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Candice Nielsen

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Vertex polygons

Candice Nielsen

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We look at hexagons whose vertex triangles have equal area, and identify necessary conditions for these hexagons to also have vertex quadrilaterals with equal area. We discover a method for creating a hexagon whose vertex quadrilaterals have equal area without necessarily having vertex triangles of equal area. Finally, we generalize the process to build any polygon with an even number of sides to have certain vertex polygons with equal area.

1. Introduction

In the article "Polygons whose vertex triangles have equal area," Harel and Rabin [2003] discuss the properties of polygons with the very special characteristic described in the title. To clarify, the authors offer the following definitions:

Definition 1. A triangle formed using three adjacent vertices of any polygon is called a *vertex triangle*.

Definition 2. A polygon $V_1 V_2 \cdots V_n$ for which all vertex triangles have the same nonzero area is called an *equal-area polygon*.

Harel and Rabin take an algebraic approach, assigning direction and magnitude to each side of the polygon. In this article, we take a geometric approach, using area formulas and triangle congruencies to identify properties of certain polygons.

To extend from triangles, we offer the following definitions:

Definition 3. A polygon of n sides, formed using n adjacent vertices of any m-sided polygon (with $m \geq n$), is called a *vertex n-gon*.

Definition 4. A polygon $V_1 V_2 \cdots V_m$ for which all vertex n-gons have the same nonzero area is called an *equal-n-gon polygon*.

It is clear that every equal-area quadrilateral is also an equal-quadrilateral polygon, since any vertex quadrilateral is the whole quadrilateral. Furthermore, every equal-area pentagon is an equal-quadrilateral polygon because, for P equal to the area of

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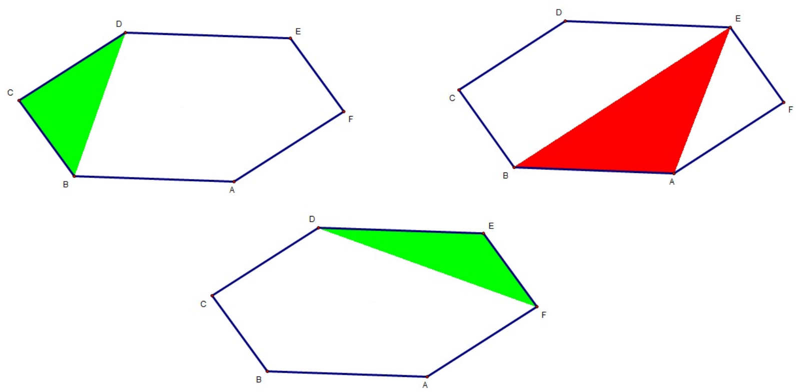

Figure 1. $\triangle BCD$ and $\triangle DEF$ are vertex triangles of hexagon $ABCDEF$, but $\triangle BEA$ is not.

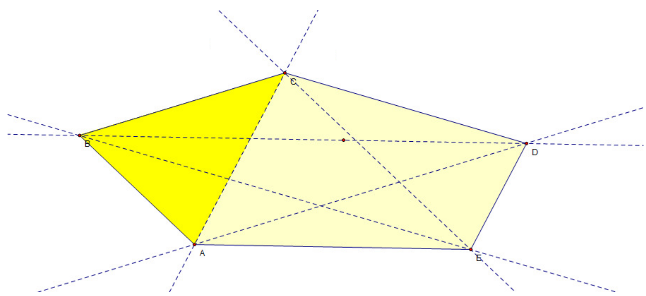

Figure 2. An equal-area pentagon is always an equal-quadrilateral pentagon.

the pentagon, and T equal to the area of any vertex triangle, the area of every vertex quadrilateral is equal to $P - T$ (Figure 2). This means every equal-quadrilateral pentagon is also an equal-area pentagon. For this reason, we begin with hexagons.

