involve
a journal of mathematics

A new series for π via polynomial approximations to arctangent

Colleen M. Bouey, Herbert A. Medina and Erika Meza

msp

2012 vol. 5, no. 4

INVOLVE 5:4 (2012)
dx.doi.org/10.2140/involve.2012.5.421

A new series for π via polynomial approximations to arctangent

Colleen M. Bouey, Herbert A. Medina and Erika Meza

(Communicated by Kenneth S. Berenhaut)

Using rational functions of the form

$$\left\{\frac{t^{12m}(t-(2-\sqrt{3}))^{12m}}{1+t^2}\right\}_{m\in\mathbb{N}}$$

we produce a family of efficient polynomial approximations to arctangent on the interval $[0, 2-\sqrt{3}]$, and hence provide approximations to π via the identity $\arctan(2-\sqrt{3}) = \pi/12$. We turn the approximations of π into a series that gives about 21 more decimal digits of accuracy with each successive term.

1. Introduction

Two of the best-known series for π are

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!\,(1103+26390k)}{(k!)^4\,396^{4k}},$$

devised by Ramanujan about a century ago (see [Baruah et al. 2007; 2009] for history), and

$$\frac{1}{\pi} = \frac{\sqrt{10005}}{4270934400}\sum_{k=0}^{\infty}\frac{(-1)^k(6k)!\,(13591409+545140134k)}{(3k)!\,(k!)^3},$$

from the 1980s [Chudnovsky and Chudnovsky 1988]. These series are interesting and important because they converge so rapidly. Indeed, the Ramanujan series gives about 6 more decimal places for π with each successive term and the Chudnovsky series about 13 more decimal places per term [Weisstein n.d.]. The Chudnovsky series was in fact the formula used recently by Yee and Kondo [2011] to compute 10

MSC2010: primary 41A10; secondary 26D05.
Keywords: polynomial approximations to arctangent, approximations of π, series for π.
This work was supported by a 2010–11 mini-grant from the Center for Undergraduate Research in Mathematics (CURM) at Brigham Young University. CURM is funded by the National Science Foundation (DMS-063664).

trillion digits of π, and a modified version of it is used by *Mathematica* to compute a large number of digits of π [Vardi 1991].

Here, in Theorem 2, we present a new series for π that yields about 21 more decimal places per term. The new series is derived from polynomial approximations to the classical arctangent function that come from the integration of rational functions.

2. Polynomial approximations to arctangent

The integration of certain rational functions has proven useful in the approximation of the classical arctangent function, and, because of identities such as $\arctan 1 = \pi/4$, these can produce approximations to π. For example, the family

$$\left\{ \frac{t^{4m}(t-1)^{4m}}{1+t^2} \right\}_{m\in\mathbb{N}}$$

was recently studied in [Medina 2006], where it is shown that it can be used to produce polynomial approximations to $\arctan x$ on the interval [0, 1] whose error is governed by the size of the rational functions on that interval. In this section, we use these methods to produce polynomial approximations to $\arctan x$ on a smaller interval where the size of the integrand is much smaller, and hence the approximations converge much faster.

Consider the sequence of rational functions

$$\frac{t^{a_n}(t-(2-\sqrt{3}))^{b_n}}{1+t^2},$$

where a_n and b_n are integers chosen so that the polynomial division yields a constant remainder, and hence after integration, the arctangent function. We use $2-\sqrt{3}$ because $\arctan(2-\sqrt{3}) = \pi/12$; thus, if we can approximate arctangent at that value, we can approximate π.

Through trial and error, one finds that 12 is the smallest integer value of the b_n above that yields a constant remainder when the polynomial division is performed.[1] The smallest value for a_n is 2, but in what follows we choose 12 for the sake of symmetry. As Lemma 2 will show, the same is true for multiples of 12; thus, we explore the family of functions

$$\left\{ \frac{t^{12m}(t-\alpha)^{12m}}{1+t^2} \right\}_{m\in\mathbb{N}} \tag{1}$$

where we let $\alpha = 2-\sqrt{3}$ to facilitate the notation.

[1] All computations were done using *Mathematica* 7.0.

The following two lemmas, whose proofs are immediate via initial computations and induction, will facilitate our exploration of the family of rational functions.

