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Boolean elements in the Bruhat order on twisted involutions

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We prove that a permutation in the Bruhat order on twisted involutions is Boolean if and only if it avoids the following patterns: 4321, 3421, 4312, 4231, 32541, 52143, 351624, 456123, 426153, 321654, 561234, 345612, 3416275, 3561274, 1532746, 4517236, 34127856, 35172846, and 36712845. This result answers a question proposed by Hultman and Vorwerk. Our technique provides an application of the pictorial representation of the Bruhat order given by Incitti.

1. Introduction

In this paper, we answer the following question proposed by Hultman and Vorwerk [2009, Problem 5.1].

Problem. A permutation $w \in \mathfrak{S}_n$ is said to be a *twisted involution* if ww_0 is an involution, where $w_0 = n, n-1, \ldots, 1$. Let $Tw(\mathfrak{S}_n)$ denote the Bruhat order on twisted involutions. With pattern avoidance, classify all twisted involutions whose principal order ideal in $Tw(\mathfrak{S}_n)$ is Boolean.

We first define some requisite terms in the problem statement (see [Björner and Brenti 2005] for background reading).

Definition. Let

$$l(w) = |\{\, \{i, j\} : i < j \text{ and } w(i) > w(j)\}|$$

denote the number of inversions of w. The *Bruhat order of the symmetric group*, denoted by $Br(\mathfrak{S}_n)$, is a partial order on $\mathfrak{S}_n$ defined as follows: w covers w' if and only if $l(w) = l(w') + 1$ and w is obtained from w' by a transposition of $w'(i)$ and $w'(j)$ for some $1 \le i, j \le n$.

MSC2010: 05E15.
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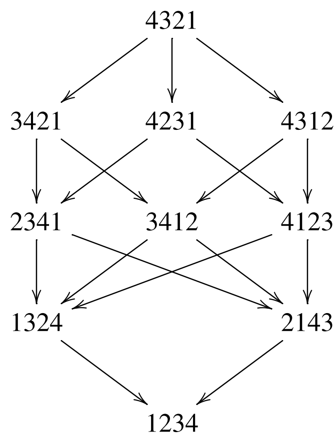
Definition. The *Bruhat order on twist involutions*, denoted by $Tw(\mathfrak{S}_n)$, is the poset on twisted involutions defined by $u \leq v$ in $Tw(\mathfrak{S}_n)$ if and only if $u \leq v$ in $Br(\mathfrak{S}_n)$.

Let Q be a poset. The *principal order ideal* of $w \in Q$, denoted by $P_Q(w)$ (or $P(w)$ when the context is clear), is the subposet of Q induced by the set of elements less than or equal to w. The Boolean poset B_k is the poset on the subsets of $\{1, 2, \ldots, k\}$ partially ordered by inclusion. A twisted involution w is said to be *Boolean* if its principal order ideal in $Tw(\mathfrak{S}_n)$ is isomorphic to a Boolean poset.

Example. The Boolean elements of $Tw(\mathfrak{S}_4)$ are 2341, 3412, 4123, 1324, 2143, and 1234.



We now present a brief history of this problem. Tenner [2007] brought pattern avoidance to the study of the Bruhat order. She classified Boolean elements of $Br(\mathfrak{S}_n)$ as 321- and 3412-avoiding permutations. Hultman and Vorwerk [2009] studied an analogue of Tenner's result for involutions: 4321-, 45312-, and 456123-avoiding permutations. At the end of [Hultman and Vorwerk 2009], the authors asked whether a similar result exists for twisted involutions. We settle this question with the following theorem.

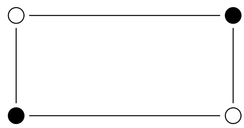
Theorem 1.1. *A twisted involution is Boolean if and only if it avoids all forbidden patterns. The forbidden patterns are* 4321, 3421, 4312, 4231, 32541, 52143, 351624, 456123, 426153, 321654, 561234, 345612, 3416275, 3561274, 1532746, 4517236, 34127856, 35172846, *and* 36712845.

