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Irreducible divisor graphs for numerical monoids

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The factorization of an element x from a numerical monoid can be represented visually as an irreducible divisor graph $G(x)$. The vertices of $G(x)$ are the monoid generators that appear in some representation of x, with two vertices adjacent if they both appear in the same representation. In this paper, we determine precisely when irreducible divisor graphs of elements in monoids of the form $N = \langle n, n+1, \dots, n+t \rangle$ where $0 \le t < n$ are complete, connected, or have a maximum number of vertices. Finally, we give examples of irreducible divisor graphs that are isomorphic to each of the 31 mutually nonisomorphic connected graphs on at most five vertices.

1. Introduction and preliminaries

Irreducible divisor graphs related to commutative rings were introduced and studied in [Coykendall and Maney 2007] and later studied in [Maney 2008; Axtell and Stickles 2008; Axtell et al. 2011]. In these papers, the authors represent elements of commutative rings using graphs which provide information about factorization properties of these elements. The general goal is to use graph-theoretic information to study factorization properties in the ring. As a notable example, it was shown in [Coykendall and Maney 2007; Axtell et al. 2011] that an atomic domain is a unique factorization domain precisely when every irreducible divisor graph over that ring is complete (equivalently, connected). We note that graphical representations of numerical semigroups have also been useful in computing a minimal set of relations, as in [Rosales 1996].

In this paper, we study irreducible divisor graphs of elements in numerical monoids — additive submonoids of the nonnegative integers. Our results indirectly apply to irreducible divisor graphs of elements of the form x^n in a polynomial ring of the form $\mathbb{F}[x^{n_1}, x^{n_2}, \dots, x^{n_t}]$ where $\mathbb{F}$ is a field, x is an indeterminate and $n_1 < n_2 < \cdots < n_t$ are positive integers. By considering a specific family of monoids (and hence commutative rings) we are able to provide more precise information

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about which graphs can be realized as irreducible divisor graphs of elements in various monoids and hence rings.

In this section we formally introduce irreducible divisor graphs of elements in numerical monoids and give some preliminary results that both motivate and provide useful tools for later sections. In Section 2 we consider numerical monoids generated by intervals of positive integers. Using the results of [García-Sánchez and Rosales 1999], where numerical monoids generated by intervals were thoroughly studied, we are able to classify exactly when the irreducible divisor graph of an element is complete and/or connected. We conclude Section 2 by presenting a method that can be used to determine whether or not a connected graph can be realized as the irreducible divisor graph of an element in a numerical monoid generated by a given interval. In Section 3 we show, by way of examples, that every connected graph with between one and five vertices can be realized as the irreducible divisor graph of an element in some numerical monoid. This leads us to ask the following question:

Question 1.1. Can every connected graph be realized as the irreducible divisor graph of an element in some numerical monoid?

Throughout, $\mathbb{N}$ will denote the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Recall that a numerical monoid is an additive submonoid of $\mathbb{N}_0$. More precisely, if $0 < n_1 < n_2 < \cdots < n_t$ are t positive integers such that for all $i \in \{2, \dots, t\}$, $n_i = a_1 n_1 + \cdots + a_{i-1} n_{i-1}$ has no nonnegative integer solutions $\{a_1, a_2, \dots, a_{i-1}\}$, then

$$N = \langle n_1, n_2, \dots, n_t \rangle = \{a_1 n_1 + \cdots + a_t n_t \ : \ a_i \in \mathbb{N}_0\} \subseteq \mathbb{N}_0$$

is the *numerical monoid* minimally generated by the set $\{n_1, n_2, \dots, n_t\}$. We now give a formal definition of the irreducible divisor graph of an element in a numerical monoid, mimicking the definition of the irreducible divisor graph of a nonzero nonunit of an atomic domain.

Definition 1.2. Let $N = \langle n_1, n_2, \dots, n_t \rangle$ be a minimally generated numerical monoid. If $x \in \mathbb{N}$, the *irreducible divisor graph of* x, denoted by $G_N(x)$, is defined as follows:

(1) The vertex set $V[G_N(x)]$ of $G_N(x)$ consists of the n_i for all i such that there exist $a_1, a_2, \dots, a_t \in \mathbb{N}_0$ with $x = \sum_{j=1}^{t} a_j n_j$ and $a_i \neq 0$.

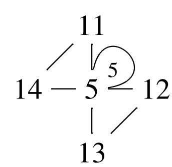
(2) The *edge set* $E[G_N(x)]$ of $G_N(x)$ has an edge from n_i to n_j for all pairs (i, j) for which there exist $a_1, a_2, \dots, a_t \in \mathbb{N}_0$ with $x = \sum_{k=1}^{t} a_k n_k$, and $a_i, a_j \neq 0$.

(3) We put $A_i - 1 \geq 0$ loops on vertex n_i, where $A_i = \max\{a_i : x = \sum_{k=1}^{t} a_k n_k$ for some $a_1, \dots, a_t \in \mathbb{N}_0\}$.

Thus, if $x \notin N$, the graph $G_N(x)$ is empty (has no vertices or edges). We write $G(x)$ in place of $G_N(x)$ if N is clear from context. Although we represent an edge as (n_i, n_j), this is not to be considered as an ordered pair and (n_j, n_i) represents the same edge.

