formula. In this section we prove the main result of this note, namely, a *Giambelli formula for Peterson varieties*.

As recalled above, the Peterson variety Y is an S^1-space for a subtorus S^1 of T^n and it can be checked that $Y^{S^1} = (\mathcal{F}\ell ags(\mathbb{C}^n))^{T^n} \cap Y$. There is a forgetful map from T^n-equivariant cohomology to S^1-equivariant cohomology obtained by the inclusion $S^1 \hookrightarrow T$, so there is a commutative diagram

$$\begin{tikzcd}
H^*_{T^n}(\mathcal{F}\ell ags(\mathbb{C}^n)) \arrow[r] \arrow[d] & H^*_{T^n}((\mathcal{F}\ell ags(\mathbb{C}^n))^{T^n}) \arrow[d] \\
H^*_{S^1}(\mathcal{F}\ell ags(\mathbb{C}^n)) \arrow[r] \arrow[d] & H^*_{S^1}((\mathcal{F}\ell ags(\mathbb{C}^n))^{T^n}) \arrow[d] \\
H^*_{S^1}(Y) \arrow[r] & H^*_{S^1}(Y^{S^1}).
\end{tikzcd} \tag{3-1}$$

The *equivariant Schubert classes* $\{\sigma_w\}$ in $H^*_{T^n}(\mathcal{F}\ell ags(\mathbb{C}^n))$ are well-known to form a $H^*_{T^n}(\mathrm{pt})$-module basis for $H^*_{T^n}(\mathcal{F}\ell ags(\mathbb{C}^n))$. We call the image of σ_w under the projection map $H^*_{T^n}(\mathcal{F}\ell ags(\mathbb{C}^n)) \to H^*_{S^1}(Y)$ the *Peterson Schubert class corresponding to w*. For the permutations $v_{\mathcal{A}}$ defined in (2-4), we denote by $p_{\mathcal{A}}$ the corresponding Peterson Schubert class, that is the image of $\sigma_{v_{\mathcal{A}}}$. (This is slightly different notation from that used in [Harada and Tymoczko 2011].) We denote by $p_{\mathcal{A}}(w) \in H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$ the restriction of the Peterson Schubert class $p_{\mathcal{A}}$ to the fixed point $w \in Y^{S^1}$.

One of the main results of [Harada and Tymoczko 2011] is that the set of 2^{n-1} Peterson Schubert classes $\{p_{\mathcal{A}}\}_{\mathcal{A} \subseteq \{1,2,\ldots,n-1\}}$ form a $H^*_{S^1}(\mathrm{pt})$-module basis for $H^*_{S^1}(Y)$ where $v_{\mathcal{A}}$ is defined in (2-4). (The fact that $H^*_{S^1}(Y)$ is a free module of rank 2^{n-1} over $H^*_{S^1}(\mathrm{pt})$ fits nicely with the result [Sommers and Tymoczko 2006, Theorem 10.2] that the Poincaré polynomial of Y is given by $(q^2+1)^{n-1}$.) It is also shown in [Harada and Tymoczko 2011] that the $n-1$ degree-2 classes $\{p_i := p_{s_i}\}_{i=1}^{n-1}$ form a multiplicative set of generators for $H^*_{S^1}(Y)$. These classes p_i are also (equivariant) Chern classes of certain line bundles over Y. Moreover, there is a *Monk formula* [Harada and Tymoczko 2011, Theorem 6.12] which expresses a product

$$p_i p_{\mathcal{A}}$$

for any $i \in \{1, 2, \ldots, n-1\}$ and any $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$ as a $H^*_{S^1}(\mathrm{pt})$-linear combination of the additive module basis $\{p_{\mathcal{A}}\}$. Since the p_i multiplicatively generate the ring, this Monk formula completely determines the ring structure of $H^*_{S^1}(Y)$. Furthermore it is in principle possible to express any $p_{\mathcal{A}}$ in terms of the p_i. Our Giambelli formula is an explicit formula which achieves this (cf. for example [Fulton 1997] for the version in classical Schubert calculus).

We begin by recalling the Monk formula, for which we need some terminology. Fix $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$. We define $\mathcal{H}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{H}_{\mathcal{A}}(j) = \text{the maximal element in the maximal consecutive substring of } \mathcal{A} \text{ containing } j.$$

Similarly, we define $\mathcal{T}_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ by

$$\mathcal{T}_{\mathcal{A}}(j) = \text{the minimal element in the maximal consecutive substring of } \mathcal{A} \text{ containing } j.$$

We say that the maps $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{T}_{\mathcal{A}}$ give the "head" and "tail" of each maximal consecutive substring of $\mathcal{A}$. For an example see [Harada and Tymoczko 2011, Example 5.6]. We recall the following.

Theorem 3.1 (Monk formula for Peterson varieties [Harada and Tymoczko 2011, Theorem 6.12]). *Fix a positive integer n. Let Y be the Peterson variety in $\mathcal{F}\ell ags(\mathbb{C}^n)$ with the natural S^1-action defined by (2-2). For $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$, let $v_{\mathcal{A}} \in S_n$ be the permutation in (2-4), and let $p_{\mathcal{A}}$ be the corresponding Peterson Schubert class in $H^*_{S^1}(Y)$. Then*

$$p_i \cdot p_{\mathcal{A}} = p_i(w_{\mathcal{A}}) \cdot p_{\mathcal{A}} + \sum_{\mathcal{A} \subsetneq \mathcal{B} \text{ and } |\mathcal{B}|=|\mathcal{A}|+1} c^{\mathcal{B}}_{i,\mathcal{A}} \cdot p_{\mathcal{B}}, \tag{3-2}$$

where, for a subset $\mathcal{B} \subseteq \{1, 2, \ldots, n-1\}$ which is a disjoint union $\mathcal{B} = \mathcal{A} \cup \{k\}$,

