involve
a journal of mathematics

Commutation classes of double wiring diagrams

Patrick Dukes and Joe Rusinko

msp

2012 vol. 5, no. 2

INVOLVE 5:2 (2012)
dx.doi.org/10.2140/involve.2012.5.207

Commutation classes of double wiring diagrams

Patrick Dukes and Joe Rusinko

(Communicated by Ravi Vakil)

We describe a new method for computing the graph of commutation classes of double wiring diagrams. Using these methods we compute the graph for five strings or less which allows us to confirm a positivity conjecture of Fomin and Zelevinsky when $n \leq 4$.

1. Introduction

In the theory of cluster algebras, the term Laurent phenomenon describes the mysterious instances in which recursively defined rational functions simplify to Laurent polynomials [Fomin and Zelevinsky 2002]. In many instances of the Laurent phenomenon, it is conjectured that the coefficients of the resulting Laurent polynomials are all positive.

One of the first examples of the Laurent phenomenon was found by Fomin and Zelevinsky [2000] when studying the relationships among minors of an $n \times n$ matrix with real coefficients. In this work they showed that every minor of a matrix was positive if and only if a particular subset of minors known as *chamber minors* was positive. These chamber minors were indexed by regions of a combinatorial object known as a double wiring diagram. Further, the relationships among minors were described in terms of a graph that describes the relationships among classes of double wiring diagrams.

As an example of the Laurent phenomenon, Fomin and Zelevinsky proved that every minor of an $n \times n$ matrix can be written as a Laurent polynomial in the chamber minors. They stated the following conjecture which remains open:

Conjecture 1.1 [Fomin and Zelevinsky 2000]. *For any n-string double wiring diagram w, every minor of an $n \times n$ matrix can be written as a Laurent polynomial with nonnegative coefficients in terms of the chamber minors of w.*

In this paper we confirm the conjecture for $n \leq 4$. To do so, we develop a new method of computing the graph of relationships among classes of double wiring

MSC2010: primary 05E15; secondary 15B48.
Keywords: wiring diagram, matrix positivity, Laurent phenomenon.

diagrams. We then use an original program for computing this graph based on this method, along with the algebra software Fermat [Lewis 2007] to compute the aforementioned Laurent polynomials.

In Section 2 of this paper we define *commutation classes of double wiring diagrams* and a graph which displays the relationships among these classes. In Section 3 we describe a new quiver representation of the commutation classes. This representation greatly simplifies the computation of the associated graph which we describe in Section 4. Finally, we return in Section 5 to the Laurent phenomenon and use our new computations to confirm the positivity conjecture for $n \leq 4$.

2. Double wiring diagrams

Fomin and Zelevinsky [2000] define an *n-stringed double wiring diagram* as two sets of n piecewise linear lines (black and gray) such that each line intersects every other line of the same color exactly once. We number gray lines from 1 to n with 1 on the top left and n on the bottom left. The black lines are labeled in the reverse order. In addition, for each chamber of the double wiring diagram we define the *chamber label* to be a pair of subsets (g, b) where g (respectively b) is the subset of $\{1, 2, \ldots, n\}$ identifying the gray (respectively black) strings that pass below the chamber. See Figure 1 for an example of a double wiring diagram with the chamber labels.

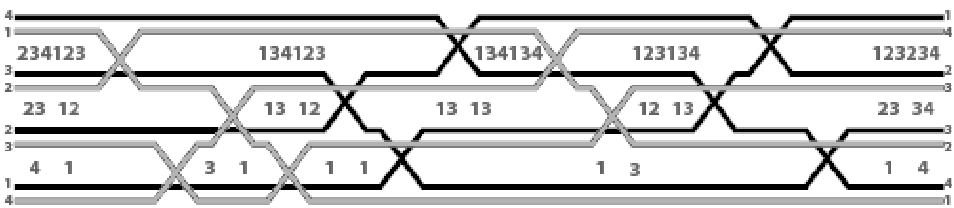

Figure 1. A four string double wiring diagram with chamber labels.

It is possible that slightly different wiring diagrams yield the same collection of chamber labels. Following Fomin and Zelevinsky, we consider two wiring diagrams that share the same collection of chamber labels *isotopic*. For single wiring diagrams, such collections of diagrams are called commutation classes, which have been studied in [Bédard 1999; Carter and Marsh 2000].