As a side note, the poset $Tw(\mathfrak{S}_n)$ is isomorphic to the dual of $I(\mathfrak{S}_n)$, the Bruhat order on involutions. Consequently, our result also characterizes Boolean principal order filters of $I(\mathfrak{S}_n)$.

Previous work [Hultman and Vorwerk 2009; Tenner 2007] relied heavily on the algebraic properties of Coxeter groups. We prove Theorem 1.1 using permutation diagrams that represent the cover relations in the Bruhat order. Such diagrams were introduced by Incitti [2003; 2004; 2005].

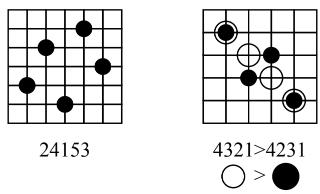
Let $w \in \mathfrak{S}_n$. The *permutation diagram* of w is the set of lattice points $(i, w(i))$, where $1 \leq i \leq n$. Permutation diagrams of twisted involutions of $\mathfrak{S}_n$ are symmetric about $x + y = n + 1$.

Incitti [2005] shows that w covers w' in $Br(\mathfrak{S}_n)$ if and only if their permutation diagrams differ by the following rectangle.



White dots belong to w and black to w', and no points lie inside the rectangle. Call the above rectangle a *cover block*.

Example. Left is the permutation diagram of 24153. The picture to the right shows that 4321 covers 4231 in $Br(\mathfrak{S}_4)$.



Incitti [2004] classifies the cover relations of $I(\mathfrak{S}_n)$ with six types of cover blocks. Reflecting them about the line $x = (n+1)/2$ gives us the cover blocks of $Tw(\mathfrak{S}_n)$ (see Section 3).

The key idea of our proof of Theorem 1.1 is to examine pairs of cover blocks on the same permutation diagram. To illustrate our technique, we start with an alternative and arguably simpler proof of Tenner [2007, Theorem 5.3] (see Section 2). In particular, we remove the need for Tenner's characterization of vexillary permutations [Tenner 2006, Theorem 3.8].

Section 3 contains the proof of Theorem 1.1. We discuss further directions in Section 4.

The Bruhat order on Coxeter groups is an extensively studied subject in combinatorics (see [Björner and Brenti 2005]). Following the work of Richardson and Springer [1990], there has been a surge of interest in the Bruhat order on twisted involutions because of its application to algebraic geometry and its resemblance to the Bruhat order on the symmetric group. For example, Hultman [2005; 2007] showed that $Tw(\mathfrak{S}_n)$ satisfies the deletion property and the subword property known for $Br(\mathfrak{S}_n)$. The similarity between $Tw(\mathfrak{S}_n)$ and $Br(\mathfrak{S}_n)$ inspired Hultman and Vorwerk [2009] to propose this problem.

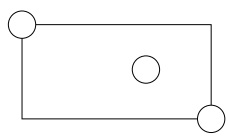
2. Boolean elements of the symmetric group

In this section, we give an alternative proof of Tenner [2007, Theorem 5.3] to illustrate our approach to Theorem 1.1.

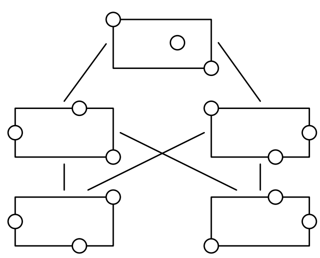
Definition. Elements of a poset are called *vertices* and cover relations *edges*. An edge uv is called an *upward move from* u if v covers u. Define a *downward move* similarly. Two edges are said to *commute uniquely* if they lie on a unique 4-cycle.

Theorem [Tenner 2007, Theorem 5.3]. *A permutation $w \in Br(\mathfrak{S}_n)$ is Boolean if and only if w is* 321- *and* 3412-*avoiding.*

Proof of necessity. Suppose w contains a 321- or 3412-pattern. Let u denote the minimal element of $P(w)$ that contains a 321- or 3412-pattern. Then the permutation diagram of u contains the following figure, where no other points lie inside the rectangle.