- *if $i \notin \mathcal{B}$ then $c^{\mathcal{B}}_{i,\mathcal{A}} = 0$,*
- *if $i \in \mathcal{B}$ and $i \notin [\mathcal{T}_{\mathcal{B}}(k), \mathcal{H}_{\mathcal{B}}(k)]$, then $c^{\mathcal{B}}_{i,\mathcal{A}} = 0$,*
- *if $k \leq i \leq \mathcal{H}_{\mathcal{B}}(k)$, then*

$$c^{\mathcal{B}}_{i,\mathcal{A}} = (\mathcal{H}_{\mathcal{B}}(k) - i + 1) \cdot \binom{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1}{k - \mathcal{T}_{\mathcal{B}}(k)}, \tag{3-3}$$

- *if $\mathcal{T}_{\mathcal{B}}(k) \leq i \leq k-1$,*

$$c^{\mathcal{B}}_{i,\mathcal{A}} = (i - \mathcal{T}_{\mathcal{B}}(k) + 1) \cdot \binom{\mathcal{H}_{\mathcal{B}}(k) - \mathcal{T}_{\mathcal{B}}(k) + 1}{k - \mathcal{T}_{\mathcal{B}}(k) + 1}. \tag{3-4}$$

We also recall that [Harada and Tymoczko 2011, Lemma 6.7] implies that if $\mathcal{B}, \mathcal{B}'$ are two disjoint subsets of $\{1, 2, \ldots, n-1\}$ such that there is no i in $\mathcal{B}$ and j in $\mathcal{B}'$ with $|i - j| = 1$, then $p_{\mathcal{B} \cup \mathcal{B}'} = p_{\mathcal{B}} p_{\mathcal{B}'}$. It follows that for any $\mathcal{A}$ we have

$$p_{\mathcal{A}} = p_{[a_1,a_2]} \cdot p_{[a_3,a_4]} \cdots p_{[a_{m-1},a_m]} \tag{3-5}$$

where $\mathcal{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m]$ is the decomposition of $\mathcal{A}$ into maximal consecutive substrings. In particular, in order to give an expression for $p_{\mathcal{A}}$ in terms of the elements p_i, from (3-5) we see that it suffices to give a formula only for the special case in which $\mathcal{A}$ consists of a single maximal consecutive string.

We now state and prove our Giambelli formula.

Theorem 3.2. *Fix n a positive integer. Let Y be the Peterson variety in $\mathcal{F}\ell ags(\mathbb{C}^n)$ with the S^1-action defined by (2-2). Suppose $\mathcal{A} = \{a, a+1, a+2, \dots, a+k\}$ where $1 \le a \le n-1$ and $0 \le k \le n-1-a$. Let $v_{\mathcal{A}}$ be the permutation corresponding to $\mathcal{A}$ defined in (2-4) and let $p_{\mathcal{A}}$ be the associated Peterson Schubert class. Then*

$$p_{\mathcal{A}} = \frac{1}{(k+1)!} \prod_{j \in \mathcal{A}} p_j.$$

We use the following lemma.

Lemma 3.3. *Suppose $i \in \{1, 2, \dots, n-1\}$ and $\mathcal{A} \subseteq \{1, 2, \dots, n-1\}$. Suppose further that $i \notin \mathcal{A}$. Then the Monk relation*

$$p_i \cdot p_{\mathcal{A}} = p_i(w_{\mathcal{A}}) \cdot p_{\mathcal{A}} + \sum_{\mathcal{A} \subset \mathcal{B} \text{ and } |\mathcal{B}| = |\mathcal{A}|+1} c_{i,\mathcal{A}}^{\mathcal{B}} \cdot p_{\mathcal{B}}$$

simplifies to

$$p_i \cdot p_{\mathcal{A}} = c_{i,\mathcal{A}}^{\mathcal{A} \cup \{i\}} \cdot p_{\mathcal{A} \cup \{i\}}. \tag{3-6}$$

Proof. First observe that the Monk relation simplifies to

$$p_i \cdot p_{\mathcal{A}} = \sum_{\mathcal{A} \subset \mathcal{B} \text{ and } |\mathcal{B}| = |\mathcal{A}|+1} c_{i,\mathcal{A}}^{\mathcal{B}} \cdot p_{\mathcal{B}} \tag{3-7}$$

if $i \notin \mathcal{A}$, since in that case $p_i(w_{\mathcal{A}}) = 0$ by [Harada and Tymoczko 2011, Lemma 6.4]. Moreover, from Theorem 3.1 we also know that $c_{i,\mathcal{A}}^{\mathcal{B}} = 0$ if $i \notin \mathcal{B}$. Hence the summands appearing in (3-7) correspond to $\mathcal{B}$ satisfying $\mathcal{A} \subseteq \mathcal{B}$, $|\mathcal{B}| = |\mathcal{A}| + 1$, and $i \in \mathcal{B}$. On the other hand, since $i \notin \mathcal{A}$ by assumption, this means that there is only one nonzero summand in the right hand side of (3-7), namely, the term corresponding to $\mathcal{B} = \mathcal{A} \cup \{i\}$. Then (3-6) follows. □

Proof of Theorem 3.2. We proceed by induction on k. First consider the base case where $k = 0$. Then $A = \{a\}$, so $p_{v_{\mathcal{A}}} = p_a$. On the right hand side, we have $\frac{1}{(0+1)!} \prod_{j \in \mathcal{A}} p_j = p_a$. This verifies the base case.