This definition is consistent with the definition from [Coykendall and Maney 2007], in that if R is the semigroup ring $R = \mathbb{F}[y^{n_1}, y^{n_2}, \ldots, y^{n_t}]$ for some field $\mathbb{F}$ and some variable y, the graphs $G_N(x)$ and $G_R(y^x)$ are isomorphic.

Example 1.3. Let N have minimal generating set $\{5, 11, 12, 13, 14\}$ and let $x = 30$. In N we can express x only as $x = 5+11+14$, $x = 5+12+13$ and $x = 6\cdot 5$. Thus $G(30)$ contains 5, 11, 12, 13 and 14 as vertices, with edges connecting vertices 5 and 11, 5 and 12, 5 and 13, 5 and 14, 11 and 14, and 12 and 13. Moreover, there are 5 loops on vertex 5, since $30 = 6\cdot 5$. Thus, the irreducible divisor graph of $x = 30$ in $N = \langle 5, 11, 12, 13, 14\rangle$ is as follows:



The following equivalent definition of an irreducible divisor graph will be useful when determining which vertices and edges occur in an irreducible divisor graph $G(x)$ and will be used extensively in the following sections.

Definition 1.4. Let $N = \langle n_1, n_2, \ldots, n_t\rangle$ be a numerical monoid. If $x \in N$, the *irreducible divisor graph of* x, denoted by $G_N(x)$, is defined as follows:

(1) $n_i \in V[G(x)]$ if and only if $x - n_i \in N$.

(2) $(n_i, n_j) \in E[G(x)]$ if and only if $x - (n_i + n_j) \in N$.

Remark 1.5. Let $x \in N$, where $N = \langle n_1, n_2, \ldots, n_t\rangle$ and $\{n_1, n_2, \ldots, n_t\}$ is a minimal generating set for N, and let $M = \langle rn_1, rn_2, \ldots, rn_t\rangle$.

(1) Clearly $rx \in M$, and $\{rn_1, rn_2, \ldots, rn_t\}$ is a minimal generating set for M.

(2) For any i,
$$n_i \in V[G_N(x)] \iff rn_i \in V[G_M(rx)].$$

(3) For any distinct i and j,
$$(n_i, n_j) \in E[G_N(x)] \iff (rn_i, rn_j) \in E[G_M(rx)].$$

Thus it is sensible, when studying irreducible divisor graphs of numerical monoids, to study only *primitive* numerical monoids — those for which the generating set is relatively prime. For the balance of this article (except for some examples in Section 3) we consider numerical monoids of the form $\langle n, n+1, \ldots, n+t\rangle$. These are primitive, and the relationship described above allows results to be applied to associated nonprimitive numerical monoids as well.

For a primitive numerical monoid N, the *Frobenius number*, $\mathcal{F}(N)$, of N is the largest natural number not in N. The following easy proposition, whose proof we leave to the reader, gives extreme conditions for when an irreducible divisor graph is either complete (all pairs of vertices are adjacent) or is completely devoid of edges. This result tells us is that the problem of describing $G_N(x)$ for a given numerical monoid N is finite — once x is large enough, it is obvious that $G_N(x)$ contains all possible vertices and edges. We will improve this result for certain classes of numerical monoids in Section 2.

Proposition 1.6. *Let $N = \langle n_1, n_2, \ldots, n_t\rangle$ be a primitive minimally generated numerical monoid with $n_1 < n_2 < \cdots < n_t$.*

(1) *If $x > \mathcal{F}(N) + n_{t-1} + n_t$, then $G(x)$ is complete.*

(2) *If $x < 2n_1$, then $G(x)$ has no edges.*

An example shows that the converses of (1) and (2) in Proposition 1.6 are false. Let $N = \langle 12, 13, 14\rangle$. Then $G(65)$ is complete because $65 = (12) + 3(13) + (14)$. However, $\mathcal{F}(N) = 71$ and $65 < \mathcal{F}(N) + 13 + 14$. Moreover, $G(29)$ is an empty graph since $29 = 12a + 13b + 14c$ has no nonnegative integer solutions (a, b, c). However, $29 \geq 2 \cdot 12$.

2. Numerical monoids generated by intervals

In this section we study numerical monoids generated by intervals; that is, minimally generated by the set $\{n, n+1, \ldots, n+t\}$, where $n \geq 1$ and $0 \leq t \leq n-1$. For the balance of this paper we will use the notation $[a, b]$ (where $a \leq b$) to represent the interval of natural numbers $\{a, a+1, \ldots, b\}$. We start with two results that we will apply often.

Proposition 2.1 [García-Sánchez and Rosales 1999, Lemma 1 and Corollary 5]. *Let $n, t \in \mathbb{N}$ and let $N = \langle n, \ldots, n+t\rangle$.*

(1) *$x \in N$ if and only if $x \in [pn, p(n+t)]$ for some $p \in \mathbb{N}$.*

(2) $\mathcal{F}(N) = \left\lceil \frac{n-1}{t} \right\rceil n - 1.$

For ease of discussion, we name the intervals (as subsets of the natural numbers) contained in the monoid. We let $N_p = [pn, p(n+t)]$, where $p \in \mathbb{N}_0$, and note that $|N_p| = pt + 1$. We define a *gap* in N to be a maximal (with respect to set containment) nonempty interval of natural numbers that is not contained in N. In order to help with visualization, Figure 1 shows the intervals and gaps for the monoid $N = \langle n, \ldots, n+t\rangle$. Notice in particular that the first gap (between N_0 and N_1) has size $n-1$, and that subsequent gaps decrease in size by t.