The two downward edges from u that act on this rectangle do not commute uniquely, as shown below.



Therefore, w is not Boolean. □

Proof of sufficiency. We start with the following lemma.

Lemma 2.1. *Let P be a graded and connected poset. If every pair of edges that share a vertex in P commute uniquely, then P is a Boolean poset.*

Proof. Let M denote the maximal element of P (the maximum exists because upward moves commute uniquely). Suppose M covers $m_1, m_2, \ldots, m_k$. Define $f(u) = \{i : m_i \geq u\}$. We prove that f bijectively maps the i-th row of P to the i-th row of B_k using strong induction on i. Base case is trivial.

Suppose f bijectively maps the first i rows of P to the first i rows of B_k. Let u be an element in the $(i+1)$st row. We claim that an element v covers u if and only if $f(v) = f(u)\backslash x$ for some $x \in f(u)$.

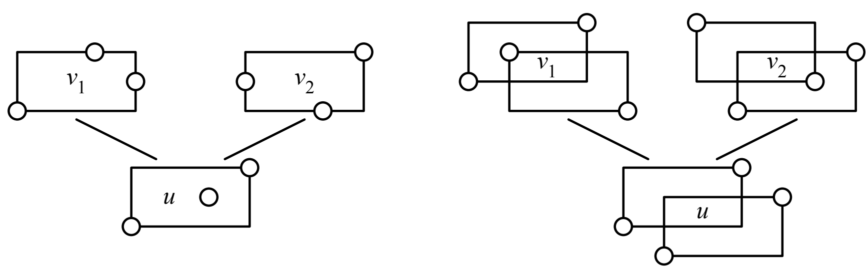
If u is covered with $v_1, v_2, \ldots, v_j$, then $f(u) = \bigcup_{1\le a\le j} f(v_a)$. Since for all $1 \le a, b \le j$, the upward moves uv_a and uv_b commute uniquely, we have $f(v_a)$ and $f(v_b)$ differ by exactly one element. Therefore, for all $1 \le a \le j$, we have $f(v_a) = f(u)\backslash x$ for some $x \in f(u)$. Conversely, suppose $f(v) = f(u)\backslash x$ for some $x \in f(u)$. Let

$$v' := f^{-1}(f(v_j) \cap f(v)).$$

Since $v_j v'$ and $v_j u$ commute uniquely, there exists a v'' in the i-th row such that $v'v''uv_j$ is a four cycle. Then $f(v'') = f(u)\backslash x$. Since the i-th row of P is isomorphic to the i-th row of B_k, we have $v = v''$.

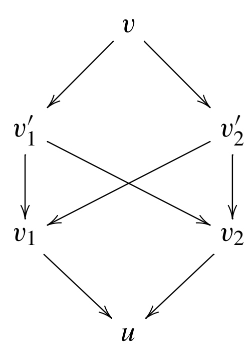
Therefore, f maps the $(i+1)$st row of P to the $(i+1)$st row of B_k, and P is isomorphic to the Boolean post B_k. □

Suppose w is 321- and 3412-avoiding. It is easy to check that all $u \le w$ are 321- and 3412-avoiding. If u is 321- and 3412-avoiding, then all pairs of downward moves from u commute uniquely. An upward move from u commute uniquely with a downward move from u. Suppose two upward moves uv_1 and uv_2 do not commute uniquely, then these two moves must be applied to one of the following figures.



In the right diagram, the element greater than both v_1 and v_2 must contain a 321-pattern, so only one of v_1 or v_2 belongs to $P(w)$.

In the left diagram, there exist v_1' and v_2' that cover both v_1 and v_2. Let v be the element that covers v_1' and v_2'. Then vv_1' and vv_2' are two downward edges that do not commute uniquely, as shown below.



Thus, one of v'_1 and v'_2 does not belong to $P(w)$, and uv_1 and uv_2 do commute uniquely in $P(w)$.

Therefore, all pairs of edges commute uniquely in $P(w)$, and w is Boolean by Lemma 2.1. □

Remark. The key idea of the proof of necessity is to identify a pair of downward moves that do not commute uniquely. The proof of sufficiency follows from Lemma 2.1.