Lemma 1. *For any $m \in \mathbb{N}$,*

$$\frac{t^{12m}}{1+t^2} = t^{12m-2} - t^{12m-4} + t^{12m-6} - t^{12m-8} + \cdots - 1 + \frac{1}{1+t^2} = \sum_{n=0}^{6m-1} (-1)^{n+1} t^{2n} + \frac{1}{1+t^2}.$$

Lemma 2. *For any $m \in \mathbb{N}$,*

$$\frac{t^{12m}(t-\alpha)^{12m}}{1+t^2} = q_m(t) + \frac{r_m}{1+t^2}, \tag{2}$$

where $r_m = (-1)^m (4\alpha)^{6m} = (-1)^m (5533696 - 3194880\sqrt{3})^m$, and the q_m are polynomials given recursively by

$$q_m(t) = t^{12}(t-\alpha)^{12}\, q_{m-1}(t) + r_{m-1}\, q_1(t),$$

with the initial quotient

$$\begin{aligned} q_1(t) = {} & -(4\alpha)^6 + (4\alpha)^6 t^2 - (4\alpha)^6 t^4 + (4\alpha)^6 t^6 - (4\alpha)^6 t^8 + (4\alpha)^6 t^{10} \\ & + (9184097 - 5302440\sqrt{3}) t^{12} + 12(564719\sqrt{3} - 978122) t^{13} \\ & + (8113645 - 4684416\sqrt{3}) t^{14} + 8(267909\sqrt{3} - 464032) t^{15} \\ & + (1200770 - 693264\sqrt{3}) t^{16} + 208(780\sqrt{3} - 1351) t^{17} \\ & + (47554 - 27456\sqrt{3}) t^{18} + 8(411\sqrt{3} - 712) t^{19} + (461 - 264\sqrt{3}) t^{20} \\ & + 12(\sqrt{3} - 2) t^{21} + t^{22}. \end{aligned}$$

The following proposition provides a closed-form formula for the quotients.

Proposition 1. *For each $m \in \mathbb{N}$, define the polynomial quotient $q_m(t) = \sum_{n=0}^{24m-2} a_n t^n$ and the polynomial remainder $r_m \in \mathbb{R}$ via* (2). *Then*

- (i) *$a_{2n} = (-1)^{m+1+n}(4\alpha)^{6m}$ and $a_{2n+1} = 0$ for $0 \le n \le 6m-1$;*
- (ii) *$a_{24m-2} = 1$ and $a_{24m-3} = -\binom{12m}{1}\alpha$ (these being the coefficients of the two highest powers of t in the quotient);*
- (iii) *$a_{24m-3-2n} = -a_{24m-3-2(n-1)} - \binom{12m}{2n+1}\alpha^{2n+1}$ for $1 \le n \le 6m-1$; and*
- (iv) *$a_{24m-2-2n} = -a_{24m-2-2(n-1)} + \binom{12m}{2n}\alpha^{2n}$ for $1 \le n \le 6m-1$.*

Proof. (i) We can rewrite and simplify the function to get

$$\frac{t^{12m}(t-\alpha)^{12m}}{1+t^2} = t^{12m}\left(\frac{(t-\alpha)^{12m}}{1+t^2}\right) = t^{12m}\left(p_m(t) + \frac{(-1)^m(4\alpha)^{6m}}{1+t^2}\right),$$

where $p_m(t)$ is some other quotient polynomial; we also note that Lemmas 1 and 2 together imply that the remainder $(-1)^m(4\alpha)^{6m}$ is indeed correct. Using Lemma 1, we make another substitution and obtain

$$t^{12m}p_m(t)+(-1)^m(4\alpha)^{6m}\left(t^{12m-2}-t^{12m-4}+t^{12m-6}-t^{12m-8}+\cdots-1+\frac{1}{1+t^2}\right),$$

which is the result of (i).

(ii) We write $\frac{t^{12m}(t-\alpha)^{12m}}{1+t^2}=\frac{t^{12m}}{1+t^2}(t-\alpha)^{12m}$. Use Lemma 1 to obtain

$$\left(t^{12m-2}-t^{12m-4}+-\cdots-1+\frac{1}{1+t^2}\right)(t-\alpha)^{12m},$$

and the binomial theorem to arrive at

$$\left(t^{12m-2}-t^{12m-4}+-\cdots-1+\frac{1}{1+t^2}\right)\sum_{k=0}^{12m}\binom{12m}{k}t^k\alpha^{12m-k}(-1)^k. \quad (3)$$

The coefficients of the two highest powers of t will come from multiplying the two highest powers of t in $(t-\alpha)^{12m}$ with t^{12m-2} in the first factor above.

