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Robert A. Beeler and Tony K. Rodriguez

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In recent work by Beeler and Hoilman, the game of peg solitaire is generalized to arbitrary boards. These boards are treated as graphs in the combinatorial sense. Normally, the goal of peg solitaire is to minimize the number of pegs remaining at the end of the game. In this paper, we consider the open problem of determining the *maximum* number of pegs that can remain at the end of the game, under the restriction that we must jump whenever possible. In this paper, we give bounds for this number. We also determine it exactly for several well-known families of graphs. Several open problems regarding this number are also given.

1. Introduction

Peg solitaire is a table game which traditionally begins with "pegs" in every space except for one which is left empty (i.e., a "hole"). If in some row or column two adjacent pegs are next to a hole (as in Figure 1), then the peg in x can jump over the peg in y into the hole in z. The peg in y is then removed. Usually, the goal is to remove every peg but one. If this is achieved, then the board is considered solved [Beasley 1985; Berlekamp et al. 2003]. However, in this paper we consider the open problem of determining the *maximum* number of pegs that can remain at the end of the game under the caveat that we jump whenever possible. We refer to this variation as the *fool's solitaire problem*.

In [Beeler and Hoilman 2011], the notion of peg solitaire was generalized to graphs. A graph, $G = (V, E)$, is a set of vertices, V, and a set of edges, E. Because of the restrictions of peg solitaire, we will assume that all graphs are finite undirected graphs with no loops or multiple edges. In particular, we will *always* assume that

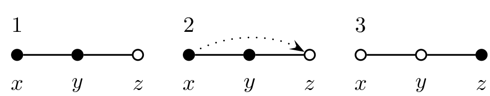

Figure 1. A typical jump in peg solitaire.

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graphs are connected. For all undefined graph theory terminology, refer to [West 1996]. In particular, $n(G)$ denotes the *order* of the graph G, that is, the number of vertices in the graph.

If there are pegs in vertices x and y and a hole in z, then we allow x to jump over y into z provided that xy, $yz \in E$. The peg in y is then removed. In general, the game begins with a *starting state* $S \subset V$ which is a set of vertices that are empty. A *terminal state* $T \subset V$ is a set of nonadjacent vertices that have pegs at the end of the game. A terminal state T is *associated* with starting state S if T can be obtained from S by a series of jumps. We will assume that S consists of a single vertex.

The *fool's solitaire number* of a graph G, denoted by $\mathrm{Fs}(G)$, is the cardinality of the largest terminal state T that is associated with a starting state consisting of a single hole. A terminal state T is a *fool's solitaire solution* if the cardinality of T is equal to $\mathrm{Fs}(G)$. The *dual* of a peg configuration T, denoted by T', is the state resulting from reversing the roles of pegs and holes.

The objective of this paper is to gain insight on the fool's solitaire number for graphs. To do this, we will determine bounds of the fool's solitaire number for graphs and find the fool's solitaire number for various classes of graphs. In analyzing the terminal states of a graph, the following theorem is useful.

Theorem 1.1 [Beeler and Hoilman 2011]. *Suppose that S is a starting state of G with associated terminal state T. Let S' and T' be the duals of S and T, respectively. It follows that T' is a starting state of G with associated terminal state S'.*

The following is an immediate corollary that will prove useful.

Corollary 1.2. *On a graph G, there exists some vertex $s \in V(G)$ such that, when $S = \{s\}$, there exists some series of jumps that will yield T as a terminal state if and only if the dual T' of T is solvable to one peg.*

This result provides an alternative method of checking if a suspected terminal state is obtainable. Generally, to determine if a terminal state T of a graph G is obtainable, you simply solve the dual.

2. Upper bounds on Fs(*G*)

In this section, we present upper bounds for $\mathrm{Fs}(G)$. We begin with a simple, but useful, theorem involving the independence number of a graph. An *independent set* of vertices is a set of mutually nonadjacent vertices. The *independence number* is the maximum size of an independent set in a graph [West 1996].

Theorem 2.1. *For any graph G, $\mathrm{Fs}(G) \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G.*

Proof. By definition, any terminal state is an independent set of vertices. Thus the maximum independent set has at least as many vertices as the largest terminal state. Ergo, $\mathrm{Fs}(G) \leq \alpha(G)$. □

While Theorem 2.1 seems almost trivial, the bound given is sharp for many graphs, as will be discussed in Section 4. Another upper bound involving the domination number follows. In a graph G, a set $S \subseteq V(G)$ is a *dominating set* if every vertex not in S has a neighbor in S. The *domination number* is the minimum size of a dominating set in G.