By induction, suppose the claim holds for $k-1$. We now show that the claim holds for k. Consider $\mathcal{A}' := \{a, a+1, \dots, a+k-1\}$ and consider the product $p_{a+k} \cdot p_{\mathcal{A}'}$. From the Monk formula in Theorem 3.1 we know that

$$p_{a+k} \cdot p_{\mathcal{A}'} = p_{a+k}(w_{\mathcal{A}'}) \cdot p_{\mathcal{A}'} + \sum_{\substack{\mathcal{A}' \subseteq \mathcal{B} \\ |\mathcal{B}| = |\mathcal{A}'|+1}} c_{a+k,\mathcal{A}'}^{\mathcal{B}} \cdot p_{\mathcal{B}}. \tag{3-8}$$

On the other hand since by definition $a+k \notin \mathscr{A}'$, by Lemma 3.3 the equality (3-8) further simplifies to

$$p_{a+k} \cdot p_{\mathscr{A}'} = c^{\mathscr{A}}_{a+k,\mathscr{A}'} \cdot p_{\mathscr{A}}.$$

Moreover, since $\mathscr{A} = \mathscr{A}' \cup \{a+k\}$, we have $\mathscr{H}_{\mathscr{A}}(a+k) = a+k$ and $\mathscr{T}_{\mathscr{A}}(a+k) = a$. Hence, by Theorem 3.1,

$$\begin{aligned} c^{\mathscr{A}}_{a+k,\mathscr{A}'} &= (\mathscr{H}_{\mathscr{A}}(a+k) - (a+k) + 1) \binom{\mathscr{H}_{\mathscr{A}}(a+k) - \mathscr{T}_{\mathscr{A}}(a+k) + 1}{(a+k) - \mathscr{T}_{\mathscr{A}}(a+k)} \\ &= ((a+k) - (a+k) + 1) \binom{a+k-a+1}{(a+k)-a} \\ &= k+1. \end{aligned} \tag{3-9}$$

Therefore

$$p_{a+k} \cdot p_{\mathscr{A}'} = (k+1) \cdot p_{\mathscr{A}}.$$

By the inductive hypothesis we have for the set $\mathscr{A}' = \{a, a+1, \dots, a+k-1\}$

$$p_{\mathscr{A}'} = \frac{1}{k!} \prod_{j \in \mathscr{A}'} p_j.$$

Substituting into the above equation yields

$$p_{\mathscr{A}} = \frac{1}{(k+1)!} \prod_{j \in \mathscr{A}} p_j$$

as desired. This completes the proof. □

Remark 3.4. We thank the referee for the following observation. The formula in Theorem 3.2 suggests that the classes p_i behave like a normal crossings divisor (up to quotient singularities), with all other classes arising (up to rational coefficients) as intersections of the components. It would certainly be of interest to understand more precisely the underlying geometry which gives rise not only to the Giambelli relation in Theorem 3.2 but also to the original Monk formula [Harada and Tymoczko 2011, Theorem 6.12].

From Theorem 3.2 it immediately follows that for any subset

$$\mathscr{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m]$$

with its decomposition into maximal consecutive substrings, we have

$$p_{\mathscr{A}} = \frac{1}{(a_2 - a_1 + 1)!} \cdot \frac{1}{(a_4 - a_3 + 1)!} \cdots \frac{1}{(a_m - a_{m-1} + 1)!} \prod_{j \in \mathscr{A}} p_j. \tag{3-10}$$

For the purposes of the next section we introduce the notation

$$\sigma(\mathcal{A}) := \frac{1}{(a_2 - a_1 + 1)!} \cdot \frac{1}{(a_4 - a_3 + 1)!} \cdots \frac{1}{(a_m - a_{m-1} + 1)!} \tag{3-11}$$

for the rational coefficient appearing in (3-10). The following is an immediate corollary of this discussion.

Corollary 3.5. *Let*

$$\mathcal{A} = [a_1, a_2] \cup [a_3, a_4] \cup \cdots \cup [a_{m-1}, a_m].$$

Then

$$p_{\mathcal{A}} = \sigma(\mathcal{A}) \prod_{j \in \mathcal{A}} p_j.$$

Simplification of the Monk relations. In this section we explain how to use the Giambelli formula to simplify the ring presentation of $H^*_{S^1}(Y)$ given in [Harada and Tymoczko 2011, Section 6]. Recall that the Peterson Schubert classes $\{p_{\mathcal{A}}\}$ form an additive module basis for $H^*_{S^1}(Y)$ and the degree-2 classes $\{p_i\}_{i=1}^{n-1}$ form a multiplicative basis, so the Monk relations give a presentation of the ring $H^*_{S^1}(Y)$ via generators and relations as follows.

Theorem 3.6 [Harada and Tymoczko 2011, Corollary 6.14]. *Fix n a positive integer. Let Y be the Peterson variety in $\mathcal{F}\ell ags(\mathbb{C}^n)$ with the S^1-action defined by (2-2). For $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$, let $v_{\mathcal{A}} \in S_n$ be the permutation given in (2-4), and let $p_{\mathcal{A}}$ be the corresponding Peterson Schubert class in $H^*_{S^1}(Y)$. Let $t \in H^*_{S^1}(\mathrm{pt}) \cong \mathbb{C}[t]$ denote both the generator of $H^*_{S^1}(\mathrm{pt})$ and its image $t \in H^*_{S^1}(Y)$. Then the S^1-equivariant cohomology $H^*_{S^1}(Y)$ is given by*

$$H^*_{S^1}(Y) \cong \mathbb{C}[t, \{p_{\mathcal{A}}\}_{\mathcal{A} \subseteq \{1,2,\ldots,n-1\}}]/\mathcal{J}$$

where $\mathcal{J}$ is the ideal generated by the relations (3-2).