2. Equal-area hexagons

The first nontrivial case of the equal-area and equal-quadrilateral polygon is the hexagon. The first task is to construct an equal-area hexagon. We can show that, for any equal-area polygon $V_1 V_2 \cdots V_n$, the line $V_i V_{i+1}$ is parallel to the line $V_{i-1} V_{i+2}$. In other words, each side is parallel to the line formed by the surrounding two vertices.

Proof. Let $V_1 V_2 \cdots V_n$ be an equal-area polygon. Then $\text{Area}(\triangle V_{i-1} V_i V_{i+1}) = \text{Area}(\triangle V_i V_{i+1} V_{i+2})$. Let $b = V_i V_{i+1}$, $h_1 = d(V_{i-1}, V_i V_{i+1})$, $h_2 = d(V_{i+2}, V_i V_{i+1})$. So $\text{Area}(\triangle V_{i-1} V_i V_{i+1}) = \frac{1}{2} b h_1 = \frac{1}{2} b h_2 = \text{Area}(\triangle V_i V_{i+1} V_{i+2})$. Therefore, $h_1 = h_2$ and $V_{i-1} V_{i+2}$ is parallel to $V_i V_{i+1}$. $\square$

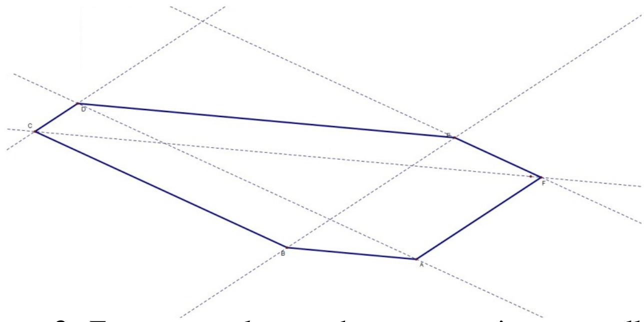

Figure 3. Every equal-area hexagon enjoys parallelism betweenopposite sides and corresponding main diagonals.

With this property, an equal-area hexagon can be uniquely determined by any trapezoid. As we build an equal-area hexagon, it is important to note that the diagonals of the hexagon need not intersect at a single point. This is a key observation as we transform the equal-area hexagon into an equal-quadrilateral hexagon.

Using the hexagon *CDEFGH* in Figure 3, certain geometric properties arise. First, the sides of the hexagon, along with the diagonals, divide the hexagon into four triangles and three trapezoids. Let us define these as follows:

Definition 5. Let *ABCDEF* be any hexagon, with $\overline{AD} \cap \overline{BE} = J$, $\overline{BE} \cap \overline{CF} = L$, $\overline{CF} \cap \overline{AD} = K$. The triangle $\triangle JKL$ is called the *center triangle*. The triangles $\triangle ABJ$, $\triangle CKD$, and $\triangle ELF$ are called *interior triangles*, and *BCKJ*, *DELK*, and *FAJL* are called *interior trapezoids* (see Figure 4).

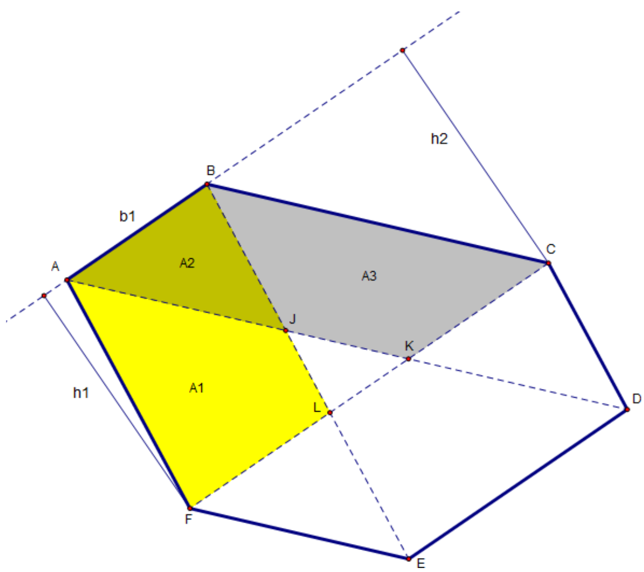

Figure 4. Equal-area hexagon *ABCDEF* and two of the associated trapezoids.