Definition 2.1. A *commutation class of double wiring diagrams* is the collection of all double wiring diagrams that share the same collection of chamber labels.

Any two commutation classes of double wiring diagrams can be linked by a sequence of the braid moves pictured in Figure 2 [Fomin and Zelevinsky 2000]. Note that in each exchange only one chamber label changes. In a braid move from wiring diagram w to wiring diagram w', we call the chamber label of w that changes under the braid move, the *center* of the braid move. A braid move is *centered* at a chamber label if that label changes under the braid move.

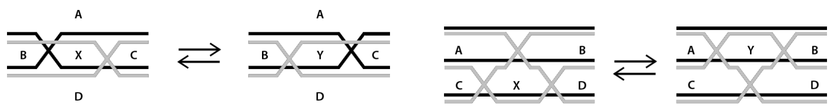

Figure 2. Braid moves. Left: 2-move; right: 3-move.

Since any two commutation classes of wiring diagrams can be connected by a sequence of braid moves, it is natural to construct a graph describing these relationships.

Definition 2.2. The graph of commutation classes of wiring diagrams, Φ_n, has a unique vertex for every commutation class of double wiring diagram with n strings. Two vertices are connected by an edge if their wiring diagrams differ by a single braid move.

Fomin and Zelevinsky [2000] prove that Φ_n is a finite connected graph and compute Φ_3. In this paper we present a method for computing Φ_n and use it to construct Φ_4 and Φ_5. We use these calculations to verify a positivity conjecture of Fomin and Zelevinsky when $n \leq 4$.

3. Using quivers to compute Φ_n

We have found that it is easier to compute Φ_n from the relationships among chamber labels than through the graphical structure of the wiring diagrams. This avoids the difficulty of keeping track of which wiring diagrams are in the same commutation class.

We introduce a quiver that describes the relevant relationships among the chamber labels of the double wiring diagram. This quiver is similar to a dual graph. The dual graph itself, however, is not an adequate data structure, as double wiring diagrams that are in the same commutation class may have differing dual graph structures.

Definition 3.1. For any double wiring diagram ω, define the quiver $Q(w)$ with vertices corresponding to chamber labels and an arrow from (g, b) to (g', b') if $g' = g \cup \{g_j\}$ and $b' = b \cup \{b_k\}$ for $g_j, b_k \in \{1, 2, \ldots, n\}$. We label the arrows of the quiver with the pair of numbers (g_j, b_k). We refer to g_j (respectively b_k) as the gray (respectively black) labels of the arrow.

Figure 3 shows $Q(w)$ for the wiring diagram pictured in Figure 1. To keep track of the geometric relationship between the wiring diagram and the quiver we define the height and position of a vertex of a quiver, which roughly describe the location in the double wiring diagram of the corresponding chamber label.

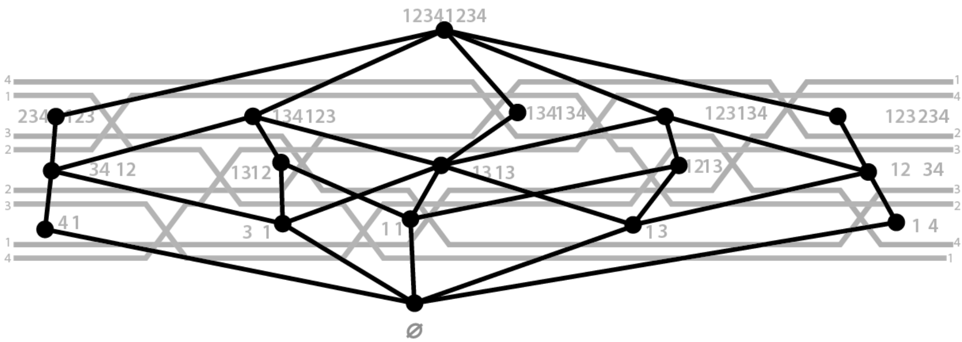

Figure 3. Quiver diagram.

Definition 3.2. Let v be a vertex of $Q(w)$ corresponding to the chamber label (g, b). We define the *height* of v to be the cardinality of g; $h(v) = |g|$. We define the position of v denoted $p(v) = \sum_{x \in b} x - \sum_{y \in g} y$.