In order to state the main result of this section we introduce some notation. For i with $1 \leq i \leq n-1$ and $\mathcal{A} \subseteq \{1, 2, \ldots, n-1\}$ define

$$m_{i,\mathcal{A}} := p_i \cdot p_{\mathcal{A}} - p_i(w_{\mathcal{A}}) \cdot p_{\mathcal{A}} - \sum_{\substack{\mathcal{A} \subseteq \mathcal{B} \\ |\mathcal{B}| = |\mathcal{A}|+1}} c^{\mathcal{B}}_{i,\mathcal{A}} \cdot p_{\mathcal{B}},$$

thought of as an element in $\mathbb{C}[t, \{p_{\mathcal{A}}\}_{\mathcal{A} \subseteq \{1,2,\ldots,n-1\}}]$, where the $c^{\mathcal{B}}_{i,\mathcal{A}} \in \mathbb{C}[t]$ are the coefficients computed in Theorem 3.1. Motivated by the Giambelli formula we also define the following elements in $\mathbb{C}[t, p_1, p_2, \ldots, p_{n-1}]$:

$$q_{i,\mathcal{A}} := p_i \cdot \sigma(\mathcal{A}) \cdot \Big(\prod_{j \in \mathcal{A}} p_j\Big) - p_i(w_{\mathcal{A}}) \cdot \sigma(\mathcal{A}) \cdot \Big(\prod_{j \in \mathcal{A}} p_j\Big) - \sum_{\substack{\mathcal{A} \subseteq \mathcal{B} \\ |\mathcal{B}| = |\mathcal{A}|+1}} c^{\mathcal{B}}_{i,\mathcal{A}} \cdot \sigma(\mathcal{B}) \Big(\prod_{k \in \mathcal{B}} p_k\Big),$$

where $\sigma(\mathscr{A}) \in \mathbb{Q}$ is the constant defined in (3-11).

Example 3.7. Let $n = 4$ and $i = 1$ and $\mathscr{A} = \{1, 2\}$. Consider

$$m_{1,\{1,2\}} = p_1 p_{v_{\{1,2\}}} - 2t\, p_{v_{\{1,2\}}} + p_{v_{\{1,2,3\}}}.$$

Expanding in terms of the Giambelli formula, we obtain

$$q_{1,\{1,2\}} = \tfrac{1}{2} p_1^2 p_2 - 2t \cdot \left(\tfrac{1}{2} p_1 p_2\right) + \tfrac{1}{6} p_1 p_2 p_3 = t\, p_1 p_2 + \tfrac{1}{6} p_1 p_2 p_3.$$

The main theorem of this section gives a ring presentation of $H^*_{S^1}(Y)$ using fewer generators and fewer relations than that in Theorem 3.6. More specifically let $\mathscr{K}$ denote the ideal in $\mathbb{C}[t, p_1, \dots, p_{n-1}]$ generated by the $q_{i,\mathscr{A}}$ for which $i \notin \mathscr{A}$, that is,

$$\begin{aligned} \mathscr{K} := \big\langle q_{i,\mathscr{A}} \,\big|\, 1 \le i \le n-1,\ \mathscr{A} \subseteq \{1, 2, \dots, n-1\},\ i \notin \mathscr{A} \big\rangle \\ \subseteq \mathbb{C}[t, p_1, \dots, p_{n-1}]. \end{aligned} \tag{3-12}$$

Theorem 3.8. *Fix n a positive integer. Let Y be the Peterson variety in $\mathscr{F}\ell ags(\mathbb{C}^n)$ equipped with the action of the S^1 in (2-2). Then the S^1-equivariant cohomology $H^*_{S^1}(Y)$ is isomorphic to the ring*

$$\mathbb{C}[t, p_1, p_2, \dots, p_{n-1}]/\mathscr{K}$$

where $\mathscr{K}$ is the ideal in (3-12).

To prove the theorem we need the following lemma.

Lemma 3.9. *Let $i \in \{1, 2, \dots, n-1\}$ and $\mathscr{A} \subseteq \{1, 2, \dots, n-1\}$. Suppose $i \notin \mathscr{A}$. Then $q_{i,\mathscr{A}} = 0$ in $\mathbb{C}[t, p_1, p_2, \dots, p_{n-1}]$.*

Proof. Since $i \notin \mathscr{A}$ by assumption, Lemma 3.3 implies that

$$m_{i,\mathscr{A}} = p_i \cdot p_{\mathscr{A}} - p_i(w_{\mathscr{A}}) \cdot p_{\mathscr{A}} - \sum_{\substack{\mathscr{A} \subseteq \mathscr{B} \\ |\mathscr{B}| = |\mathscr{A}|+1}} c^{\mathscr{B}}_{i,\mathscr{A}} \cdot p_{\mathscr{B}}$$

simplifies to

$$m_{i,\mathscr{A}} = p_i \cdot p_{\mathscr{A}} - c^{\mathscr{A}\cup\{i\}}_{i,\mathscr{A}} \cdot p_{\mathscr{A}\cup\{i\}}. \tag{3-13}$$

Thus in order to compute the corresponding $q_{i,\mathscr{A}}$ it remains to compute $c^{\mathscr{A}\cup\{i\}}_{i,\mathscr{A}}$ and apply the Giambelli formula.

Let $\mathscr{A} = [a_1, a_2] \cup [a_3, a_4] \cup \dots \cup [a_{m-1}, a_m]$ be the decomposition of $\mathscr{A}$ into maximal consecutive substrings. Consider the decomposition of $\mathscr{A}\cup\{i\}$ into maximal consecutive substrings. There are several cases to consider:

(1) The singleton set $\{i\}$ is a maximal consecutive substring of $\mathscr{A} \cup \{i\}$, that is, $i - 1 \notin \mathscr{A}$ and $i + 1 \notin \mathscr{A}$.

(2) The inclusion of i extends a maximal consecutive substring to its right by 1 element, that is, there exists a maximal consecutive string $[a_\ell, a_{\ell+1}] \subseteq \mathcal{A}$ such that $i = a_{\ell+1} + 1$ and that $[a_\ell, i]$ is a maximal consecutive substring of $\mathcal{A} \cup \{i\}$.

(3) The inclusion of i extends a maximal consecutive substring to its left by 1 element, that is, there exists a maximal consecutive string $[a_\ell, a_{\ell+1}] \subseteq \mathcal{A}$ such that $i = a_\ell - 1$ and that $[i, a_{\ell+1}]$ is a maximal consecutive substring of $\mathcal{A} \cup \{i\}$.