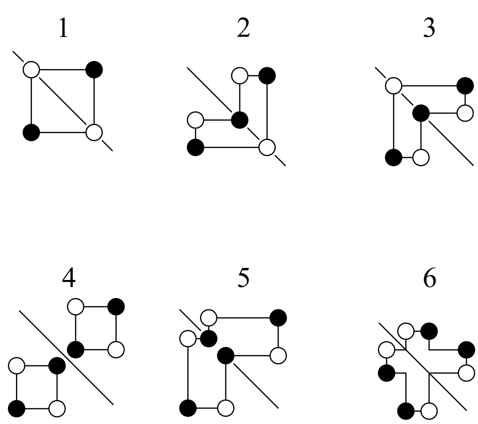
We can also use this idea to prove Hultman and Vorwerk [2009, Theorem 1.1], with the caveat that there are six types of cover blocks in the Bruhat order on involutions.

3. Proof of the main theorem

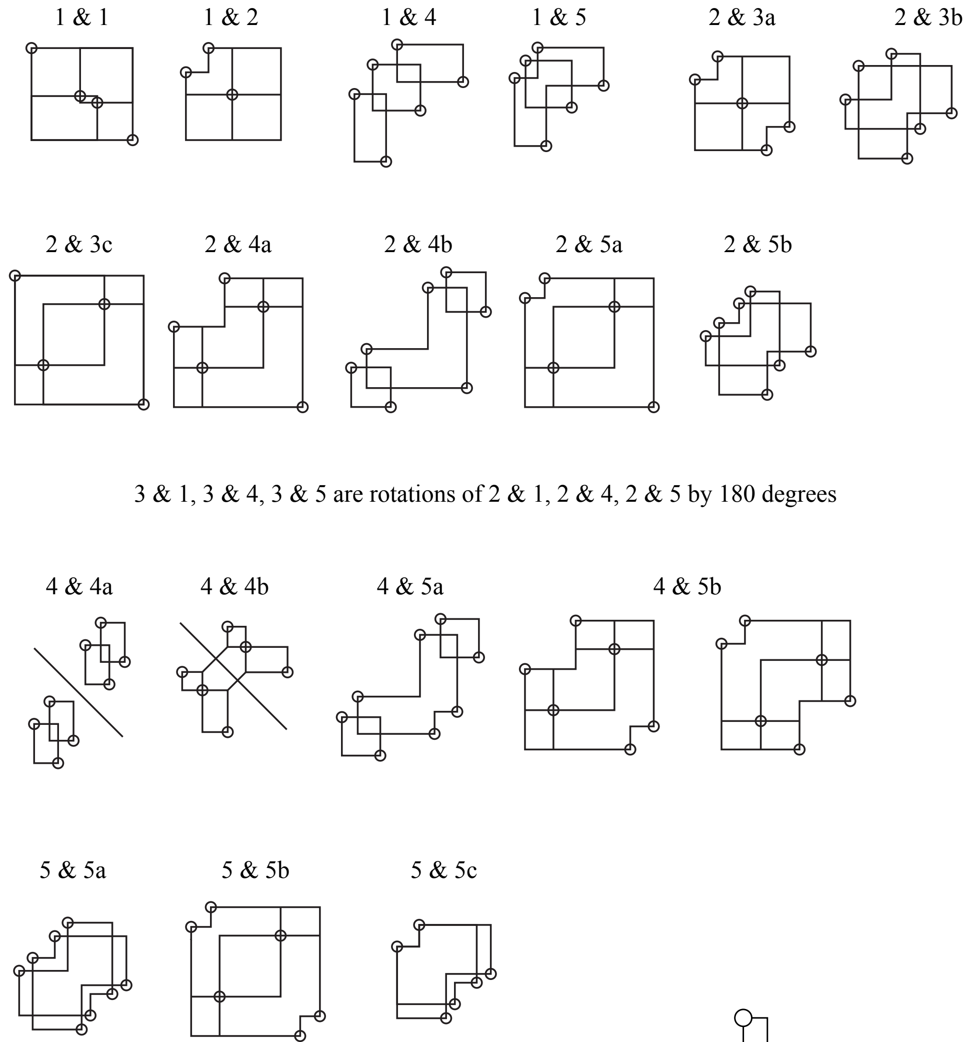
We wish to apply the same technique as the previous section, so we need to identify pairs of edges that do not commute uniquely.