Notice that the height increases the higher up one moves in the diagram while the position increases from left to right.

In order to construct Φ_n one needs to be able to identify the edges that are incident to a given vertex. This information is local in nature so we introduce language that allows us to discuss pieces of $Q(w)$.

Definition 3.3. A *subquiver* of $Q(w)$ is any subset of the vertices of $Q(w)$ together with a (possibly empty) set of arrows whose corresponding vertices are in the subset.

The following definitions provide the language needed to discuss the subquivers that are fundamental to the identifying braid moves.

Definition 3.4. A subquiver S of $Q(w)$ is *complete* if it contains every arrow of Q that connects two vertices of S.

Definition 3.5. A subquiver S of $Q(w)$ is *full* if, given that (g_1, b_1) and (g_2, b_2) are elements of S with height h, S contains the vertices corresponding to all chamber labels with height h and position between the positions of (g_1, b_1) and (g_2, b_2).

Notice that complete full subquivers completely determine a portion of a wiring diagram without missing arrows or vertices.

Using the language of complete full subquivers we can describe all of the edges that are incident to a vertex of Φ_n. Recall, each edge of Φ_n corresponds to a particular braid move centered at a particular chamber of the double wiring diagram.

Theorem 3.6. *There exists a* 3*-move centered at label* (g, b) *if and only if* $Q(w)$ *contains a complete, full subquiver of one of the two types shown in* Figure 4.

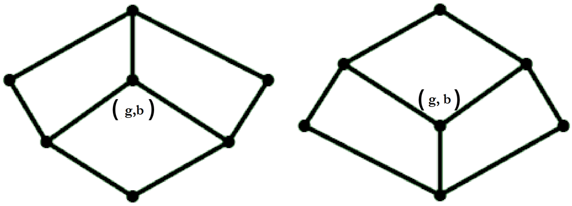

Figure 4. Subquivers for 3-move.

Proof. Assume a 3-move exists. Then there must be a region of the wiring diagram isomorphic to Figure 2 (right). Constructing the subquiver from this diagram yields Figure 4.

For the other direction, assume $Q(w)$ has a compete full subquiver isomorphic to Figure 4 (left). We examine the possible gray labels for this subquiver. Since the bottom vertex is connected to the top by a path of length three, we know that only three distinct edge labels may appear in this subquiver. We label the leftmost path from the bottom to the top that passes through (g, b), x, y, z as pictured in Figure 5.

For each four-cycle in Figure 5 only two distinct edge labels may be used since the bottom and top vertices are connected by a path of length two. This limits the potential labelings to those in Figure 6. The case of picture d) in the figure cannot exist because strings z and y are exchanged twice, which contradicts the definition of a double wiring diagram.

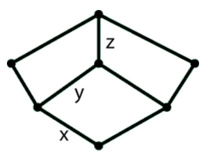

Figure 5. Grey labels for subquiver.

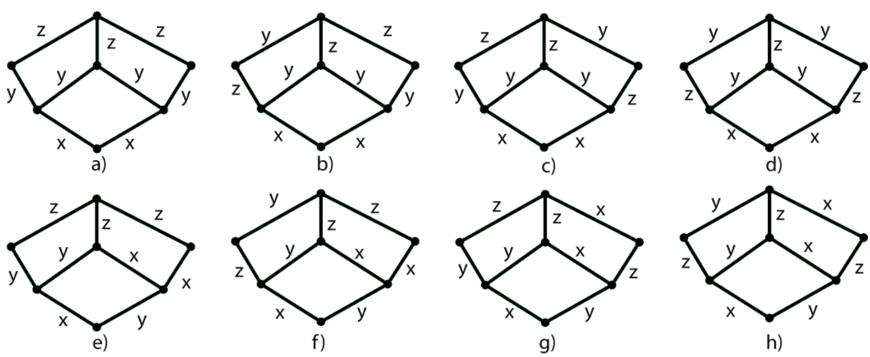

Figure 6. Possible labeled subquivers.

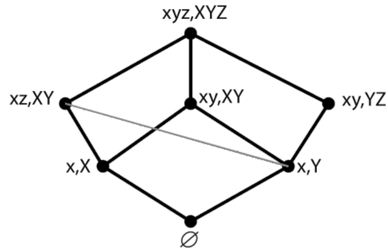

Figure 7. Invalid quiver labeling.