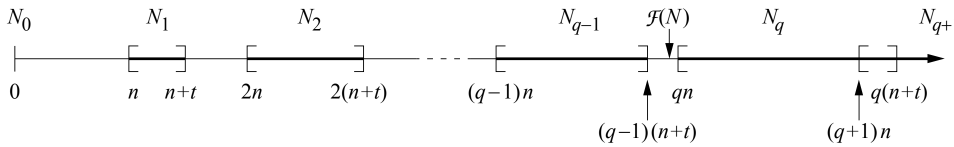


Figure 1. Intervals contained in the monoid $N = \langle n, \ldots, n+t \rangle$.

Graph-theoretic properties of $G(x)$. The next result shows that the only irreducible divisor graphs of elements in a numerical monoid generated by an interval containing no loops are disjoint unions of components each isomorphic to K_1 or K_2, the complete graphs on one and two vertices. Since loops almost always occur, we omit consideration of loops in the sequel.

Proposition 2.2. *Let $n \in \mathbb{N}$ and $N = \langle n, n+1, n+2, \ldots, n+t \rangle$ where $0 \le t \le n-1$. If $x \in N$, then $G(x)$ has no loops if and only if $G(x)$ is isomorphic to a disjoint union of components each isomorphic to K_1 or K_2.*

Proof. If $x \in N$, then by Proposition 2.1 $x \in [pn, p(n+t)]$ for some positive integer p. Thus $x = pn+k$ where $0 \le k \le pt$. First assume $p \ge 3$ and write $x = pn+ps+r$ where either $0 \le s < t$ and $0 \le r < p$ or else $s = t$ and $r = 0$. If $0 \le s < t$ and $0 \le r < p$, then $x = r(n+s+1)+(p-r)(n+s)$. Since $p \ge 3$, either $r \ge 2$ or $p-r \ge 2$. Thus there is at least one loop on either the vertex $n+s$ or the vertex $n+s+1$. If $x = p(n+t)$ then there are $p-1 \ge 2$ loops on vertex $n+t$. Therefore, if $p \ge 3$, $G(x)$ contains at least one loop.

If $p = 1$, then $x = n+i$ where $i \in [0, t]$ and $G(x)$ is isomorphic to K_1. If $p = 2$, then $x = 2n+j$ where $1 \le j \le 2t-1$. If j is even, then $x = 2n+j = 2\,(n+j/2)$, resulting in a loop on the vertex $n+j/2$. If j is odd, then note that, for any $n+i \in V[G(x)]$, $x-(n+i) = 2n+j-(n+i) = n+j-i$ and hence $0 \le j-i \le t$. Thus $x-[(n+i)+(n+j-i)] = 0$ and so $n+i$ is adjacent only to $n+j-i$. As this holds for all i with $n+i \in V[G(x)]$, $G(x)$ consists of multiple components isomorphic to K_2, which by definition has no loops. □

The next set of theorems — our main results — give complete classifications of when $G(x)$ has $t+1$ vertices, is connected with $t+1$ vertices, or is complete with $t+1$ vertices whenever $x \in \langle n, n+1, \ldots, n+t \rangle$.

Proposition 2.3. *Let $N = \langle n, \ldots, n+t \rangle$, where $n > 1$ and $0 < t < n$. Then $G(x)$ has $t+1$ vertices if and only if $x \in [(p+1)n+t, (p+1)n+pt]$ with $p > 0$. Moreover, if $x > \mathcal{F}(N)+n+t$ then $G(x)$ has $t+1$ vertices.*

Proof. By Definition 1.4, vertex $n+i$ is in the graph if and only if $x-n-i \in N$. Thus the $t+1$ vertices $\{n, \ldots, n+t\}$ are in the graph if and only if

$$S := [x-n-t, x-n] \subset N.$$

Since $N = \bigcup_{p\geq 0} N_p$ (by Proposition 2.1) and since $|N_p| \geq t+1$ for $p > 0$, we have $S \subset N_p$ for some $p > 0$ when $pn \leq x-n-t$ and $x-n \leq p(n+t)$, i.e., $x \in [(p+1)n+t,\, (p+1)n+pt]$.