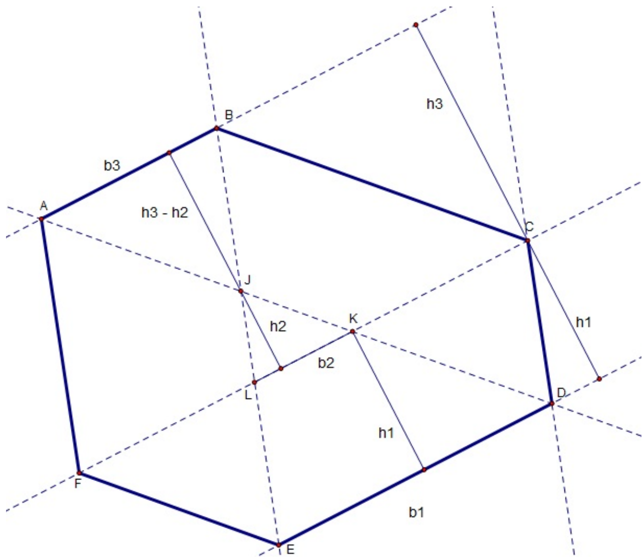

Figure 5. Equal-area, equal-quadrilateral hexagon $ABCDEF$.

Lemma. *For an equal-area hexagon, each interior trapezoid has the same nonzero area and each interior triangle has the same nonzero area.*

Proof. Let $ABCDEF$ be an equal-area hexagon, with $\overline{AD} \cap \overline{BE} = J$, $\overline{BE} \cap \overline{CF} = L$, $\overline{CF} \cap \overline{AD} = K$ (see Figure 5). From the previous proof, $\overline{AF} \parallel \overline{BE}$ and $\overline{AB} \parallel \overline{FL}$, so $ABLF$ is a parallelogram. Likewise, $\overline{AK} \parallel \overline{BC}$ and $\overline{AB} \parallel \overline{CK}$, so $ABCK$ is a parallelogram, and the parallelograms share a base, $\overline{AB}$. Let $b_1 = AB$, $h_1 = \text{height}(ABLF) = \text{height}(\triangle BAF)$, and $h_2 = \text{height}(ABCK) = \text{height}(\triangle ABC)$. Then $\text{Area}(ABLF) = b_1 h_1 = 2\text{Area}(\triangle BAF)$ and $\text{Area}(ABCK) = b_1 h_2 = 2\text{Area}(\triangle ABC)$. Since $ABCDEF$ is an equal-area hexagon, we have $\text{Area}(\triangle BAF) = \text{Area}(\triangle ABC)$, so $\text{Area}(ABLF) = \text{Area}(ABCK)$. Let $A_1 = \text{Area}(AJLF)$, $A_2 = \text{Area}(\triangle ABJ)$, and $A_3 = \text{Area}(BCKJ)$. Then $\text{Area}(ABLF) = A_1 + A_2$ and $\text{Area}(ABCK) = A_2 + A_3$. This implies that $A_1 = A_3$. Similar argument supports that all interior trapezoids have the same nonzero area, as do all interior triangles. $\square$

Definition 6. For any integer $n > 1$ and any polygon having n sides with vertices $V_1, V_2, \dots, V_{2n}$, a *true diagonal* has endpoints V_i and V_{i+n}, where $i \in \{1, 2, \dots, n\}$.