Repeat this argument for the black strings and label those cases A through H. We now determine which gray and black cases can be paired together. Since the labels must be distinct, the only potential pairs are (a, H), (b, G) and (c, F), and their opposites (h, A), (g, B) and (f, C).

If we draw a subquiver with the labels in the case (b, G), as in Figure 7, we recover an extra arrow, which contradicts the hypothesis that the subquiver was complete. The pairs (c, F), (g, B) and (f, C) are symmetric to (b, G), so they are also eliminated. This leaves only (a, H) and (h, A) as possible labelings.

By symmetry of the labelings we may assume the edge labels are of type (h, A). Since this subquiver is full, there are no missing vertices. This means that changes in chamber labels of the same cardinality indicate a unique braid crossing as pictured in Figure 8 (left). No other crossings may occur in this region because the quiver is complete. Therefore, the strings must connect without creating any other crossings. This yields the 3-move pictured in Figure 8 (right). The proof for Figure 4 (right) follows the same argument with reflected labels. □

Theorem 3.7. *There exists a* 2*-move centered at label* (g, b) *if and only if* $Q(w)$ *contains the full subquiver shown in* Figure 9*.*

Proof. Assume a 2-move exists. Then there must be a region of the wiring diagram isomorphic to Figure 2 (left). Constructing the quiver from this diagram yields the subquiver in Figure 9.

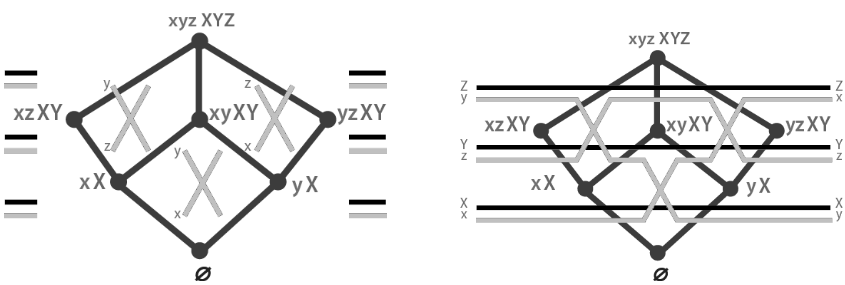

Figure 8. Reconstructed 3-move.

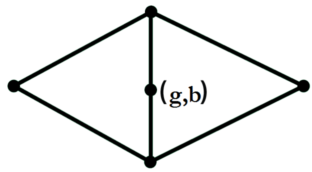

Figure 9. Subquiver for 2-move.

Now assume $Q(w)$ contains the full subquiver in Figure 9. We examine the possible gray labels for the subquiver. Label the arrows to and from (g, b) as x and y. Since there is a path from the bottom vertex to the top vertex of length two, all arrows in the subquiver must be labeled x or y. Figure 10 shows the possible labelings. The case corresponding to picture d) can be eliminated because it would require strings x and y to be exchanged twice.

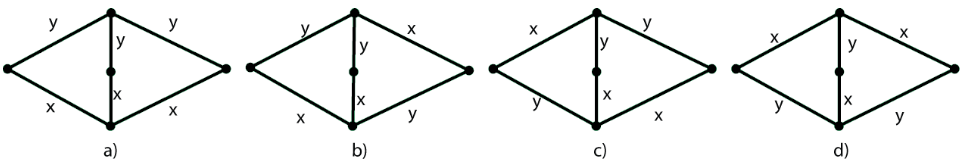

Figure 10. Quiver labelings.

We construct a similar pattern of possibilities for the black strings by labeling the arrows with X and Y. We need to determine which gray and black cases can be paired together. Since all of the labelings are distinct, the only potential pairs of cases are (b, C) and (c, B); see Figure 11.

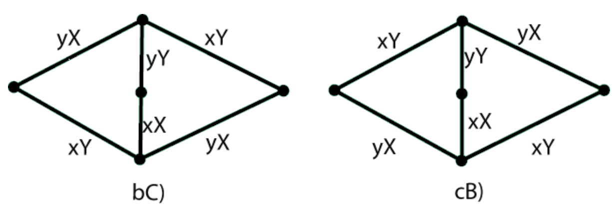

Figure 11. Potential quiver labelings.