(4) The inclusion of i glues together two maximal consecutive substrings of $\mathcal{A}$, that is, there exist two maximal consecutive substrings $[a_\ell, a_{\ell+1}]$, $[a_{\ell+2}, a_{\ell+3}]$ such that $i = a_{\ell+1} + 1 = a_{\ell+2} - 1$ and hence $[a_\ell, a_{\ell+3}] = [a_\ell, a_{\ell+1}] \cup \{i\} \cup [a_{\ell+2}, a_{\ell+3}]$ is a maximal consecutive substring of $\mathcal{A} \cup \{i\}$.

We consider each case separately.

Case (1): Suppose $\{i\}$ is a maximal consecutive substring in $\mathcal{A} \cup \{i\}$. In this case, the coefficient $c_{i,\mathcal{A}}^{\mathcal{A}\cup\{i\}}$ is 1. Hence we have

$$m_{i,\mathcal{A}} = p_i \, p_{\mathcal{A}} - p_{v_{\mathcal{A}\cup\{i\}}}.$$

Since $\{i\}$ is a maximal consecutive substring in $\mathcal{A} \cup \{i\}$, we have $\sigma(\mathcal{A}) = \sigma(\mathcal{A} \cup \{i\})$. We conclude that

$$q_{i,\mathcal{A}} = p_i \cdot \left(\sigma(\mathcal{A}) \cdot \left(\prod_{j\in\mathcal{A}} p_j\right)\right) - \sigma(\mathcal{A}\cup\{i\}) \cdot \left(\prod_{j\in\mathcal{A}\cup\{i\}} p_j\right) = 0,$$

as desired.

Cases (2) and (3) are very similar, so we only present the argument for Case (2). Suppose i extends a maximal consecutive substring $[a_\ell, a_{\ell+1}]$ of $\mathcal{A}$ to its right. Then

$$m_{i,\mathcal{A}} = p_i \cdot p_{\mathcal{A}} - (i - a_\ell + 1)\, p_{v_{\mathcal{A}\cup\{i\}}},$$

since $k = i = \mathcal{H}_{\mathcal{B}}(i)$ and $\mathcal{T}_{\mathcal{B}}(i) = a_\ell$ so $c_{i,\mathcal{A}}^{\mathcal{A}\cup\{i\}} = i - a_\ell + 1$. We compute

$$\begin{aligned} q_{i,\mathcal{A}} = p_i &\left(\left(\prod_{\substack{1\le s\le m-1\\ s \text{ odd}}} \frac{1}{(a_{s+1}-a_s+1)!}\right)\cdot\left(\prod_{j\in\mathcal{A}} p_j\right)\right) \\ &- (i-a_\ell+1)\cdot\left(\prod_{\substack{1\le s\le m-1\\ s \text{ odd},\, s\ne\ell}} \frac{1}{(a_{s+1}-a_s+1)!}\right)\cdot\left(\frac{1}{(i-a_\ell+1)!}\right)\cdot\left(\prod_{j\in\mathcal{A}\cup\{i\}} p_j\right), \end{aligned}$$

where one of the factors in the product in the second expression has changed because the maximal consecutive string $[a_\ell, a_{\ell+1}]$ has been extended in $\mathcal{A} \cup \{i\}$. Since

$$(i - a_\ell + 1)\left(\frac{1}{(i - a_\ell + 1)!}\right) = \frac{1}{(a_{\ell+1} - a_\ell + 1)!}$$

by assumption on i, we conclude $q_{i,\mathscr{A}} = 0$ as desired.

Case (4). Here the inclusion of i glues together two maximal consecutive substrings $[a_\ell, a_{\ell+1}]$, $[a_{\ell+2}, a_{\ell+3}]$ in $\mathscr{A}$. We then have $k = i$, $\mathscr{H}_{\mathscr{B}}(i) = a_{\ell+3}$, $\mathscr{T}_{\mathscr{B}}(i) = a_\ell$. Hence the coefficient $c_{i,\mathscr{A}}^{\mathscr{A}\cup\{i\}}$ is

$$c_{i,\mathscr{A}}^{\mathscr{A}\cup\{i\}} = (a_{\ell+3} - i + 1)\binom{a_{\ell+3} - a_\ell + 1}{i - a_\ell} = \frac{(a_{\ell+3} - a_\ell + 1)!}{(i - a_\ell)!\,(a_{\ell+3} - i)!}.$$

The expansion of $p_i \cdot p_{\mathscr{A}}$ is the same as in the previous cases. The term corresponding to $c_{i,\mathscr{A}}^{\mathscr{A}\cup\{i\}} \cdot p_{\mathscr{A}\cup\{i\}}$ is

$$\frac{(a_{\ell+3}-a_\ell+1)!}{(i-a_\ell)!\,(a_{\ell+3}-i)!} \cdot \Biggl(\prod_{\substack{1\le s\le m-1\\ s \text{ odd and } s\neq \ell,\ell+2}} \frac{1}{(a_{s+1}-a_s+1)!}\Biggr)\cdot\Biggl(\frac{1}{(a_{\ell+3}-a_\ell+1)!}\Biggr)\cdot\Biggl(\prod_{j\in\mathscr{A}\cup\{i\}} p_j\Biggr).$$

Since by assumption on i we have $i = a_{\ell+1} + 1 = a_{\ell+2} - 1$, we obtain the simplification

$$\begin{aligned}\frac{(a_{\ell+3} - a_\ell + 1)!}{(i - a_\ell)!\,(a_{\ell+3} - i)!}\Biggl(\frac{1}{(a_{\ell+3} - a_\ell + 1)!}\Biggr) &= \frac{1}{(i - a_\ell)!\,(a_{\ell+3} - i)!}\\ &= \frac{1}{(a_{\ell+1} - a_\ell + 1)!}\cdot\frac{1}{(a_{\ell+3} - a_{\ell+2} + 1)!}\end{aligned}$$

from which it follows that $q_{i,\mathscr{A}} = 0$ also in this case. The result follows. □

Example 3.10. Let $n = 5$, $i = 4$ and let $\mathscr{A} = \{1, 2\}$. Consider

$$m_{4,\{1,2\}} = p_4 \cdot p_{v_{\{1,2\}}} - c_{4,\{1,2\}}^{\{1,2,4\}} \cdot p_{v_{\{1,2,4\}}}.$$