Theorem 1. *An equal-area hexagon is equal-quadrilateral if and only if all its true diagonals intersect at a single point.*

Proof. Let $ABCDEF$ be an equal-area, equal-quadrilateral hexagon, with the following properties: $\overline{AD} \cap \overline{BE} = J$, $\overline{BE} \cap \overline{CF} = L$, $\overline{CF} \cap \overline{AD} = K$. Suppose, for the sake of

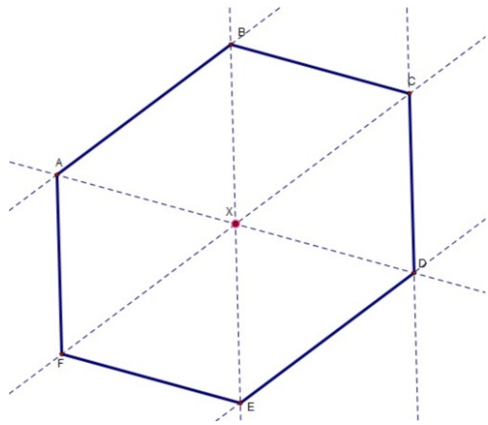

Figure 6. Equal-area hexagon *ABCDEF* with main diagonals intersecting.

contradiction, that J, K, and L are three distinct points. Let A_1 be the area of the interior triangles, A_2 be the area of the interior trapezoids, and A_c be the area of the center triangle. Consider the vertex quadrilaterals *ABCD* and *BCDE*. Since *ABCDEF* is an equal-quadrilateral hexagon, the areas of the vertex quadrilaterals are equal to each other. Thus, $\text{Area}(ABCD) = 2A_1 + A_2 = A_1 + 2A_2 + A_c = \text{Area}(BCDE)$.

Let $b_1 = DE$, and let h_1 equal the height of trapezoid *ELKD*, which is equal to the height of vertex triangle *EDC*.

Let $b_2 = LK$, and let h_2 equal the height of center triangle JKL.

Let $b_3 = AB$ and let h_3 equal the height of vertex triangle ABC, so the height of interior triangle ABJ is $h_3 - h_2$. Then $2A_1 + A_2 = A_1 + 2A_2 + A_c$ implies

$$2\big(\tfrac{1}{2}b_3(h_3 - h_2)\big) + \tfrac{1}{2}(b_1 + b_2)h_1 = \tfrac{1}{2}b_3(h_3 - h_2) + 2\big(\tfrac{1}{2}(b_1 + b_2)h_1\big) + \tfrac{1}{2}b_2h_2$$

This simplifies to

$$b_2h_2 + b_1h_1 + b_2h_1 = b_3h_3 - b_3h_2. \tag{1}$$

Since *ABCDEF* is equal-area, the vertex triangles have the same area, and $\frac{1}{2}b_1h_1 = \frac{1}{2}b_3h_3$, so $b_1h_1 = b_3h_3$ and (1) becomes

$$b_2h_2 + b_2h_1 = -b_3h_2. \tag{2}$$

Since b_2h_2, b_2h_1, b_3h_2 are all positive values, this is a contradiction. Therefore, $J = K = L$, and the diagonals of *ABCDEF* intersect at a single point.

For the other direction, let *ABCDEF* be an equal-area hexagon, satisfying $\overline{AD} \cap \overline{BE} \cap \overline{CF} = X$.

Without loss of generality, consider $\triangle ABX$. Since the height of $\triangle ABX$ is equal to the height of $\triangle ABC$, their areas are equal. Thus, the area of each interior triangle is the area of a vertex triangle. Since all vertex triangles share an equal area, so do the interior triangles. Each vertex quadrilateral is made up of three interior triangles, so each vertex quadrilateral shares an equal area. Therefore, $ABCDEF$ is an equal-quadrilateral hexagon. □

3. Equal-quadrilateral hexagons

While constructing an equal-quadrilateral hexagon out of an equal-area hexagon is helpful, the question arose: if a hexagon is equal-quadrilateral, is it necessarily equal-area? We are able to observe, through interior triangle congruencies, that the intersection of the three diagonals is the midpoint of each diagonal. Since the three diagonals are diameters of three concentric circles, we have a new way to construct the equal-quadrilateral hexagon.