As the labelings are symmetric, we can assume without loss of generality that the diagram has edge labels of type (b, C). Since this subquiver is full there are no missing vertices. This means that changes in chamber labels of the same height indicate a unique braid crossing as pictured in Figure 12 (left). Since no other crossings may occur in this region we connect the strings without creating any other crossings. Doing so yields the 2-move pictured in Figure 12 (right). $\square$

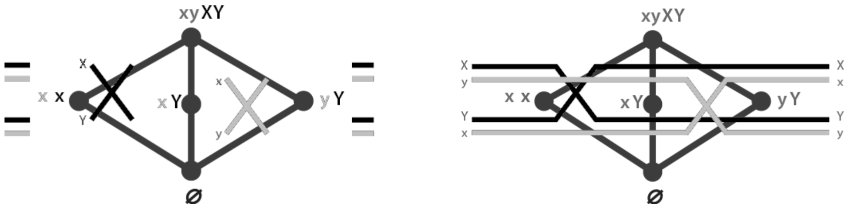

Figure 12. Reconstructed 2-move.

4. Describing Φ_n

Theorems 3.6 and 3.7 allow for the computation of the graph Φ_n for any n using the following algorithm:

(1) Choose an n-stringed double wiring diagram w with quiver $Q(w)$.

(2) Using Theorems 3.6 and 3.7 find and connect all vertices incident to the vertex corresponding to w.

(3) Repeat the previous step with the new set of vertices.

(4) Repeat this process until no new vertices can be added.

Since Φ_n is finite and connected this process will terminate and compute the entire graph. This process has been implemented in C++ using algorithms available at [Dukes 2011].

The smallest graph Φ_2 consists of two vertices connected by an edge. The graph of Φ_3 first appeared in [Fomin and Zelevinsky 2000]. Figure 13 shows a new

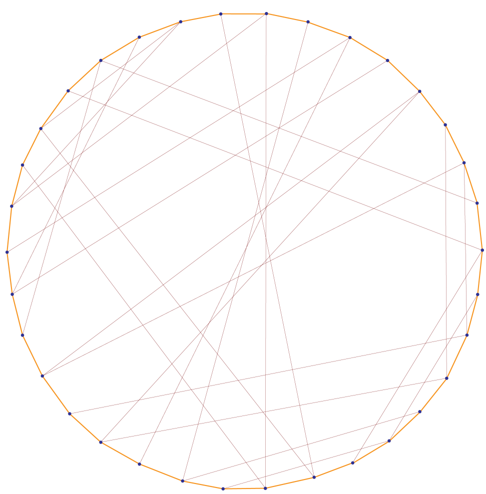

Figure 13. Φ_3 with Hamiltonian cycle highlighted.

	Φ_2	Φ_3	Φ_4	Φ_5
Vertices	2	34	4894	5520372
Edges	1	120	33300	60930112

Table 1. Φ_n edge and vertex data.

	1	2	3	4	5	6	7	8	9
Φ_2	2								
Φ_3			16	18					
Φ_4				2	522	1362	1754	1054	200

Table 2. Number of vertices of given degree for Φ_2 through Φ_4.

	6	7	8	9	10	11
Φ_5	84	28584	198596	632028	1165732	1402756
	12	13	14	15	16	
Φ_5	1165888	651188	227520	44452	3544	

Table 3. Number of vertices of given degree for Φ_5.

representation of Φ_3 which indicates the presence of a Hamiltonian path. Tables 1, 2 and 3 summarize information about Φ_n for $n \leq 5$.

5. Total positivity conjecture

With these explicit computations of Φ_n in hand, we return to the positivity conjecture of Fomin and Zelevinsky described in the introduction. We briefly review the setup of their work here. See [Fomin and Zelevinsky 2000] for a more complete description.

Definition 5.1. An $n \times n$ matrix M with entries in $\mathbb{R}$ is called *totally positive* if all minors of M are positive.

Definition 5.2 (Fomin and Zelevinsky). Let w be an n-stringed double wiring diagram. For each chamber label (b, g) of w we define the minor $\Delta_{g,b}$ to be the determinant of the matrix with rows of M corresponding to g and columns of M corresponding to b. We call the collection of all such minors the *chamber minors of w*.