From (3-3) it follows that $c_{4,\{1,2\}}^{\{1,2,4\}} = 1$. The corresponding $q_{4,\{1,2\}}$ can be computed to be

$$q_{4,\{1,2\}} = p_4\left(\tfrac{1}{2!}p_1p_2\right) - \left(\tfrac{1}{2!}p_1p_2\right)p_4 = 0.$$

Proof of Theorem 3.8. By Theorem 3.6 we know that

$$H^*_{S^1}(Y) \cong \mathbb{C}[t, \{p_{\mathscr{A}}\}_{\mathscr{A}\subseteq\{1,2,\dots,n-1\}}]/\mathscr{J},$$

where $\mathscr{J}$ is the ideal generated by the relations (3-2) so we wish to prove

$$\mathbb{C}[t, p_1, \dots, p_{n-1}]/\mathscr{K} \cong \mathbb{C}[t, \{p_{\mathscr{A}}\}_{\mathscr{A}\subseteq\{1,2,\dots,n-1\}}]/\mathscr{J}.$$

The content of the Giambelli formula (Theorem 3.2) is that the expressions

$$p_{\mathscr{A}} - \sigma(\mathscr{A})\prod_{j\in\mathscr{A}} p_j$$

are elements of $\mathcal{J}$. Hence

$$\begin{aligned}\mathcal{J} &= \big\langle m_{i,\mathcal{A}} \;\big|\; 1 \le i \le n-1,\, \mathcal{A} \subseteq \{1, 2, \dots, n-1\}\big\rangle \\ &\qquad + \Big\langle p_{\mathcal{A}} - \sigma(\mathcal{A}) \prod_{j \in \mathcal{A}} p_j \;\Big|\; 1 \le i \le n-1,\, \mathcal{A} \subseteq \{1, 2, \dots, n-1\}\Big\rangle \\ &= \big\langle q_{i,\mathcal{A}} \;\big|\; 1 \le i \le n-1,\, \mathcal{A} \subseteq \{1, 2, \dots, n-1\}\big\rangle \\ &\qquad + \Big\langle p_{\mathcal{A}} - \sigma(\mathcal{A}) \prod_{j \in \mathcal{A}} p_j \;\Big|\; 1 \le i \le n-1,\, \mathcal{A} \subseteq \{1, 2, \dots, n-1\}\Big\rangle.\end{aligned}$$

We therefore have

$$\frac{\mathbb{C}[t, \{p_{\mathcal{A}}\}_{\mathcal{A} \subseteq \{1,2,\dots,n-1\}}]}{\mathcal{J}} \cong \frac{\mathbb{C}[t, p_1, \dots, p_{n-1}]}{\big\langle q_{i,\mathcal{A}} \;\big|\; 1 \le i \le n-1,\, \mathcal{A} \subseteq \{1, 2, \dots, n-1\}\big\rangle};$$

but since $q_{i,\mathcal{A}} = 0$ if $i \notin \mathcal{A}$ by Lemma 3.9 we conclude that

$$\begin{aligned}&\big\langle q_{i,\mathcal{A}} \;\big|\; 1 \le i \le n-1,\, \mathcal{A} \subseteq \{1, 2, \dots, n-1\}\big\rangle \\ &\qquad = \big\langle q_{i,\mathcal{A}} \;\big|\; 1 \le i \le n-1,\, \mathcal{A} \subseteq \{1, 2, \dots, n-1\} \text{ and } i \notin \mathcal{A}\big\rangle,\end{aligned}$$

from which the result follows. □

Example 3.11. Let $n = 4$ and Y the Peterson variety in $\mathcal{F}\ell ags(\mathbb{C}^4)$. The degree-2 multiplicative generators are p_1, p_2, and p_3. Then the statement of Theorem 3.8 yields a presentation of the equivariant cohomology ring of Y as

$$H^*_{S^1}(Y) \cong \mathbb{C}[t, p_1, p_2, p_3]/\mathcal{K},$$

where $\mathcal{K}$ is the ideal generated by the following 12 elements:

$$\begin{gathered}2p_1^2 - 2t\,p_1 - p_1p_2,\\ 2p_2^2 - 2t\,p_2 - p_1p_2 - p_2p_3,\\ 2p_3^2 - 2t\,p_3 - p_2p_3,\\ 3p_1^2p_2 - 6t\,p_1p_2 - p_1p_2p_3,\\ 3p_1p_2^2 - 6t\,p_1p_2 - 2p_1p_2p_3,\\ 2p_1^2p_3 - 2t\,p_1p_3 - p_1p_2p_3,\\ 2p_1p_3^2 - 2t\,p_1p_3 - p_1p_2p_3,\\ 3p_2^2p_3 - 6t\,p_2p_3 - 2p_1p_2p_3,\\ 3p_2p_3^2 - 6t\,p_2p_3 - p_1p_2p_3,\\ p_1^2p_2p_3 - 3t\,p_1p_2p_3,\\ p_1p_2^2p_3 - 4t\,p_1p_2p_3,\\ p_1p_2p_3^2 - 3t\,p_1p_2p_3.\end{gathered}$$

This list is not minimal: for instance, one can immediately see the sixth and seventh expressions in this list are multiples of the first and third ones, so evidently they are unnecessary for defining the ideal $\mathcal{K}$. In fact, more is true: a Macaulay 2 computation shows that the ideal $\mathcal{K}$ is in fact generated by just the *quadratic* relations, that is, the first three elements in the above list. (We thank the referee for pointing this out.) Note that the original presentation given in Theorem 3.6 uses 8 generators and 24 relations, so this discussion shows that our presentation indeed gives a simplification of the description of the ring.