Theorem 2. *A hexagon whose true diagonals are diameters of concentric circles is an equal-quadrilateral hexagon.*

Proof. Let AD, BE, CF be diameters of three concentric circles with center X and also be diagonals of hexagon $ABCDEF$ (see Figure 7). Without loss of generality, consider $ABCD$ and $BCDE$. We have $ABCD \cap BCDE = BCDX$. We also have $EX = XB$ and $AX = XD$ because they are radii of the same respective circles. Furthermore, $\angle EXD \cong \angle BXA$ because they are vertical angles. Thus, by the side-angle-side condition, $\triangle EXD \cong \triangle BXA$. Since $BCDX$ is congruent to itself, $ABCD$ and $BCDE$ are congruent, and therefore share an equal, nonzero area. With this argument, every vertex quadrilateral of $ABCDEF$ shares the same, nonzero area. Thus, $ABCDEF$ is an equal-quadrilateral hexagon. □

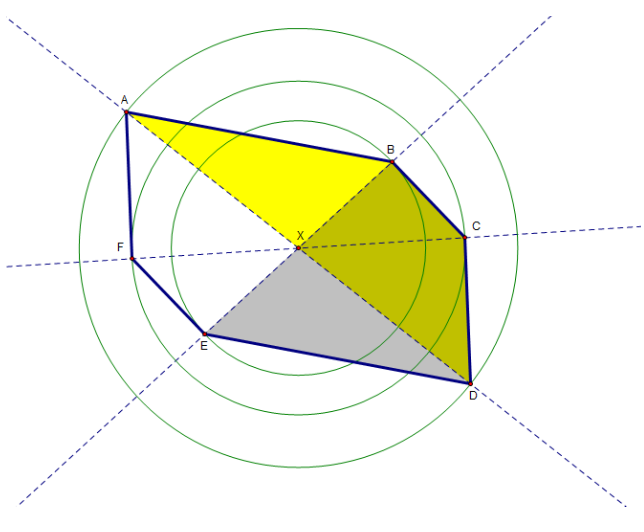

Figure 7. Equal-quadrilateral hexagon $ABCDEF$.

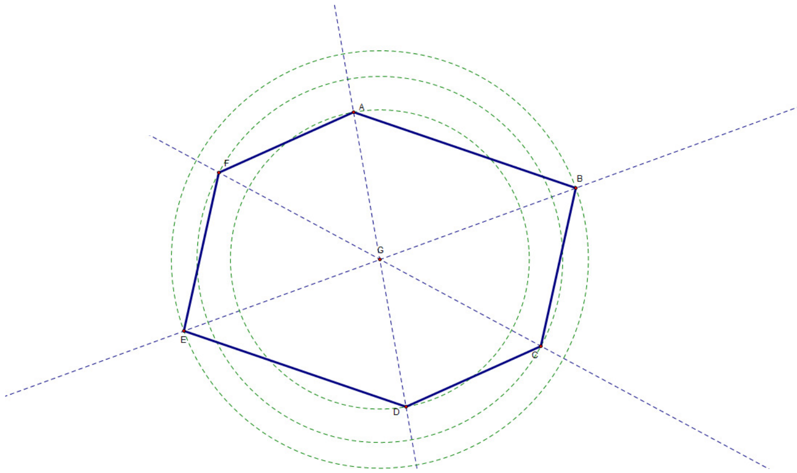

Figure 8. *ABCDEF* is an equal-quadrilateral hexagon, but it is not an equal-area hexagon: Area($\triangle ABC$), Area($\triangle BCD$), and Area($\triangle CDE$) are all different (and each is equal to the area of the symmetrically placed triangle).

To answer the question posed at the beginning of this section, Figure 8 offers a counterexample. All vertex quadrilaterals share an equal area, while the vertex triangles have varying areas.

4. Equal-(n+1)-gon polygons

Corollary. *For any integer $n > 1$, a polygon with $2n$ sides is an equal-$(n+1)$-gon polygon if its true diagonals are diameters of n concentric circles* (see Figure 9).

Proof. Let $n \in Z$, with $n > 1$. Let P_0 be a $2n$-sided polygon constructed using the endpoints of diameters of n concentric circles. Call the center of the circles B, and denote the vertices of P_0 by $V_1, V_2, V_3, \ldots, V_{2n}$.