Fomin and Zelevinsky [2000] proved that for any commutation class of double wiring diagrams w, a matrix M is totally positive if and only if all of its chamber minors are positive. In addition they conjectured that every minor could be written as a Laurent polynomial in the chamber minors with nonnegative coefficients.

Using our computations from Section 3, we can confirm this conjecture for $n \le 4$.

Theorem 5.3. *For $n \le 4$ and any n-stringed double wiring diagram w, every minor of an $n \times n$ matrix can be written as a Laurent polynomial with nonnegative coefficients in terms of the chamber minors of w.*

Proof. Fomin and Zelevinsky [2000] show that if w and w' are linked by a braid move as pictured in Figure 2, then their chamber minors satisfy the equation

$$AD + BC = XY. \tag{1}$$

Using the program Fermat [Lewis 2007] and a C++ program written by the first author, we verify Theorem 5.3 using the following algorithm:

(1) For each vertex $v \in \Phi_n$ and minor Δ, find a path from v to a vertex v' such that Δ is a chamber minor of v'. This is possible since Φ_n is connected and every minor appears as the chamber minor for some double wiring diagram.

(2) At each edge of this path use Fermat to compute the new minor as a Laurent polynomial in terms of the previous minors using Equation (1). The Laurent theorem [Fomin and Zelevinsky 1999] guarantees the result will be a Laurent polynomial in the chamber minors of v. Repeat the process until Δ is written as a Laurent polynomial in terms of the chamber minors of v.

(3) Verify that the corresponding Laurent polynomial has all positive coefficients.

The relevant code and data files can be found in [Dukes 2011]. □

Example 5.4. In this example we demonstrate that $\Delta_{14,12}$ can be written as a Laurent polynomial in the chamber minors of w as in the wiring diagram in Figure 1 with nonnegative coefficients. First, the diagrams in Figure 14 determine a path in Φ_4 from w to a vertex corresponding to a diagram with $\Delta_{14,12}$ as a chamber minor.

Each exchange along this path introduces a new chamber minor. Using (1), we compute the new chamber minor as a Laurent polynomial in terms of the chamber minors of w. The results of the Fermat computations of these Laurent polynomials are listed below.

$$\begin{aligned}
\Delta_{34,13} &= \frac{\Delta_{34,12}\Delta_{13,13} + \Delta_{134,123}\Delta_{3,1}}{\Delta_{13,12}} \\
&= \Delta_{134,123}\Delta_{13,12}^{-1}\Delta_{3,1} + \Delta_{34,12}\Delta_{13,12}^{-1}\Delta_{13,13}, \\
\Delta_{14,13} &= \frac{\Delta_{1,1}\Delta_{34,13} + \Delta_{4,1}\Delta_{13,13}}{\Delta_{3,1}} \\
&= \Delta_{134,123}\Delta_{13,12}^{-1} + \Delta_{34,12}\Delta_{13,12}^{-1}\Delta_{13,13}\Delta_{3,1}^{-1}\Delta_{1,1} + \Delta_{13,13}\Delta_{4,1}\Delta_{3,1}^{-1}, \\
\Delta_{14,12} &= \frac{\Delta_{14,13}\Delta_{34,12} + \Delta_{134,123}\Delta_{4,1}}{\Delta_{34,13}} = \Delta_{34,12}\Delta_{3,1}^{-1}\Delta_{1,1} + \Delta_{13,12}\Delta_{4,1}\Delta_{3,1}^{-1}.
\end{aligned}$$