The last condition expresses the case when x is sufficiently large that the integers in S are all larger than $\mathcal{F}(N)$; since there are no gaps above this point, $S \subseteq N$. This is also the point above which

$$[(p+1)n+t,\, (p+1)n+pt] \cap [(p+2)n+t,\, (p+2)n+(p+1)t] \neq \varnothing. \qquad \square$$

Proposition 2.4. *Let $N = \langle n, \dots, n+t\rangle$, where $n > 1$ and $0 < t < n$. Then $G(x)$ is complete on $t+1$ vertices if and only if $x \in [(p+2)n+2t-1,\, (p+2)n+pt+1]$ for $p \geq 0$ (if $t = 1$), $p > 0$ (if $t = 2$) and $p > 1$ otherwise. Moreover, if $x > \mathcal{F}(N)+2n+2t+1$ then $G(x)$ is complete on $t+1$ vertices.*

Proof. By Definition 1.4 the graph is complete if and only if $x-(n+i)-(n+j) \in N$ for each pair of distinct i and j in $[0, t]$, that is, when $S = [x-(n+t)-(n+t-1),\, x-n-(n+1)] \subset N$. Note that $|S| = 2t-1$ and $|N_p| = pt+1 \geq 2t-1$ when $p \geq (2t-2)/t$, which produces the bounds on p. When N_p is large enough to contain S, it is also required that $pn \leq x-(n+t)-(n+t-1)$ and $x-n-(n+1) \leq p(n+t)$ which implies $x \in [(p+2)n+2t-1,\, (p+2)n+pt+1]$.

As in Proposition 2.3, the second condition occurs when all elements of S are larger than $\mathcal{F}(N)$, that is, $S \subset N$ whenever $x-2n-2t+1 > \mathcal{F}(N)$. $\square$

The goal now is to give a result analogous to Propositions 2.3 and 2.4 for connected graphs with $t+1$ vertices. First we require two technical lemmas which relate the vertex degrees of $G(x)$ to the set $S = [x-2n-2t+1,\, x-2n-1]$. We then use this set to characterize when $G(x)$ is connected on $t+1$ vertices.

Lemma 2.5. *Let $N = \langle n, \dots, n+t\rangle$, where $n > 1$ and $0 < t < n$, and let $S = [x-2n-2t+1,\, x-2n-1]$. Then*

(1) *If S contains an interval of length $t+1$ that is contained in N then $G(x)$ has a vertex of degree t.*

(2) *If S contains an interval of length $t+1$ that is disjoint from N then $G(x)$ has a vertex of degree 0.*

Proof. Let $S_k = [x-2n-k-t,\, x-2n-k]$ be an interval of length $t+1$ in S.

For the first statement, we can find k so that $S_k \subset N$. The edge $(n+k, n+j)$ is in $E[G(x)]$ if and only if $x-2n-k-j \in N$. Since $S_k \subset N$, $x-2n-k-j \in N$ for $0 \leq j \leq t$. Thus (ignoring loops on $n+k$) the vertex $n+k$ has degree t.

For the second statement, we can find k so that S_k is disjoint from N. As above, we see that vertex $n+k$ is not adjacent to any other vertex. $\square$

If not for the vertices n and $n+t$, the preceding statements could each be made into equivalences. In fact, these vertices will require examination during the course of the next proof; we did not complicate the statement of Lemma 2.5 because these special cases each occur only once.

Lemma 2.6. *Let* $N = \langle n, \dots, n+t\rangle$, *where* $n > 1$ *and* $0 < t < n$, *and let* $S = [x-2n-2t+1, x-2n-1]$. *Then* $G(x)$ *is connected on* $t+1$ *vertices if and only if* $|S \cap N| \geq t$.

Proof. We note that an edge $(n+i, n+j)$ is in $E(G(x))$ when $x-(n+i)-(n+j) \in N$. Since $x-2n-2t+1 \leq x-2n-i-j \leq x-2n-1$, $E(G(x))$ is characterized by the intersection of S and N.

Furthermore, either $S \cap N \subset N_p$ or $S \cap N \subset N_p \cup N_{p+1}$ for some p. To see this, we assume that $S \cap N_p \neq \varnothing$. Then $x-2n-2t+1 \leq p(n+t)$, and hence $x-2n-1 \leq (p+1)(n+t)-n+t-2 < (p+1)(n+t)$. Thus the largest element of S is smaller than the largest element of N_{p+1}. We may therefore consider two cases:

Case 1: $S \cap N = S \cap N_p$ for some p. We divide this case into three subcases: $|S \cap N| > t$, $|S \cap N| = t$ or $|S \cap N| < t$.

In the first subcase, we notice that there is an interval of length at least $t+1$ in $S \cap N$ (in fact, $S \cap N$ is a single interval), so by Lemma 2.5 there is a vertex of degree t and hence $G(x)$ is connected on $t+1$ vertices.

For the second subcase we assume $|S \cap N| = t$. Since $|N_p| = pt+1$, we certainly have $|N_p| \neq t$ unless $p = 0$ and $t = 1$, in which case $G(x) = K_2$, which is connected on two vertices. Otherwise, $S \cap N \subset N_p$, so $|N_p| > t$. Since both N_p and S are intervals, $S \cap N_p$ comprises precisely either the first t elements of N_p or the last. If $S \cap N_p = [x-2n-2t+1, x-2n-t]$ then $\deg(n+t) = t$, while if $S \cap N_p = [x-2n-t, x-2n-1]$ then $\deg(n) = t$.

In the last subcase, we note that if $S \cap N = \varnothing$ then there is an interval of length at least $t+1$ (namely, all of S) that is not contained in N, so by Lemma 2.5 $G(x)$ is not connected. We assume for the balance of this case that $S \cap N$ is nonempty.

