involve
a journal of mathematics

Dynamic impact of a particle

Jeongho Ahn and Jared R. Wolf

msp

2013 vol. 6, no. 2

INVOLVE 6:2 (2013)
dx.doi.org/10.2140/involve.2013.6.147

Dynamic impact of a particle

Jeongho Ahn and Jared R. Wolf

(Communicated by John Baxley)

In this work, we consider a moving particle which drops down onto a stationary rigid foundation and bounces off after its contact. The equation of its motion is formulated by a second-order ordinary differential equation. The particle satisfies the Signorini contact conditions which can be interpreted in terms of complementarity conditions. The existence of weak solutions is shown by using a finite time step and the necessary a priori estimates which allow us to pass to the limit. The uniqueness of the solutions can be proved under some additional assumptions. Conservation of energy is also investigated theoretically and numerically. Numerical solutions are computed via both finite- and infinite-dimensional approaches.

1. Introduction

Contact between two bodies happens in our life everyday. Consider, for example, the contact between a floor and an elastic ball such as a basketball or a volleyball, or contact between a brake pad and a disc of a car's wheel. These contact phenomena may seem to be simple from physical or engineering points of view. However, proving the existence of solutions for these contact models requires very sophisticated mathematical analysis and is a mathematical challenge.

Historically, the study of contact mechanics may have originated with [Hertz 1881], where the physicist analyzed a *static* contact problem of two elastic bodies. Mathematical research on contact problems has become more active since Signorini [1933] formulated the general static contact problem of linearly elastic bodies. Most mathematical research on contact mechanics has focused on *static* or *quasistatic* problems and relatively little research on *dynamic* contact problems has been carried out. This has started to change, as mathematical tools and numerical methods for dynamic contact problems have been developed.

MSC2010: primary 65L20; secondary 74H20.

Keywords: Signorini contact conditions, conservation of energy, complementarity conditions, time discretization.

This work has been supported by a SURF (Student Undergraduate Research Fellowship) from the Arkansas Department of Higher Education.

Readers interested in contact problems may refer to the remarkable paper [Stewart 2000] for *rigid-body* dynamics with friction and impact which is described by ordinary differential equations (ODEs) and [Kikuchi and Oden 1988] for contact in *elasticity* which deals with elliptic, parabolic, or hyperbolic types of partial differential equations (PDEs).

The study of one-dimensional contact problems is of considerable importance, in its own right and because it provides a foundation for higher-dimensional problems. There are many one-dimensional dynamic contact models involving vibrating strings, elastic rods, and elastic beams modeled in various ways: Euler–Bernoulli beams (linear), Timoshenko beams, and many kinds of nonlinear beams. Nonlinear Gao beams [Gao 1996] are especially noteworthy, as their model allows for buckling, and their contact problems have recently been the subject of many interesting studies; see [Ahn et al. 2012], for example.

There are many open questions in dynamic contact problems. For example, showing the uniqueness of solutions for rigid dynamics models or dynamic contact models between an elastic body and a rigid foundation with Signorini contact conditions is a challenging problem. In addition, proving the existence of solutions for dynamic contact between a purely elastic body and a rigid foundation over more than three dimensions is still an open question. Indeed, the dynamic contact problem has been studied in [Ahn and Stewart 2009], where the viscosity is added to prove the existence of solutions. Mathematically speaking, inserting the viscosity into the equation of motion is a great idea to obtain more regularized solutions and to show the existence of solutions for almost elastic bodies. "Almost elastic" implies that a viscous quantity dealing with the viscosity is chosen by a very small number, which enables viscoelastic bodies of Kelvin–Voigt type to get closer to elastic bodies.

One of the major concerns of dynamic contact problems is to show conservation of energy for the elastic case or energy balance for the viscoelastic case. This is an open question and in [Ahn 2007; 2008; 2012] it has been investigated theoretically and numerically. However, proving it for the general case may be a very difficult task. In rigid-body dynamic problems with frictionless impact, showing conservation of energy depends on the coefficient of restitution (COR). If COR $= 1$, that is, for the elastic case, energy conserves, but if $0 \leq$ COR < 1, that is, for the inelastic case, energy decreases. Furthermore, considering COR for particles results in showing the uniqueness of solutions, which is stated at the end of Section 4.

This work is motivated by the three-dimensional dynamic contact problem, although particles are neither elastic nor viscoelastic. Our dynamic contact model may be basic but will provide a great opportunity to think about significant issues on higher-dimensional dynamic contact problems with elastic bodies or rigid bodies.

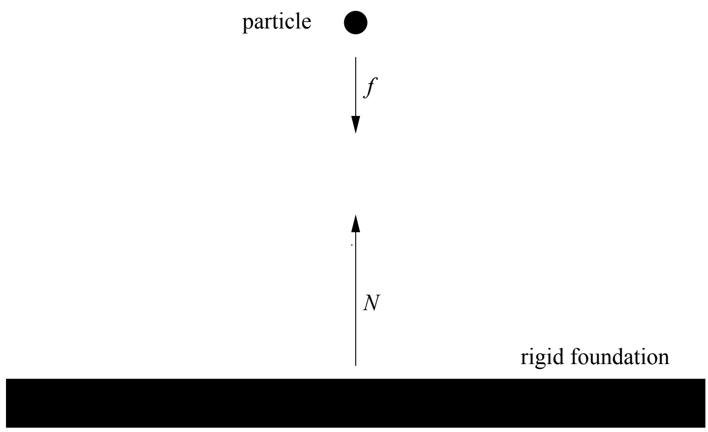


Figure 1. Dynamic contact of a particle.