(iii) To find each new odd coefficient we take the coefficient of the previous highest-order odd term and pair it with one lower power of t on the left of (3); since the signs of t alternate, we negate this. Each new coefficient will have a new lower-order term from the right paired with the highest power on the left. Adding these two, we get the coefficients of the new odd power of t.

(iv) The same argument as in (iii) gives the coefficients of the even powers. □

Since the functions (1) are small in the interval $[0,\alpha]$, integration of (2), after division by r_m, will yield approximations to arctangent on $[0,\alpha]$. That is,

$$\frac{1}{r_m}\int_0^x \frac{t^{12m}(t-\alpha)^{12m}}{1+t^2}\,dt=\frac{1}{r_m}\int_0^x q_m(t)\,dt+\arctan x, \quad (4)$$

and hence

$$P_m(x)=\frac{-1}{r_m}\int_0^x q_m(t)\,dt$$

will approximate arctangent on $[0,\alpha]$ with the error of the approximation given by the integral on the left side of (4), the maximum error occurring when $x=\alpha$. Proposition 1 provides a way to directly compute (after integration) these approximating polynomials; we will provide examples after we analyze their accuracy.

Substituting the largest and smallest values of t into the denominator of the left side of (4), we arrive at the inequality

$$\frac{1}{r_m}\int_0^\alpha \frac{t^{12m}(t-\alpha)^{12m}}{1+\alpha^2}\,dt<\frac{1}{r_m}\int_0^\alpha \frac{t^{12m}(t-\alpha)^{12m}}{1+t^2}\,dt<\frac{1}{r_m}\int_0^\alpha t^{12m}(t-\alpha)^{12m}\,dt. \quad (5)$$

It is now evident that, to further analyze the approximation, we need to compute

$$I_m := \int_0^\alpha t^{12m}(t-\alpha)^{12m}\,dt.$$

This is done via repeated integration by parts:

$$I_m = \int_0^\alpha t^{12m}(t-\alpha)^{12m}\,dt = \frac{((12m)!)^2}{(24m+1)!}\alpha^{24m+1}. \tag{6}$$

Since, as already noted, the left side of (4) is the error when $P_m(x)$ approximates $\arctan x$ on $[0,\alpha]$, we will use

$$e_m = \frac{1}{r_m}\int_0^\alpha \frac{t^{12m}(t-\alpha)^{12m}}{1+t^2}\,dt;$$

that is, e_m denotes the error when $P_m(\alpha)$ is used to approximate $\arctan\alpha = \pi/12$. Using this notation, we use (5) with m and $m+1$ to get

$$\frac{1}{(1+\alpha^2)r_m}I_m < e_m < \frac{1}{r_m}I_m \quad\text{and}\quad \frac{1}{(1+\alpha^2)r_{m+1}}I_{m+1} < e_{m+1} < \frac{1}{r_{m+1}}I_{m+1}. \tag{7}$$

Combining these two inequalities we arrive at

$$\frac{e_{m+1}}{e_m} < \frac{(1+\alpha^2)\,r_m\,I_{m+1}}{r_{m+1}\,I_m}, \tag{8}$$

which provides the estimate on how much better the next iterate is compared to the previous one.

Theorem 1. *Define $e_m = \left|\pi/12 - P_m(\alpha)\right|$, the error produced in approximating $\pi/12$ by the m-th iterate of the new sequence of approximating polynomials. Then, as $m\to\infty$,*

$$\frac{e_{m+1}}{e_m} < \frac{\alpha^{19}}{2^{34}} \approx 7.9063628967\times 10^{-22} = 0.00000000000000000000079063628967.$$

That is, each iterate gives about 21 more decimal places of accuracy in approximating $\pi/12$.