If $|N_p| > t$, that is, $p > 0$, then since $|S \cap N_p| < t$, S cannot extend the interval N_p in two directions, hence the intersection of S with the complement of N is a single interval. Thus there is an interval of length at least $t+1$ that is not in N, so by Lemma 2.5 there is a vertex of degree 0 and $G(x)$ is not connected.

If $p = 0$, then $S \cap N = \{0\}$, and the degree of each vertex is at most 1. If $t > 1$, this shows that $G(x)$ is not connected. If $t = 1$, then the hypothesis of the subcase $|S \cap N| < t$ is not satisfied.

Case 2: S intersects the two intervals N_{p-1} and N_p, as shown in Figure 2. We choose k so that $x-2n-k = pn$, the smallest element of N_p, and we let $S \cap N = [x-2n-2t+1, x-2n-k-j] \cup [x-2n-k, x-2n-1]$.

We divide this case into the three subcases $|S \cap N| > t-1$ (i.e., $|S \cap N| \geq t$), $|S \cap N| = t-1$ and $|S \cap N| < t-1$.

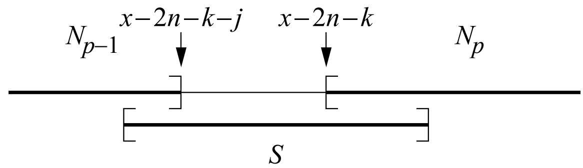


Figure 2. Case 2: S overlaps two intervals.

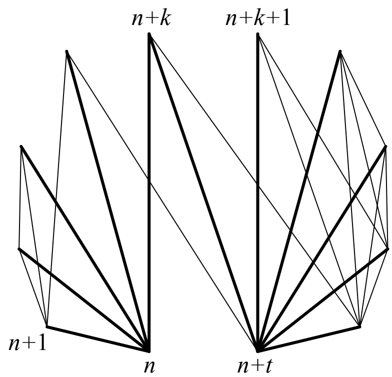


Figure 3. $G_N(x)$, in Case 2 when $|S \cap N| \geq t$, with a connected subgraph highlighted.

The graph for the first subcase is shown in Figure 3. In this case $j \leq t$, and the verification that the darkened subgraph exists is straightforward. In particular, the element of S associated with the edge $(n+k, n+t)$ is $x-2n-k-t$, so this element and the ones associated with the other darkened edges involving $n+t$ are contained in the lower portion of $S \cap N$, while the ones associated with the edges involving n are contained in the upper portion.

If $|S \cap N| < t-1$, the gap between N_{p-1} and N_p contains at least $t+1$ consecutive integers, so by Lemma 2.5 there is a vertex of degree 0, and $G(x)$ is not connected.

We are left with the subcase $|S \cap N| = t-1$. The graph for this case is shown in Figure 4, and we verify that the subgraph on vertices $\{n, \ldots, n+k\}$ and that on $\{n+k+1, \ldots, n+t\}$ have no edges between them. Indeed, the missing edges between the subgraphs are associated with the elements $x-(n+k)-(n+t) = x-2n-k-t$ through $x-n-(n+k+1) = x-2n-k-1$, none of which is in N. $\square$

Proposition 2.7. *Let* $N = \langle n, \ldots, n+t \rangle$, *where* $n > 1$ *and* $0 < t < n$. *Then* $G(x)$ *is connected on* $t+1$ *vertices if and only if at least one of the following conditions holds*:

(1) $x \in [(p+2)n+t, (p+2)n+(p+1)t]$ *for* $p \geq 0$ (*if* $t = 1$) *and* $p > 0$ *otherwise.*

(2) $x > C(N)$, *where* $C(N) = \mathcal{F}(N) + 2n + t + 1$ *if* t *divides* $n-1$, *and* $C(N) = \mathcal{F}(N) + n + t + 1$ *otherwise.*

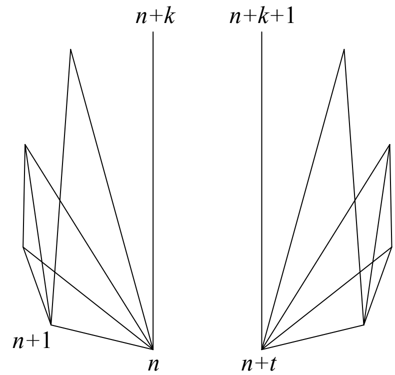


Figure 4. $G_N(x)$, in Case 2 when $|S \cap N| = t-1$.

Proof. We define $S = [x-2n-2t+1, x-2n-1]$ as before and recall that by Lemma 2.6, $G(x)$ is connected on $t+1$ vertices if and only if $|S \cap N| \geq t$.

If S intersects exactly one interval N_p, then $|S \cap N| \geq t$ when the smallest element of S is close enough to the left end of the interval, that is, $x-2n-2t+1 \geq pn-(t-1)$, or is not too close to the right end, that is, $x-2n-2t+1 \leq p(n+t)-(t-1)$. These inequalities give the first condition, and the conditions on p follow from the requirement that $|N_p| \geq t$.