We first classify the cover blocks of the Bruhat order on twisted involutions. Incitti [2004] characterizes the six types of cover blocks of the Bruhat order on involutions. Since permutation diagrams of twisted involutions are equivalent to those of involutions reflected about $x = (n + 1)/2$, we obtain the following characterization of cover relations of $Tw(\mathfrak{S}_n)$.

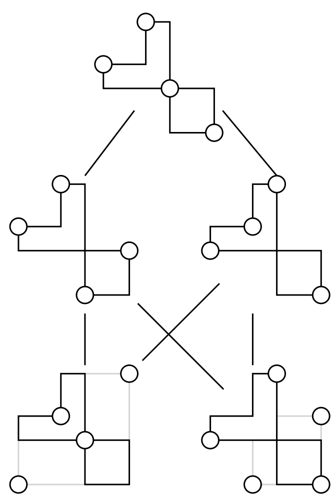
Definition. Let w and w' denote two twisted involutions. We have w covers w' if and only if their permutation diagrams differ by one of the following *cover blocks*. The white dots belong to w and black to w', and no points lie inside these shapes.



The six types of cover blocks induce fifteen types of intersecting pairs. The pairs of downward moves that do not commute uniquely define the *forbidden pairs* shown below, where no points lie inside the area enclosed by the lines.



These forbidden pairs will give us the forbidden patterns in Theorem 1.1. Note that cover relation 6 never appears in the forbidden pairs because any forbidden pair with relation 6 induces a forbidden pattern already contained in the earlier ones.

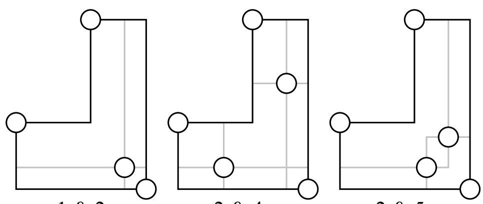


Example. Two downward moves applied to the forbidden pair 1&2 do not commute uniquely. (See figure on the right.)

We now prove Theorem 1.1.

Proof of necessity. Suppose w contains one of the forbidden patterns. If this forbidden pattern in the permutation diagram of w is not symmetric about $x+y=n+1$, then we can treat the corresponding points as the full Bruhat order. Since all forbidden patterns contain either 321 or 3412, the permutation cannot be Boolean.

We now assume that the forbidden pattern is symmetric about $x+y=n+1$. Let u denote the minimal element of $P(w)$ that contains a forbidden pattern. Then the permutation diagram of u contains a forbidden pair. (The forbidden pattern of u with the smallest area in the permutation diagram cannot contain other points. For example, if there are other points inside cover block 2, then we obtain a forbidden pattern with smaller area, as shown below.)



1 & 2 2 & 4a 2 & 5a

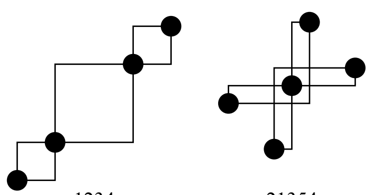
The two downward moves applied to this forbidden pair of u do not commute uniquely. Thus, w is not Boolean. □

Proof of sufficiency. To apply Lemma 2.1, we check two things:

1. If w avoids all forbidden patterns, then any $u \leq w$ also avoids all forbidden patterns.
2. If u avoids all forbidden patterns, then all pairs of edges emanating from u commute uniquely.

Any $u \leq w$ avoids all forbidden patterns because, after every downward move, the number of cover blocks and the rank of u decrease by exactly one, which means no new cover block can be created.