Proof. Use $|r_m| = (4\alpha)^6$, $1+\alpha^2 = 4\alpha$, (6) and (8) to get

$$\begin{aligned}\frac{e_{m+1}}{e_m} &< \frac{((12(m+1))!)^2\alpha^{24(m+1)+1}}{(4\alpha)^{6(m+1)}(24(m+1)+1)!}\cdot\frac{(4\alpha)^{6m+1}\,(24m+1)!}{((12m)!)^2\,\alpha^{24m+1}}\\ &= \frac{((12m+12)(12m+11)\cdots(12m+1))^2\alpha^{24}}{(4\alpha)^5\,(24m+25)(24m+24)\cdots(24m+2).}\end{aligned}$$

As $m\to\infty$, this becomes

$$\frac{(12^{12}\,m^{12})^2\alpha^{24}}{4^5\,\alpha^5\,24^{24}\,m^{24}} = \frac{\alpha^{19}}{4^5\,2^{24}} = \frac{\alpha^{19}}{2^{34}}.$$

□

Example 1. We use the coefficient formulas of Proposition 1 to find approximating polynomials. With $m = 1$,

$$\begin{aligned}P_1(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \frac{x^{11}}{11} + \frac{(419-60\sqrt{3})x^{13}}{4096} - \frac{3(362-209\sqrt{3})x^{14}}{14336} \\ - \frac{(2916\sqrt{3}-955)x^{15}}{61440} - \frac{(172-99\sqrt{3})x^{16}}{8192} + \frac{(1255+468\sqrt{3})x^{17}}{34816} - \frac{13x^{18}}{4608} \\ - \frac{13(61+36\sqrt{3})x^{19}}{38912} - \frac{(172+99\sqrt{3})x^{20}}{10240} + \frac{(5051+2916\sqrt{3})x^{21}}{86016} \\ - \frac{3x^{22}}{22528(2-\sqrt{3})^5} + \frac{x^{23}}{94208(2-\sqrt{3})^6}.\end{aligned}$$

Then

$$P_1(2-\sqrt{3}) = \frac{57423810140 - 22529108583\sqrt{3}}{70291415040},$$

and numerically we verify that $\left|P_1(2-\sqrt{3}) - \pi/12\right| < 4.81587 \times 10^{-23}$, or, after multiplication by 12,

$$\left|\frac{57423810140 - 22529108583\sqrt{3}}{5857617920} - \pi\right| < 5.779054023 \times 10^{-22}.$$

Example 2. With $m = 2$,

$$P_2(\alpha) = \frac{3013932255372315189770935 - 1155363167301686928932166\sqrt{3}}{3868552012005059812392960},$$

and $\left|P_2(\alpha) - \pi/12\right| \approx 2.55 \times 10^{-44}$.

3. Converting the iteration into a series

Theorem 1 requires the computation of a new set of polynomial coefficients when we want to obtain an approximation to π with more accuracy. For example, if we have a polynomial that gives n digits of accuracy for π when evaluated at α, then we need to compute a whole new polynomial, and hence a new set of coefficients, in order to obtain $(n+21)$ more digits of accuracy. Following a technique first developed in [Dalzell 1944] and used recently in [Lucas 2009] to produce a rational series that gives 3–4 more decimal places of accuracy for π with each successive term, we now focus on developing a series that provides the same number of digits (i.e., about 21) per term in computing π as each iteration of the polynomial sequence.

We know that

$$\frac{t^{12}(t-\alpha)^{12}}{1+t^2} = q_1(t) - \frac{(4\alpha)^6}{1+t^2}, \tag{9}$$

which can be rewritten as

$$\frac{1}{1+t^2} = \frac{q_1(t)}{t^{12}(t-\alpha)^{12} + (4\alpha)^6}.$$

Next we factor out $(4\alpha)^6$ on the denominator to get

$$\frac{1}{1+t^2} = \frac{q_1(t)}{(4\alpha)^6} \cdot \frac{1}{1 + \left(\frac{t(t-\alpha)}{2\sqrt{\alpha}}\right)^{12}}.$$

Expanding the right side in a geometric series gives

$$\frac{1}{1+t^2} = \left(\frac{q_1(t)}{(4\alpha)^6}\right) \sum_{n=0}^{\infty} (-1)^n \left(\frac{t(t-\alpha)}{2\sqrt{\alpha}}\right)^{12n}. \tag{10}$$

We integrate both sides on $[0, \alpha]$ and bring the integral inside the sum to get

$$\arctan \alpha = \frac{1}{(4\alpha)^6} \sum_{n=0}^{\infty} \frac{(-1)^n}{(4\alpha)^{6n}} \int_0^{\alpha} q_1(t)\, t^{12n} (t-\alpha)^{12n}\, dt. \tag{11}$$

The polynomial $q_1(t)$ is of degree 22 so we need to compute integrals of the form

$$\int_0^{\alpha} t^{12n+k} (t-\alpha)^{12n}\, dt$$

for $k = 0, \ldots, 22$. This is done using repeated integration by parts; we get

$$\int_0^{\alpha} t^{12n+k} (t-\alpha)^{12n}\, dt = \frac{(12n+k)!\,(12n)!\,\alpha^{24n+k+1}}{(24n+k+1)!}. \tag{12}$$