If S spans a gap of size larger than $t-1$, then $G(x)$ is not connected, while if S spans a gap of size at most $t-1$ then $G(x)$ is connected. Since consecutive gaps decrease in size by t (refer to Figure 1), the last gap, $\mathcal{G}$, has size at most t. Assume that $S \cap \mathcal{G} \neq \varnothing$. If $|S \cap \mathcal{G}| < t$, then $G(x)$ is connected. If $|S \cap \mathcal{G}| = t$, that is, $\mathcal{F}(N) \in S$, then $G(x)$ is not connected. Moreover, the last gap has size less than t if and only if t does not divide the size of the first gap, namely that between N_0 and N_1, which has size $n-1$. In this case, $G(x)$ is connected on $t+1$ vertices for all $x > y$ satisfying $y-2n-2t+1 = np-(t-1)$ where N_p is the last interval *before* $\mathcal{F}(N)$, that is, if $\mathcal{F}(N) = qn-1$, then $p = q-1$. If the last gap is of size t, the relevant p belongs to the interval *after* $\mathcal{F}(N)$, that is, $p = q$. □

Note that Proposition 2.7 is worded differently from Propositions 2.3 and 2.4. In Proposition 2.7, when t does not divide $n-1$, there are values of x that do not satisfy the first condition, but do produce connected graphs.

The following corollary is a concise restatement of the previous results in the case $t = n-1$. Though it follows from these results, the direct proof is more straightforward, so it is sketched.

Corollary 2.8. *Let $n > 2$, $N = \langle n, \ldots, 2n-1 \rangle$ and $x \in N$.*

(1) *$G(x)$ has n vertices if and only if $x \geq 3n-1$.*

(2) *The following statements are equivalent.*

(a) *$G(x)$ is connected with n vertices.*
(b) $\deg(n) = n-1$.
(c) $x \geq 4n-1$.

(3) *$G(x)$ is complete on n vertices if and only if $x \geq 5n-3$.*

Proof. Notice that $N = \{0\} \cup [n, \infty)$.

For (1) we require that $[x-(2n-1), x-n] \subset N$, which is true precisely when $x-(2n-1) \geq n$.

For (2) we note that $\deg(n) = n-1$ (omitting loops, as usual), when $[x-n-(2n-1), x-n-(n-1)] \subset N$, which is true precisely when $x-3n+1 \geq n$, so conditions (b) and (c) are equivalent. It is clear that in this case $G(x)$ is connected. Conversely, if $G(x)$ is connected then vertex $2n-1$ is adjacent to at least one other vertex, that is, $x-(2n-1)-(n+j) \in N$ for some $j \in [0, n]$, so $x-(3n-1) \geq x-(3n-1)-j \geq n$, and the inequality (c) is established.

For (3) we note that vertices $2n-1$ and $2n-2$ must be adjacent, so $x-4n+3 = x-(2n-1)-(2n-2) \geq n$, which produces the inequality. Moreover, if the inequality is satisfied all pairs of vertices are adjacent since $x-(n+i)-(n+j) \geq x-4n+3$ if i and j are distinct integers in $[0, n-1]$. □

Remark 2.9. For $n = 2$, statements (2) and (3) in Corollary 2.8 would not quite be correct, because the set S comprises the single element $x-5$, and can thus coincide with $N_0 = \{0\}$. Thus, in addition to the ranges listed, $G(5)$ is complete (and therefore connected).

Constructions. The goal of this section is to address the following question: "When N is a numerical monoid generated by an interval, which connected graphs occur as $G(x)$ for some $x \in N$?" Throughout, we assume $N = \langle n, n+1, \ldots, n+t \rangle$ with $0 \leq t \leq n-1$ and require $G(x)$ to have $t+1$ vertices. It remains an open question as to what graphs can be realized when not all generators are required to occur as a vertex.

There are $\binom{t+1}{2}$ ways to choose two distinct values $n+i, n+j \in [n, n+t]$ and yet only $2t-1$ distinct sums $(n+i)+(n+j)$. By Definition 1.4, vertices $n+i$ and $n+j$ are adjacent in $G(x)$ if $x-[(n+i)+(n+j)] \in N$. Thus, to determine the number of edges that can occur in the irreducible divisor graph $G(x)$ for some $x \in \langle n, n+1, \ldots, n+t \rangle$ we consider the $2t-1$ possible sums in $[2n+1, 2n+2t-1]$ along with Proposition 2.1.

We have no general result for what graphs occur when $t > 4$, but the methods of this section may be extended for larger values of t. We now show how to determine which connected 5-vertex graphs with exactly four edges can be realized as $G(x)$ for $x \in N = \langle n, n+1, \ldots, n+4 \rangle$. The results of the remaining cases are outlined in Section 3.

a	Number of edges	Edges
$2n+1$	1	$(n, n+1)$
$2n+2$	1	$(n, n+2)$
$2n+3$	2	$(n, n+3),\ (n+1, n+2)$
$2n+4$	2	$(n, n+4),\ (n+1, n+3)$
$2n+5$	2	$(n+1, n+4),\ (n+2, n+3)$
$2n+6$	1	$(n+2, n+4)$
$2n+7$	1	$(n+3, n+4)$

Table 1. Edges associated with values of $x-a$.