If u avoids all forbidden patterns, then two downward moves from u commute uniquely. We can check that an upward move and a downward move always commute (this also follows from the lexicographical shellability of $Tw(\mathfrak{S}_n)$). The pairs of upward moves that do not commute uniquely induce patterns 1234, 21354, and 321654.



1234 21354

We have 321654 is a forbidden pattern. For the other two patterns, the element that covers both end points of an upward move contains a element with a forbidden pattern. Therefore, only one of the endpoints is contained in $P(w)$, and all pairs of upward moves do commute uniquely in $P(w)$. Lemma 2.1 shows that w is indeed Boolean. □

4. Further remarks

Our result provides an application of Incitti's pictorial representation of the Bruhat order [Incitti 2004]. Incitti [2003; 2004; 2005] also classify representations of cover relations for the Bruhat order on Coxeter groups of types B and D as well as involutions in these groups.

Question. What is the analogue of our result for Coxeter groups of types B and D?

Green and Losonczy [2002] classify Boolean elements of the poset on commutation classes of reduced decompositions: 4321-, 4231-, 4312-, and 3421-avoiding permutations. The author's recent work [Meng $\geq$ 2012] generalizes Green and Losonczy's work to the higher Bruhat order. Using Incitti's pictures, we can show that $Br(\mathfrak{S}_n)$ is generated by 4-cycles. Compare this with the fact that the higher Bruhat order $B(n, 2)$ is generated by 4-cycles and 8-cycles, we believe that studying the similarity between the strong Bruhat order and the higher Bruhat order is worthwhile.

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References

[Björner and Brenti 2005] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics **231**, Springer, New York, 2005. MR 2006d:05001 Zbl 1110.05001

[Green and Losonczy 2002] R. M. Green and J. Losonczy, "Freely braided elements of Coxeter groups", *Ann. Comb.* **6**:3-4 (2002), 337–348. MR 2004d:20042 Zbl 1052.20028

[Hultman 2005] A. Hultman, "Fixed points of involutive automorphisms of the Bruhat order", *Adv. Math.* **195**:1 (2005), 283–296. MR 2006a:06001 Zbl 1102.06002

[Hultman 2007] A. Hultman, "The combinatorics of twisted involutions in Coxeter groups", *Trans. Amer. Math. Soc.* **359**:6 (2007), 2787–2798. MR 2007k:20082 Zbl 1166.20030

[Hultman and Vorwerk 2009] A. Hultman and K. Vorwerk, "Pattern avoidance and Boolean elements in the Bruhat order on involutions", *J. Algebraic Combin.* **30**:1 (2009), 87–102. MR 2011c:06009 Zbl 1225.06002

[Incitti 2003] F. Incitti, "The Bruhat order on the involutions of the hyperoctahedral group", *European J. Combin.* **24**:7 (2003), 825–848. MR 2004h:05128 Zbl 1056.20027

[Incitti 2004] F. Incitti, "The Bruhat order on the involutions of the symmetric group", *J. Algebraic Combin.* **20**:3 (2004), 243–261. MR 2005h:06003 Zbl 1057.05079

[Incitti 2005] F. Incitti, "Bruhat order on classical Weyl groups: minimal chains and covering relation", *European J. Combin.* **26**:5 (2005), 729–753. MR 2006b:06005 Zbl 1083.20036

[Meng $\geq$ 2012] D. Meng, "Reduced decompositions and permutation patterns generalized to the higher Bruhat order", http://web.mit.edu/delong13/papers.html. Submitted.

[Richardson and Springer 1990] R. W. Richardson and T. A. Springer, "The Bruhat order on symmetric varieties", *Geom. Dedicata* **35**:1-3 (1990), 389–436. MR 92e:20032 Zbl 0704.20039

[Tenner 2006] B. E. Tenner, "Reduced decompositions and permutation patterns", *J. Algebraic Combin.* **24**:3 (2006), 263–284. MR 2007f:05008 Zbl 1101.05003

[Tenner 2007] B. E. Tenner, "Pattern avoidance and the Bruhat order", *J. Combin. Theory Ser. A* **114**:5 (2007), 888–905. MR 2008d:05164 Zbl 1146.05054

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