If we write $q_1(t) = \sum_{k=0}^{22} a_k t^k$, then

$$\frac{\pi}{12} = \frac{1}{(4\alpha)^6} \sum_{n=0}^{\infty} \frac{(-1)^n \alpha^{18n+1} (12n)!}{4^{6n}} \sum_{k=0}^{22} a_k \frac{(12n+k)!\alpha^k}{(24n+k+1)!}. \tag{13}$$

Simplification of the inside sum leads to the following theorem.

Theorem 2. *We have*

$$\pi = \sum_{n=0}^{\infty} \frac{(-1)^n (2-\sqrt{3})^{18n+1} ((12n)!)^2 (p_1(n) + p_2(n)\sqrt{3})}{2^{12(n+1)-1} (24n+1)!\, q(n)}, \tag{14}$$

where

$$
\begin{aligned}
p_1(n) = {} & 293063424013062144n^{11} + 1743144635880815616n^{10} \\
& + 4603477509110094336n^9 + 7113505268868220800n^8 \\
& + 7133195052290432592n^7 + 4863768060244254588n^6 \\
& + 2295600628029058188n^5 + 747948981593488485n^4 \\
& + 164336063152773014n^3 + 23098444048852896n^2 \\
& + 1859706966144526n + 64510302034815,
\end{aligned}
$$

$$
\begin{aligned}
p_2(n) = {} & 92656102528843776n^{11} + 553643573938200576n^{10} \\
& + 1466739601852815360n^9 + 2269385610499169280n^8 \\
& + 2272991576208150528n^7 + 1542973536047871648n^6 \\
& + 721853379546109560n^5 + 231741816550236960n^4 \\
& + 49765271182018546n^3 + 6762629909208426n^2 \\
& + 519049199193830n + 16879034409510, \quad \textit{and}
\end{aligned}
$$

$$
\begin{aligned}
q(n) = {} & 18786186952704n^{11} + 111934363926528n^{10} + 295980289228800n^9 \\
& + 457648310845440n^8 + 458818030927872n^7 + 312432825729024n^6 \\
& + 147050553999360n^5 + 47683923189760n^4 + 10399859469824n^3 \\
& + 1446143661248n^2 + 114720643240n + 3904125225.
\end{aligned}
$$

Moreover, if we define the error between the m-th partial sum of the series and π by $e_m = |\pi - S_m|$, then, as $m \to \infty$,

$$\frac{e_{m+1}}{e_m} < \frac{(2-\sqrt{3})^{19}}{2^{34}} \approx 7.9063628967 \times 10^{-22}.$$

Proof. Because of Theorem 1, it suffices to show that

$$\left| \frac{1}{(4\alpha)^6} \sum_{n=m}^{\infty} \frac{(-1)^n}{(4\alpha)^{6n}} \int_0^{\alpha} q_1(t)\, t^{12n} (t-\alpha)^{12n}\, dt \right| = \left| \frac{1}{r_m} \int_0^{\alpha} \frac{t^{12m}(t-\alpha)^{12m}}{1+t^2}\, dt \right|. \quad (15)$$

Using (9) to substitute for $q_1(t)$ and interchanging integration and summation in (15), we obtain

$$\frac{1}{(4\alpha)^6} \int_0^{\alpha} \sum_{n=m}^{\infty} \frac{(-1)^n}{(4\alpha)^{6n}} \left(\frac{t^{12n}(t-\alpha)^{12n}}{1+t^2} \right) (t^{12}(t-\alpha)^{12} + (4\alpha)^6)\, dt,$$

which we can simplify to

$$\frac{1}{(4\alpha)^6} \int_0^{\alpha} \left(\frac{t^{12}(t-\alpha)^{12} + (4\alpha)^6}{1+t^2} \right) \sum_{n=m}^{\infty} \left(\frac{(-1)t^{12}(t-\alpha)^{12}}{(4\alpha)^6} \right)^n dt.$$

The sum is a geometric series; after simplification, we get (15), as desired. □

The new series (14) gives about 21 more decimal places of accuracy with each successive term, though the terms are significantly more complicated and hence more "computationally expensive" than those in either the Ramanujan and Chudnovsky series. We note that all three series require the computation of a single square root, but the powers of $2-\sqrt{3}$ in the new series do slow down numerical computations. Thus, at this stage, it is fair to say that the Chudnovsky series still provides the fastest numerical tool for computing large numbers of digits of π. Nevertheless, it should be noted that the series (14) is very easy to program (in any language) and provides a viable method for computing digits of π; in fact, we have used it to compute a million digits on a desktop computer.