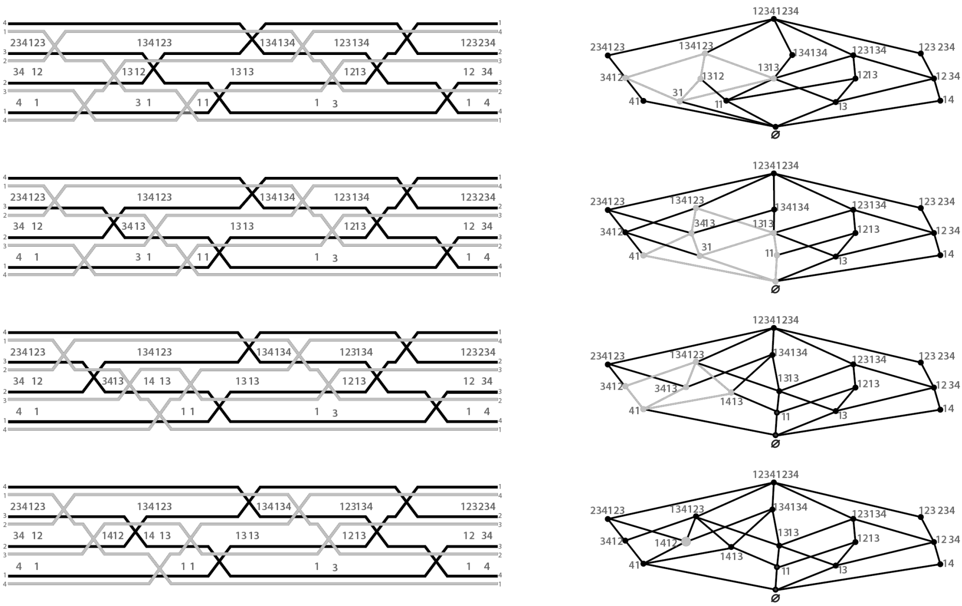

Figure 14. Path to a vertex with $\Delta_{14,12}$ as a chamber minor.

It suffices to observe that the coefficients in the expression of $\Delta_{14,12}$ in terms of the chamber minors of w, are all positive.

Remark 5.5. The example above is not indicative of the complexity of the computations. In Φ_4 the Laurent polynomials in the solution frequently had over 100 terms.

Although we were able to compute Φ_5 we were unable to confirm the conjecture for $n = 5$ because of number of computations required. There are $34 \times 14 = 476$ pairs of vertices and chamber minors in Φ_3, and $62 \times 4{,}894 = 303{,}420$ such combinations in Φ_4. To confirm the conjecture with brute force for $n = 5$ would require $242 \times 5{,}520{,}372 = 1{,}335{,}930{,}024$ computations each involving extremely large Laurent polynomials.

References

[Bédard 1999] R. Bédard, "On commutation classes of reduced words in Weyl groups", *European J. Combin.* **20**:6 (1999), 483–505. MR 2001i:05160 Zbl 0934.05126

[Carter and Marsh 2000] R. Carter and R. Marsh, "Regions of linearity, Lusztig cones, and canonical basis elements for the quantized enveloping algebra of type A_4", *J. Algebra* **234**:2 (2000), 545–603. MR 2001k:17019 Zbl 0999.17024

[Dukes 2011] P. Dukes, Double wiring diagram files, 2011, http://www.patrickdukes.com/double-wiring-diagram-files.html.

[Fomin and Zelevinsky 1999] S. Fomin and A. Zelevinsky, "Double Bruhat cells and total positivity", *J. Amer. Math. Soc.* **12**:2 (1999), 335–380. MR 2001f:20097 Zbl 0913.22011

[Fomin and Zelevinsky 2000] S. Fomin and A. Zelevinsky, "Total positivity: tests and parametrizations", *Math. Intelligencer* **22**:1 (2000), 23–33. MR 2001b:15030 Zbl 1052.15500

[Fomin and Zelevinsky 2002] S. Fomin and A. Zelevinsky, "The Laurent phenomenon", *Adv. in Appl. Math.* **28**:2 (2002), 119–144. MR 2002m:05013 Zbl 1012.05012

[Lewis 2007] R. Lewis, "Fermat: a computer algebra system for polynomial and matrix computation", 2007, http://home.bway.net/lewis.

Received: 2011-08-30 Revised: 2011-09-03 Accepted: 2011-09-14

pdukes3@gmail.com *School of Computing, Clemson University, Clemson, SC 29634, United States*

rusinkoj@winthrop.edu *Department of Mathematics, Winthrop University, Rock Hill, SC 29733, United States*

involve

msp.berkeley.edu/involve

See inside back cover or http://msp.berkeley.edu/involve for submission instructions.

The subscription price for 2012 is US $105/year for the electronic version, and $145/year (+$35 shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94704-3840, USA.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, Department of Mathematics, University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW™ from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

http://msp.org/

A NON-PROFIT CORPORATION

Typeset in LaTeX

involve

2012 vol. 5 no. 2