Using Definition 1.4 we can determine which of the $\binom{4+1}{2} = 10$ possible edges occur in $G(x)$ by considering which values $x - \big((n+i)+(n+j)\big)$ are in N as distinct i and j range over the set $\{0, 1, 2, 3, 4\}$. Since $(n+i)+(n+j) \in [2n+1, 2n+7]$, we may summarize the relationships among values $x-a$ and edges in $G(x)$ as in Table 1.

We will use this table as a guide for constructing irreducible divisor graphs $G(x)$ with $x \in \langle n, n+1, n+2, n+3, n+4\rangle$ such that $G(x)$ has exactly 5 vertices and exactly four edges. By Proposition 2.1, the smallest number of consecutive positive integers in N is 5. Moreover, the number of consecutive integers in N must be $p(n+4) - pn + 1 = 4p+1$ for some $p \in \mathbb{N}$ and the length of a sequence of consecutive integers not in N must be $(p+1)n - p(n+4) - 1 = n - 4p - 1$ for some integer p with $1 \le p \le (n-1)/4$; that is, the gap sizes are congruent to $n-1$ modulo 4.

Referring to Table 1, we see that in order to guarantee exactly 4 edges in $G(x)$, we need to have either 3 or 4 consecutive integers not in N. Indeed, the set $[x-(2n+7), x-(2n+1)]$, which we called S in Lemmas 2.5 and 2.6, must intersect N in at most two intervals; see Figure 2. In the former case we are left with 4 edges exactly when $x-(2n+5)$, $x-(2n+4)$, and $x-(2n+3)$ are not in N. In the latter case we have 4 edges exactly when either $x-(2n+7)$, $x-(2n+6)$, $x-(2n+5)$ and $x-(2n+4)$ are not in N or $x-(2n+4)$, $x-(2n+3)$, $x-(2n+2)$ and $x-(2n+1)$ are not in N.

Suppose first that $x-(2n+5)$, $x-(2n+4)$, and $x-(2n+3)$ are not in N and hence $x-(2n+1)$, $x-(2n+2)$, $x-(2n+6)$, and $x-(2n+7)$ are in N. That is

$$E[G(x)] = \big\{(n, n+1), (n, n+2), (n+2, n+4), (n+3, n+4)\big\}.$$

To guarantee exactly 3 consecutive integers not in N we need, from Proposition 2.1, $n-4p-1 = 3$ where $p \ge 1$. In order for the correct three consecutive values to be outside of N, we require $x-(2n+6) = p(n+4)$ since $x-(2n+6)$ is the largest value in N preceding this sequence. Since $n = 4p+4$, $x = \frac{1}{4}n^2 + 2n + 2$ and we obtain the graph $G\left(\frac{1}{4}n^2+2n+2\right)$ in $N=\langle n, n+1, n+2, n+3, n+4\rangle$ whenever $n=4k$ with $k>1$.

$$n+1 \text{ ——— } n \text{ ——— } n+2 \text{ —— } n+4 \text{ —— } n+3$$

Now suppose that either $x-(2n+3)$, $x-(2n+2)$, $x-(2n+1)\in N$ or $x-(2n+7)$, $x-(2n+6)$, $x-(2n+5)\in N$. In the first case,

$$E[G(x)]=\{(n,n+1),\,(n,n+2),\,(n,n+3),\,(n+1,n+2)\}$$

in which case $G(x)$ has only 4 vertices. In the second case,

$$E[G(x)]=\{(n+3,n+4),\,(n+2,n+4),\,(n+1,n+4),\,(n+2,n+3)\}$$

	N	x
1	$\langle n\rangle,\ n>0$	$pn,\ p>0$
2	$\langle n, n+1\rangle,\ n>1$	$2n+1$
3	$\langle n, n+1, n+2\rangle,\ n=2k,\ k>1$	$\frac{1}{2}n^2+2n$
4	$\langle n, n+1, n+2\rangle,\ n>3$	$x\in[pn+3,\ p(n+2)-3],\ p>3$
4	$\langle 3, 4, 5\rangle$	$x>11$
6	$\langle n, .\,., n+3\rangle,\ n=3k,\ k>1$	$\frac{1}{3}n^2+2n+2$
7	$\langle n, .\,., n+3\rangle,\ n=3k,\ k>1$	$\frac{1}{3}n^2+2n+3$
8	$\langle n, .\,., n+3\rangle,\ n=3k+2,\ k>0$	$\frac{1}{3}n^2+\frac{7}{3}n+2$
9	$\langle n, .\,., n+3\rangle,\ n=3k+2,\ k>0$	$\frac{1}{3}n^2+\frac{7}{3}n$
10	$\langle n, .\,., n+3\rangle,\ n>4$	$x\in[pn+5,\ p(n+3)-5],\ p>3$
10	$\langle 4, 5, 6, 7\rangle$	$x>16$
13	$\langle n, .\,., n+4\rangle,\ n=4k,\ k>1$	$\frac{1}{4}n^2+2n+2$
16	$\langle n, .\,., n+4\rangle,\ n=4k,\ k>1$	$\frac{1}{4}n^2+2n+3$
19	$\langle n, .\,., n+4\rangle,\ n=4k,\ k>1$	$\frac{1}{4}n^2+2n$
22	$\langle n, .\,., n+4\rangle,\ n=4k+3,\ k>0$	$\frac{1}{4}n^2+\frac{9}{4}n+2$
26	$\langle n, .\,., n+4\rangle,\ n=4k+3,\ k>0$	$\frac{1}{4}n^2+\frac{9}{4}n+4$
28	$\langle n, .\,., n+4\rangle,\ n=4k+3,\ k>0$	$\frac{1}{4}n^2+\frac{9}{4}n+5$
29	$\langle n, .\,., n+4\rangle,\ n=4k+2,\ k>0$	$\frac{1}{4}n^2+\frac{5}{2}n+4$
30	$\langle n, .\,., n+4\rangle,\ n=4k+2,\ k>0$	$\frac{1}{4}n^2+\frac{5}{2}n+6$
31	$\langle n, .\,., n+4\rangle,\ n>5$	$x\in[pn+7,\ p(n+4)-7],\ p>3$
31	$\langle 5, 6, 7, 8, 9\rangle$	$x>21$