Let P_1 be a polygon with vertices $V_i, V_{i+1}, \ldots, V_{i+n}$, and let Area$(P_1) = A_1$. Let P_2 be the polygon with vertices $V_{i+1}, V_{i+2}, \ldots, V_{i+n+1}$ and let Area$(P_2) = A_2$. We have $P_1 \cap P_2 =$ polygon$(V_{i+1}, V_{i+2}, \ldots, V_{i+n}) \cup \triangle V_{i+1}V_{n+1}B$, which we will call Q_0. Note that Area(Q_0) is equal to itself, so we need only to prove that Area$(P_1 - Q_0) =$ Area$(P_2 - Q_0)$. Since BV_{i+n} and BV_i are radii of the same circle, they are congruent, and likewise for BV_{i+1} and BV_{i+n+1}. Angles V_iBV_{i+1} and $V_{i+n}BV_{i+n+1}$ are congruent because they are vertical angles. Thus, by the side-angle-side formula, the triangles are congruent and therefore have equal area.

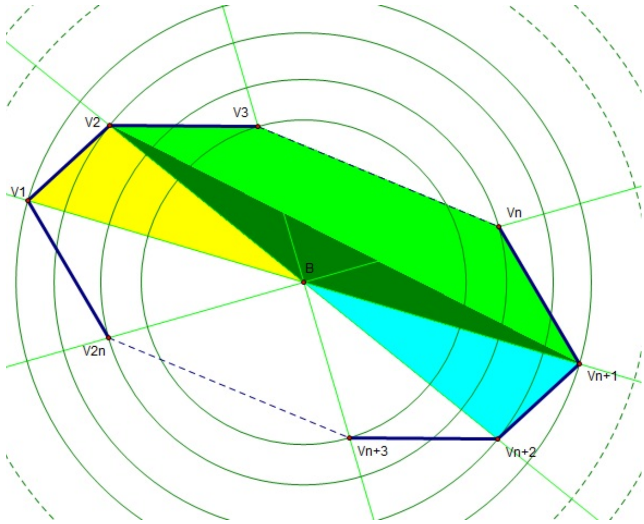

Figure 9. P_1 constructed from diameters of concentric circles.

So $\text{Area}(P_1 - Q_0) = \text{Area}(P_2 - Q_0)$, and we finally have

$$\begin{aligned}\text{Area}(P_1) &= \text{Area}(Q_0) + \text{Area}(P_1 - Q_0)\\ &= \text{Area}(P_2 - Q_0) + \text{Area}(Q_0) = \text{Area}(P_2).\end{aligned}$$

Therefore, the areas of all vertex $(n+1)$-gons are equal to each other. □

5. Results and open questions

Using known properties of equal-area polygons, we discovered properties of the equal-quadrilateral hexagon. We stated and proved a result that gives necessary conditions for an equal-area hexagon to also be equal-quadrilateral. Finally, we were able to generalize the process of constructing an equal-quadrilateral hexagon to allow construction of any equal-$(n+1)$-gon polygon.

An additional observation on the equal-area hexagon, whether convex or nonconvex, is that the area of the hexagon is equal to the sum of the areas of the vertex triangles. Likewise, the area of any equal-quadrilateral hexagon is twice the area of the vertex quadrilaterals. While this is immediately clear for a convex hexagon, it is not so when the hexagon is nonconvex. Since it is likely the proofs for these observations are simple, they were omitted from this article.

Some questions to consider in extending the idea of equal-n-gon polygons are:

(1) Given an equal-area heptagon, what are the necessary conditions to imply an equal-quadrilateral heptagon? Does equal-quadrilateral imply equal-area in heptagons? If not, how can we construct an equal-quadrilateral heptagon?

(2) Our corollary applies only to polygons with an even number of sides. Given a polygon with an odd number of sides, are there sufficient conditions to ensure vertex polygons of equal area?

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nielsenc@net.elmhurst.edu *Mathematics Department, Elmhurst College, Elmhurst, IL 60126, United States*

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