Theorem 2.2. *For any graph G*, $\mathrm{Fs}(G) \leq n(G) - \gamma(G)$, *where $\gamma(G)$ is the domination number of G.*

Proof. We begin by showing that the dual of any terminal state is a dominating set. Let T be any terminal state of a graph G. Note that T is an independent set of $V(G)$. Consider T', the dual of T. Since each vertex in a dominating set dominates itself, every vertex not in T is dominated. Also, by definition of an independent set, every vertex in T is adjacent only to vertices in T', so these vertices are dominated as well. Thus T' is a dominating set.

We now show that $\mathrm{Fs}(G) \leq n(G) - \gamma(G)$. Note that $\mathrm{Fs}(G) = |T| = n(G) - |T'|$. Since T' is a dominating set by the argument above, we have that $\gamma(G) \leq |T'|$. Hence $\mathrm{Fs}(G) = n(G) - |T'| \leq n(G) - \gamma(G)$. □

The upper bound given in Theorem 2.1 can be improved for several classes of graphs.

Theorem 2.3. *Let G be a graph. If for every maximum independent set A the dual of A is an independent set with at least two vertices*, *then* $\mathrm{Fs}(G) \leq \alpha(G) - 1$.

Proof. Suppose to the contrary that $\mathrm{Fs}(G) = \alpha(G)$. This implies that A is a terminal state for some maximum independent set A. Thus, by Corollary 1.2, G would be solvable from starting state A'. Because the dual of A is also an independent set, it follows that no moves are possible from this starting state. Hence either $|A'| = 1$ or $\mathrm{Fs}(G) \leq \alpha(G) - 1$. Since we assume that A' has at least two vertices, $\mathrm{Fs}(G) \leq \alpha(G) - 1$. □

3. Families of graphs

In this section, we present the fool's solitaire number of certain families of graphs. As usual, P_n, C_n, and K_n will denote the path, the cycle, and the complete graph on n vertices, respectively. Let $K_{n,m}$ denote the complete bipartite graph with $V = X \cup Y$, $X = \{x_1, \ldots, x_n\}$, and $Y = \{y_1, \ldots, y_m\}$, where $n \geq m$. In particular, $K_{1,n}$ is called a *star*. The n-dimensional hypercube is denoted by Q_n.

Note that, if $\mathrm{Fs}(G) = \alpha(G)$, it suffices to provide the series of peg solitaire jumps that will yield a solution. If $\mathrm{Fs}(G) = \alpha(G) - 1$, it suffices to demonstrate that

$\mathrm{Fs}(G) \neq \alpha(G)$ and to provide the series of peg solitaire jumps that will yield a terminal state with cardinality $\alpha(G) - 1$.

The following proposition is obvious, but included for the sake of completeness.

Proposition 3.1. *The fool's solitaire number for the complete graph on n vertices is one.*

We now consider complete bipartite graphs.

Proposition 3.2. *For the star $K_{1,n}$, $\mathrm{Fs}(K_{1,n}) = n$.*

Proof. Note that $\alpha(K_{1,n}) = n$. Placing the hole in the center makes it so that no moves are available. Thus $\mathrm{Fs}(G) = n$. □

Theorem 3.3. *For the complete bipartite graph $K_{n,m}$, if $n, m > 1$, then $\mathrm{Fs}(K_{n,m}) = n - 1$.*

Proof. We begin by showing that $\mathrm{Fs}(K_{n,m}) \neq n$. For the complete bipartite graph $K_{n,m}$, note that $\alpha(K_{n,m}) = n$. The only maximum independent set of $K_{n,m}$ is X, which has independent set Y as its dual. Since $|Y| = m > 1$, $\mathrm{Fs}(K_{n,m}) \leq n - 1$ by Theorem 2.3.

We claim that $T = X - \{x_1\}$ is the fool's solitaire solution. Hence we must show that $T' = Y \cup \{x_1\}$ is reducible to a single peg. For $i = 1, \ldots, \lfloor m/2 \rfloor$, we let the $(2i - 1)$-st move be from x_1 over y_{2i-1} into x_2. Similarly, the $2i$-th jump is from x_2 over y_{2i} into x_1. If m is odd, then we make an additional jump from x_1 over y_m into x_2. Since $K_{n,m}$ is solvable from starting state T', it follows that $\mathrm{Fs}(K_{n,m}) = n - 1$ by Corollary 1.2. □

We will now consider the solutions to paths and cycles. When discussing these graphs, we will label the vertices of the graphs with elements of the set $\{0, 1, \ldots, n - 1\}$ in the obvious way. Also note that P_2 and P_3 are isomorphic to $K_{1,1}$ and $K_{1,2}$, respectively. As the fool's solitaire number of these graphs was determined in Proposition 3.2, we do not consider these cases below.