Table 2. Construction families. The first column refers to the numbering in Figure 5. We use the abbreviation $\langle n, .\,., n+3\rangle$ for $\langle n, n+1, n+2, n+3\rangle$, and likewise for $\langle n, .\,., n+4\rangle$.

and again we have a graph with only 4 vertices. Therefore, the graph shown above is the only graph with 5 vertices and 4 edges that can be realized as $G(x)$ for some $x \in \langle n, n+1, n+2, n+3, n+4\rangle$.

Similar arguments can be made to determine which connected graphs on $t+1$ vertices can be realized as $G(x)$ with $x \in \langle n, n+1, \ldots, n+t\rangle$, and these conditions are listed in Table 2 on the previous page.

3. Connected graphs with at most five vertices

In this section we give examples showing that each of the 31 nonisomorphic connected graphs with one to five vertices can be realized as the irreducible divisor graph of an element in a primitive minimally generated numerical monoid. In Figure 5, if the positive integers $n_1, \ldots, n_t$ occur as vertices in the graph $G(x)$,

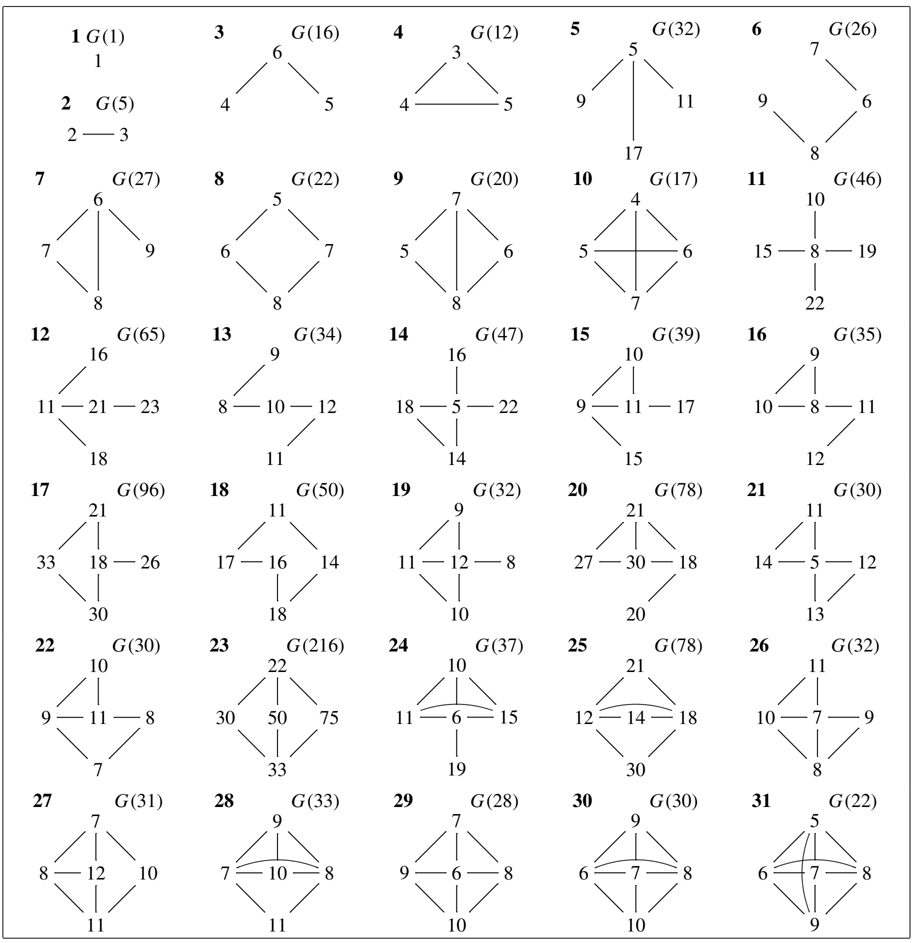


Figure 5. Connected graphs with at most five vertices.

then $x \in N = \langle n_1, \ldots, n_t \rangle$. In Table 2 we give, when possible, a family of examples realizing a given graph using the methods of Section 2. When such a family is not given, it is because that graph cannot be realized as the irreducible divisor graph of an element in a numerical monoid generated by an interval.

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