4. Further remarks

A similar process can be used with the rational functions

$$\left\{\frac{t^{4m}(t-1/\sqrt{3})^{6m}}{1+t^2}\right\}_{m\in\mathbb{N}}$$

to produce polynomial approximations to arctangent on the interval $[0, 1/\sqrt{3}]$, and hence approximations to π, because $\arctan(1/\sqrt{3})=\pi/6$. These approximations yield 5–6 more decimal places of accuracy with each iteration, and the computations are significantly "less expensive" than those of the sequence herein. (Our research in fact began with the exploration of this other family.)

It is our opinion that the series (14) should be seen as a byproduct of the approximating polynomials P_m which provide good approximations to arctangent on the entire interval $[0, 2-\sqrt{3}]$. It is possible that the P_m could prove useful for approximating π when used in conjunction with multiple-angle identities such as $\pi/4 = 5\arctan\frac{1}{7} + 2\arctan\frac{3}{79}$ [Calcut 2009].

Acknowledgments

We thank CURM's Director Michael Dorff for his leadership in facilitating, sponsoring and promoting our research.

References

[Baruah et al. 2007] N. D. Baruah, B. C. Berndt, and H. H. Chan, "Ramanujan's series for $1/\pi$: a survey", *Math. Student* Special Centenary Volume (2007), 1–24. MR 2010d:11151

[Baruah et al. 2009] N. D. Baruah, B. C. Berndt, and H. H. Chan, "Ramanujan's series for $1/\pi$: a survey", *Amer. Math. Monthly* **116**:7 (2009), 567–587. MR 2549375 Zbl 1229.11162

[Calcut 2009] J. S. Calcut, "Gaussian integers and arctangent identities for π", *Amer. Math. Monthly* **116**:6 (2009), 515–530. MR 2010f:11182 Zbl 1229.11164

[Chudnovsky and Chudnovsky 1988] D. V. Chudnovsky and G. V. Chudnovsky, "Approximations and complex multiplication according to Ramanujan", pp. 375–472 in *Ramanujan revisited* (Urbana-Champaign, 1987), Academic Press, Boston, 1988. MR 89f:11099

[Dalzell 1944] D. P. Dalzell, "On 22/7", *J. London Math. Soc.* **19** (1944), 133–134. MR 7,152b Zbl 0060.15306

[Lucas 2009] S. K. Lucas, "Approximations to π derived from integrals with nonnegative integrands", *Amer. Math. Monthly* **116**:2 (2009), 166–172. MR 2478060 Zbl 1179.26008

[Medina 2006] H. A. Medina, "A sequence of polynomials for approximating arctangent", *Amer. Math. Monthly* **113**:2 (2006), 156–161. MR 2006m:41004 Zbl 1132.41304

[Vardi 1991] I. Vardi, *Computational recreations in Mathematica*, Addison-Wesley, Redwood City, CA, 1991. MR 93e:00002 Zbl 0786.11002

[Weisstein n.d.] E. W. Weisstein, "Pi formulas", web page, Wolfram Research MathWorld, Available at http://mathworld.wolfram.com/PiFormulas.html.

[Yee and Kondo 2011] A. J. Yee and S. Kondo, "Round 2... 10 trillion digits of pi", Report, 2011, Available at http://www.numberworld.org/misc_runs/pi-10t/details.html.

Received: 2011-08-25 Revised: 2012-01-30 Accepted: 2012-03-04

cbouey@lion.lmu.edu *Mathematics Department, Loyola Marymount University, 1 LMU Drive, Los Angeles, CA 90045, United States*

hmedina@lmu.edu *Mathematics Department, Loyola Marymount University, 1 LMU Drive, Los Angeles, CA 90045, United States*

emeza2@lion.lmu.edu *Mathematics Department, Loyola Marymount University, 1 LMU Drive, Los Angeles, CA 90045, United States*

involve

msp.org/involve

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2012 is US $105/year for the electronic version, and $145/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2012 vol. 5 no. 4

1944-4176(2012)5:4;1-9