Theorem 3.4. *For the path on n vertices, if $n > 3$, then $\mathrm{Fs}(P_n) = \lfloor n/2 \rfloor$.*

Proof. Note the independence number of a path on n vertices is $\lceil n/2 \rceil$.

We begin by showing that, if n is odd, then $\mathrm{Fs}(P_n) < \lceil n/2 \rceil$. There is only one independent set with cardinality $\lceil n/2 \rceil$, namely $\{0, 2, 4, \ldots, n-3, n-1\}$. Because the dual of this set is an independent set with at least two vertices, $\mathrm{Fs}(P_{2k+1}) \leq \lfloor n/2 \rfloor$ by Theorem 2.3.

To obtain the fool's solitaire solution for P_n (regardless of whether n is even or odd), begin with the hole in 0. The i-th move will be to use the peg in $2i$ to jump over $2i - 1$ into $2i - 2$. This will remove $\lceil n/2 \rceil$ pegs. It follows that $\mathrm{Fs}(P_n) = \lfloor n/2 \rfloor$. □

Theorem 3.5. *For the cycle on n vertices, $\mathrm{Fs}(C_n) = \lfloor \frac{n-1}{2} \rfloor$.*

Proof. Note that $\alpha(C_n) = \lfloor n/2 \rfloor$. We begin by showing that if n is even, then $\mathrm{Fs}(C_n) < n/2$. Let $n = 2k$, where $k \in \mathbb{Z}$. Up to automorphism on the vertices, C_{2k} has one maximum independent set of vertices. Since the dual of this set is an independent set with at least two vertices, it follows that $\mathrm{Fs}(C_{2k}) \leq k-1$ by Theorem 2.3.

To obtain the fool's solitaire solution of C_n (regardless of whether n is even or odd), begin with the hole in 0. The i-th move will be to use the peg in $2i$ to jump over $2i-1$ into $2i-2$. This can be repeated k times, removing $\lceil n/2 \rceil$ pegs. If n is even, we make an additional jump from 0 over $n-1$ into $n-2$. In either case, $\mathrm{Fs}(C_n) = \lfloor \frac{n-1}{2} \rfloor$. □

We will now consider the hypercube on 2^n vertices, Q_n. As usual, each vertex will be labeled with an element from the set $\{0, 1, \ldots, 2^n - 1\}$, with two vertices being adjacent if and only if their binary expansions differ by one bit.

Theorem 3.6. *The fool's solitaire number of the n-dimensional hypercube for $n \geq 2$ is* $\mathrm{Fs}(Q_n) = 2^{n-1} - 1$.

Proof. We first show that $\mathrm{Fs}(Q_n) \neq \alpha(Q_n) = 2^{n-1}$. Up to automorphism on the vertices, there is a unique maximum independent set of vertices, namely the set of all vertices whose binary expansions have an even number of ones. As the dual of this set is an independent set with at least two vertices, $\mathrm{Fs}(Q_n) \leq 2^{n-1} - 1$.

Note that Q_n is Hamiltonian with an even number of vertices [Harary et al. 1988]. Relabel the vertices of Q_n along a Hamiltonian cycle with the numbers $0, 1, \ldots, 2^n - 1$ in the obvious way. Note that the odd-numbered vertices correspond to the vertices with an odd number of ones in their binary expansions. Hence, the odd-numbered vertices form a maximum independent set in Q_n. We claim that $\{1, 3, \ldots, 2^n - 3\}$ is the fool's solitaire solution. Hence we must show that the dual of this set, $\{2^n - 1, 0, 2, 4, \ldots, 2^n - 2\}$, is reducible to a single peg. Begin by jumping from $2^n - 1$ over 0 into 1. For the remaining $2^{n-1} - 1$ moves, the i-th move is from $2i - 1$ over $2i$ into $2i + 1$, where $i = 1, \ldots, 2^{n-1} - 1$. Hence $\mathrm{Fs}(Q_n) = 2^{n-1} - 1$. □

4. Lower bounds on Fs(*G*)

In Section 2, we gave several upper bounds on the fool's solitaire number. Unfortunately, lower bounds on the fool's solitaire number are more difficult to prove in general. However, a useful proposition follows.