Remark 3.12. We thank the referee for the following comment. Based on our Giambelli formula, Theorem 3.8, and the example of $n = 4$ discussed above, it seems natural to conjecture that for any value of n, the corresponding ideal $\mathcal{K}$ is generated by just the quadratic relations. Using Macaulay 2, we have verified that the conjecture holds for a range of small values of n, but we were unable to give a proof for the general case. If the conjecture is true, then it would be a very significant simplification of the presentation of this ring and would lead to many interesting geometric and combinatorial questions.

4. Stirling numbers of the second kind

In this section we prove that Stirling numbers of the second kind appear in the multiplicative structure of the ring $H^*_{S^1}(Y)$. We learned this result from H. Naruse and do not claim originality, though the proof given is our own. The *Stirling number of the second kind*, which we denote $S(n, k)$, counts the number of ways to partition a set of n elements into k nonempty subsets (see, e.g., [Knuth 1975, Section 1.2.6]). For example, $S(3, 2)$ is the number of ways to put balls labeled 1, 2, and 3 into two identical boxes such that each box contains at least one ball. It is then easily seen that $S(3, 2) = 3$.

Theorem 4.1. *Fix a positive integern. Let Y be the Peterson variety in $\mathcal{F}\ell ags(\mathbb{C}^n)$ equipped with the action of the S^1 in (2-2). For $\mathcal{A} \subseteq \{1, 2, \dots, n-1\}$, let $v_{\mathcal{A}}$, $p_{\mathcal{A}}$ be as in Theorem 3.6. The following equality holds in $H^*_{S^1}(Y)$ for any k with $1 \le k \le n-1$:*

$$p_1^k = \sum_{j=1}^{k} S(k, j) t^{k-j}\, p_{v_{[1,j]}}. \tag{4-1}$$

Proof. We proceed by induction on k. Consider the base case $k = 1$. Then (4-1) becomes the equality

$$p_1 = S(1, 1) p_1.$$

Here $S(1, 1)$ is the number of ways to put 1 ball into 1 box, so $S(1, 1) = 1$ and the claim follows.

Now assume that (4-1) holds for k. We need to show that it also holds for $k+1$, that is,

$$p_1^{k+1} = \sum_{j=1}^{k+1} S(k+1, j) t^{k+1-j} p_{v_{[1,j]}}.$$

By the inductive hypothesis this is equivalent to showing that

$$\sum_{i=1}^{k} S(k,i) t^{k-i} p_1 p_{v_{[1,i]}} = \sum_{j=1}^{k+1} S(k+1, j) t^{k+1-j} p_{v_{[1,j]}}. \tag{4-2}$$

We now expand the left-hand side using the Monk formula. For each i it can be computed that

$$p_1 \, p_{v_{[1,i]}} = it \, p_{v_{[1,i]}} + p_{v_{[1,i+1]}}$$

where we have used [Harada and Tymoczko 2011, Lemma 6.4] to compute $p_1(w_{[1,i]})$. Therefore

$$\begin{aligned}
&\sum_{i=1}^{k} S(k,i) t^{k-i} p_1 p_{v_{[1,i]}} \\
&= \sum_{i=1}^{k} S(k,i) t^{k-i} (it \, p_{v_{[1,i]}} + p_{v_{[1,i+1]}}) \\
&= \sum_{i=1}^{k} i \, S(k,i) t^{k+1-i} p_{v_{[1,i]}} + \sum_{i=1}^{k} S(k,i) t^{k-i} p_{v_{[1,i+1]}} \\
&= S(k,1) t^k p_1 + \sum_{i=2}^{k} i \, S(k,i) t^{k+1-i} p_{v_{[1,i]}} + \sum_{i=1}^{k-1} S(k,i) t^{k-i} p_{v_{[1,i+1]}} + S(k,k) p_{v_{[1,k+1]}} \\
&= S(k,1) t^k p_1 + \sum_{i=2}^{k} i \, S(k,i) t^{k+1-i} p_{v_{[1,i]}} + \sum_{i=2}^{k} S(k,i-1) t^{k+1-i} p_{v_{[1,i]}} + S(k,k) p_{v_{[1,k+1]}} \\
&= S(k+1,1) t^k p_1 + \sum_{i=2}^{k} (i \, S(k,i) + S(k,i-1)) t^{k+1-i} p_{v_{[1,i]}} + S(k+1,k+1) p_{v_{[1,k+1]}} \\
&= S(k+1,1) t^k p_1 + \sum_{i=2}^{k} S(k+1, j) t^{k+1-i} p_{v_{[1,i]}} + S(k+1,k+1) p_{v_{[1,k+1]}} \\
&= \sum_{j=1}^{k+1} S(k+1, j) t^{k+1-j} p_{v_{[1,j]}}
\end{aligned}$$

where we have used the recurrence relation

$$S(k+1, j) = j S(k, j) + S(k, j-1)$$

for Stirling numbers and the fact that

$$S(k,1) = S(k,k) = S(k+1,1) = S(k+1,k+1) = 1$$

for any k. The result follows. $\square$

5. Stability of Peterson Schubert classes

We now observe that the Peterson Schubert classes $\{p_{\mathscr{A}}\}$ for the Peterson varieties satisfy a stability property for varying n, similar to that satisfied by the classical equivariant Schubert classes. This is an observation we learned from H. Naruse; we do not claim originality. For this section only, for a fixed positive integer n we denote by Y_n the Peterson variety in $\mathscr{F}\ell ags(\mathbb{C}^n)$.