Proposition 4.1. *Suppose that H is obtained from G by appending a vertex that is not adjacent to any vertex in the fool's solitaire solution of G. It follows that* $\mathrm{Fs}(H) \geq \mathrm{Fs}(G) + 1$.

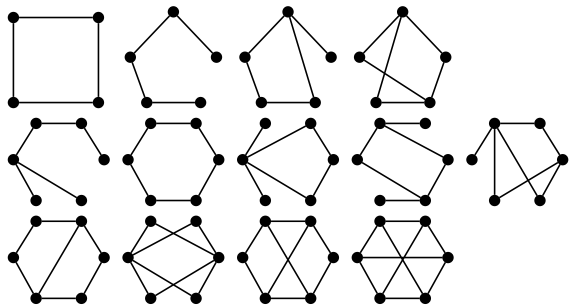


Figure 2. Graphs with $n(G) \leq 6$ such that $\text{Fs}(G) = \alpha(G) - 1$.

Proof. Suppose that H is obtained from G by appending a vertex v' to G such that $vv' \notin E(G)$ for all $v \in T$, where T is the fool's solitaire solution of G. We obtain a terminal state of H with $|T| + 1$ vertices by finding the fool's solitaire solution on the subgraph induced by the vertices of G. Since v' is not adjacent to any vertex in T, it follows that $T \cup \{v'\}$ is a valid terminal state of H. This terminal state has $\text{Fs}(G) + 1$ vertices. Hence, $\text{Fs}(H) \geq \text{Fs}(G) + 1$. □

To aid in a more general result, an exhaustive computer search of all terminal states associated with a single vertex starting state was performed on all 143 nonisomorphic connected graphs with six vertices or less. The algorithm is implemented on the first author's website [Beeler and Norwood n.d.].

Lists of graphs of small order were obtained from the appendix of [Harary 1969]. The independence numbers of these graphs were verified using the Small Graph Database [Grout n.d.].

Of the 143 connected graphs with six vertices or less, 130 of them satisfy $\text{Fs}(G) = \alpha(G)$. The remaining thirteen graphs satisfy $\text{Fs}(G) = \alpha(G) - 1$. These graphs are given in Figure 2.

Based on this and the results of Section 3, we present the following conjecture.

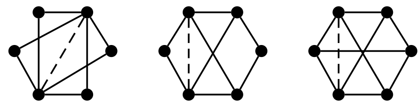
Conjecture 4.2. For all connected graphs G,

$$\alpha(G) - 1 \leq \text{Fs}(G) \leq \alpha(G).$$

While we were unable to prove this, Proposition 4.1 may prove useful for an inductive proof of this conjecture.

5. Open problems

Let H be a graph obtained from G by deleting an edge of G. We note that $\alpha(H) \geq \alpha(G)$ for all graphs G. Thus, a natural conjecture is that $\text{Fs}(H) \geq \text{Fs}(G)$ for all graphs G. However, this is not the case. Using the aforementioned exhaustive computer search on all graphs with six vertices or less, three were found in which



$\mathrm{Fs}(G) = 4$ $\mathrm{Fs}(G) = 3$ $\mathrm{Fs}(G) = 3$

Figure 3. Graphs in which edge deletion lowers $\mathrm{Fs}(G)$.

edge deletion actually *lowers* the fool's solitaire number. These graphs are given in Figure 3. In each of these cases, deleting the dashed edge will lower the fool's solitaire number by one.

Some natural open questions motivated by this observation include:

(i) How much can edge deletion lower the fool's solitaire number?

(ii) Let $\mathrm{ED}(n)$ be the number of nonisomorphic graphs with n vertices such that edge deletion lowers the fool's solitaire number. If n is large enough, does $\mathrm{ED}(n) = 0$? Let $i(n)$ be the number of nonisomorphic graphs with n vertices. What can be said about $\lim_{n\to\infty} \mathrm{ED}(n)/i(n)$?

One of the major results in [Beeler and Hoilman 2011] was to show that the cartesian product of solvable graphs was likewise solvable. What can be said about $\mathrm{Fs}(G \,\square\, H)$ in terms of $\mathrm{Fs}(G)$ and $\mathrm{Fs}(H)$?

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beelerr@etsu.edu *Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN 37614, United States*

ztkr2@goldmail.etsu.edu *Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN 37614, United States*

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