Let $X_{w,n} \subseteq \mathscr{F}\ell ags(\mathbb{C}^n)$ denote the *Schubert variety* corresponding to $w \in S_n$ in $\mathscr{F}\ell ags(\mathbb{C}^n)$. By the standard inclusion of groups $S_n \hookrightarrow S_{n+1}$, we may also consider w to be an element in S_{n+1}. Furthermore there is a natural T^n-equivariant inclusion $\iota_n : \mathscr{F}\ell ags(\mathbb{C}^n) \hookrightarrow \mathscr{F}\ell ags(\mathbb{C}^{n+1})$ induced by the inclusion of the coordinate subspace $\mathbb{C}^n$ into $\mathbb{C}^{n+1}$. Then with respect to ι_n the Schubert variety $X_{w,n}$ maps isomorphically onto the corresponding Schubert variety $X_{w,n+1}$. Since the equivariant Schubert classes are cohomology classes corresponding to the Schubert varieties, this implies that for any $w \in S_n$ there exists an infinite sequence of Schubert classes $\{\sigma_{w,m}\}_{m=n}^{\infty}$ which lift the classes $\sigma_{w,n} \in H^*_{T^n}(\mathscr{F}\ell ags(\mathbb{C}^n))$, that is,

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^*_{T^n}(\mathscr{F}\ell ags(\mathbb{C}^{n+2})) & \longrightarrow & H^*_{T^n}(\mathscr{F}\ell ags(\mathbb{C}^{n+1})) & \longrightarrow & H^*_{T^n}(\mathscr{F}\ell ags(\mathbb{C}^{n})) \\ \cdots \longmapsto & \sigma_{w,n+2} & \longmapsto & \sigma_{w,n+1} & \longmapsto & \sigma_{w,n} \end{array} \tag{5-1}$$

and furthermore for any $v \in S_n$ and any $m \geq n$, the restriction $\sigma_{w,m}(v)$ is equal to $\sigma_{w,n}(v)$. The theorem below asserts that a similar statement holds for Peterson Schubert classes. Observe that the inclusion $\iota_n : \mathscr{F}\ell ags(\mathbb{C}^n) \hookrightarrow \mathscr{F}\ell ags(\mathbb{C}^{n+1})$ mentioned above also induces a natural inclusion $j_n : Y_n \hookrightarrow Y_{n+1}$ since the principal nilpotent operator on $\mathbb{C}^{n+1}$ preserves the coordinate subspace $\mathbb{C}^n$. Moreover, since the central circle subgroup of $U(n,\mathbb{C})$ acts trivially on $\mathscr{F}\ell ags(\mathbb{C}^n)$ for any n, the inclusion j_n is equivariant with respect to the S^1-actions on Y_n and Y_{n+1} given by the two circle subgroups defined by (2-2) in $U(n,\mathbb{C})$ and $U(n+1,\mathbb{C})$ respectively. Thus there is a pullback homomorphism $j_n^* : H^*_{S^1}(Y_{n+1}) \to H^*_{S^1}(Y_n)$ analogous to the map $\iota_n : H^*_{T^n}(\mathscr{F}\ell ags(\mathbb{C}^{n+1})) \to H^*_{T^n}(\mathscr{F}\ell ags(\mathbb{C}^n))$ above.

Theorem 5.1. *Fix a positive integer n. Let Y_n denote the Peterson variety in $\mathscr{F}\ell ags(\mathbb{C}^n)$ equipped with the natural S^1-action defined by (2-2). For $w \in S_n$ let $p_{w,n} \in H^*_{S^1}(Y_n)$ denote the Peterson Schubert class corresponding to w. Then the natural inclusions $j_m : Y_m \hookrightarrow Y_{m+1}$ for $m \geq n$ induce a sequence of homomorphisms $j_m^* : H^*_{S^1}(Y_{m+1}) \to H^*_{S^1}(Y_m)$ such that $j_m^*(p_{w,m+1}) = p_{w,m}$, that is,* *there exists a infinite sequence of Peterson Schubert classes* $\{p_{w,m}\}_{m=n}^{\infty}$ *that lift* $p_{w,n} \in H^*_{T^n}(\mathcal{F}\ell ags(\mathbb{C}^n))$:

$$
\begin{array}{ccccccc}
\cdots \longrightarrow & H^*_{S^1}(Y_{n+2}) & \longrightarrow & H^*_{S^1}(Y_{n+1}) & \longrightarrow & H^*_{S^1}(Y_n) \\
\cdots \longmapsto & p_{w,n+2} & \longmapsto & p_{w,n+1} & \longmapsto & p_{w,n}
\end{array}
\tag{5-2}
$$

Moreover, for any $v \in Y_n^{S^1}$ *and any* $m \geq n$, *the restriction* $p_{w,m}(v)$ *equals* $p_{w,n}(v)$.

Proof. By naturality and the definition of Peterson Schubert classes $p_{w,n} \in H^*_{S^1}(Y_n)$ as the images of $\sigma_{w,n}$, it is immediate that (5-1) can be expanded to a commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r] & H^*_{T^n}(\mathcal{F}\ell ags(\mathbb{C}^{n+2})) \arrow[r, "\iota^*_{n+1}"] \arrow[d] & H^*_{T^n}(\mathcal{F}\ell ags(\mathbb{C}^{n+1})) \arrow[r, "\iota^*_n"] \arrow[d] & H^*_{T^n}(\mathcal{F}\ell ags(\mathbb{C}^{n})) \arrow[d] \\
\cdots \arrow[r] & H^*_{S^1}(Y_{n+2}) \arrow[r, "j^*_{n+1}"'] & H^*_{S^1}(Y_{n+1}) \arrow[r, "j^*_n"'] & H^*_{S^1}(Y_n)
\end{tikzcd}
\tag{5-3}
$$

where the vertical arrows are the projection maps obtained by the composition of $H^*_{T^n}(\mathcal{F}\ell ags(\mathbb{C}^m)) \to H^*_{S^1}(\mathcal{F}\ell ags(\mathbb{C}^m))$ with $H^*_{S^1}(\mathcal{F}\ell ags(\mathbb{C}^m)) \to H^*_{S^1}(Y_m)$, for $m = n+2, n+1, n$. In particular, for any $w \in S_n$ and $m \geq n$, the vertical maps send $\sigma_{w,m}$ to $p_{w,m}$. The result follows. □

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dbayegan@gmail.com
Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB30WA, United Kingdom

megumi.harada@math.mcmaster.ca
Department of Mathematics and Statistics, McMaster University, 1280 Main Street, West, Hamilton, Ontario L8S4K1, Canada

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