2. Continuous formulations and some mathematical backgrounds

The motion of a particle in this physical situation is described by the ordinary differential equation (ODE)

$$u_{tt} = N + f \quad \text{for all } t \in (0, T],$$

where $u = u(t)$ is the displacement of a particle, $f = f(t)$ is a given body force, and $N = N(t)$ is a contact force. The acceleration of the particle, u_{tt}, is the second derivative of u with respect to time t, and T is the final time for the motion of the particle. When the particle drops down and hits the fixed flat rigid obstacle φ and bounces off, the Signorini contact conditions are applied which can be understood in terms of complementarity conditions (CCs). In general, the CCs $0 \le a \perp b \ge 0$ mean that the scalars a and b are nonnegative and either a or b is zero. Now, we can see that the contact conditions satisfy the CCs (2-2) where the flat rigid foundation φ does not depend on time t. When there is a gap between the particle and the rigid foundation ($u(t) > \varphi$), the contact force N must be zero, and when the particle is in contact with the rigid foundation ($u(t) = \varphi$), that is, there is no gap, the contact force takes place ($N(t) \ge 0$). We note that $u(t) \ge \varphi$ implies that the particle does not penetrate the rigid foundation unless the normal compliance applies to the stationary foundation. By Newton's third law, the contact forces N are always regarded as nonnegative. The physical situation is illustrated in Figure 1.

Thus, we establish the ODE and the CCs that describe the physical situation: for all $t \in (0, T]$,

$$u_{tt}(t) = N(t) + f(t), \tag{2-1}$$

$$0 \le u(t) - \varphi \perp N(t) \ge 0, \tag{2-2}$$

$$u^0 = u(0), \tag{2-3}$$

$$u_t^0 = u_t(0), \tag{2-4}$$

where u^0 is the initial displacement and u_t^0 is the initial velocity of the particle. For our convenience, it can be assumed that the flat rigid foundation $\varphi = 0$, without loss of generality. In order to prove the existence of solutions, (2-1) has to be considered in the sense of distributions and then we will seek solutions $u : [0, T] \to \mathbb{R}$ in appropriate spaces.

Let q and g be any functions. Then we introduce the little o notation:

$$q = o(g) \quad \text{provided} \quad \lim_{t\to\infty} \frac{|q(t)|}{|g(t)|} = 0.$$

This notation implies that the function g approaches infinity even faster than the function q does as $t \uparrow \infty$.

The Laplace transform of any function w, which is a useful tool for handling ODEs, is defined by

$$(\mathscr{L}w(t))(s) = \int_0^\infty w(t)e^{-st}\,dt. \tag{2-5}$$

It is important to take a restriction of the number s (possibly complex number) into consideration, in order to see the convergence of (2-5). Lemma 1 in Section 3 requires Lerch's theorem [Widder 1941, pp. 62–63]; generally speaking, it implies that if $(\mathscr{L}w)(s) = (\mathscr{L}\varpi)(s)$ with all s in some region of convergence, then $w(t) = \varpi(t)$ for *almost all* $t \in [0, T]$. This is called Lerch's cancellation law.

3. Conservation of energy

In this dynamic contact problem, the energy function $E(t)$ is defined by

$$E(t) := E[u, u_t] = \tfrac{1}{2}[u_t(t)]^2 - f(t)u(t), \tag{3-1}$$

where the first term and the second term in (3-1) are called the kinetic energy and the potential energy, respectively, and u_t denotes the velocity of a particle. One can see that the velocity u_t is replaced by the new variable v in Section 4.

If the conservation of energy is considered in terms of the *atom* level (see [Moreau et al. 1988]), its mathematical proof will be much harder. Showing conservation of energy might be the most difficult task in the dynamic contact problems with Signorini contact conditions. However, if functions are piecewise continuous, then the Laplace transform is one-to-one, which means that we can apply Lerch's cancellation law. In order to do so, we assume that the impact time period is not instantaneous; that is, the impact time period is $(t_* - \epsilon, t_* + \epsilon)$ with sufficiently small $\epsilon > 0$. In the following lemma, the minimum requirement is that the displacement u is piecewise smooth which implies that u is differentiable almost everywhere and u_t has a jump discontinuity at a finite number of points.

Lemma 1. *Assume that there is no change of body force and the solutions u satisfying the continuous formulations* (2-1)–(2-4) *are piecewise smooth and $u(t) = \varphi$ for all $t \in (t_* - \epsilon, t_* + \epsilon)$ with the fixed $t_* \in (0, \infty)$. If $E = o(e^t)$ as $t \uparrow \infty$, then energy conserves*; *that is*, $E(0) = E(t)$ *for almost all $t \in (0, \infty)$.*

Proof. Multiplying both sides of (2-1) by the velocity u_t, we have $u_{tt}u_t - fu_t = Nu_t$. Since $(d/dt)(u_t^2/2) = u_{tt}u_t$, we can obtain

$$\frac{d}{dt}\left(\frac{u_t^2}{2} - fu\right) = Nu_t.$$

Recall the CCs $0 \le u(t) - \varphi \perp N(t) \ge 0$ with $0 < t_* < t \le T$. There are two cases; if $N(t) = 0$, then $N(t)u_t = 0$ over the interval $(0, T]$, and if $N(t) > 0$ over $(t_* - \epsilon, t_* + \epsilon)$ and $N(t) = 0$ outside of $(t_* - \epsilon, t_* + \epsilon)$, then $u(t) = \varphi$ over $(t_* - \epsilon, t_* + \epsilon)$ and thus $N(t)u_t = 0$ on $(0, T]$. So $E(0) = E(t)$ for $t \in (0, T]$. Note the velocity u_t is piecewise continuous.

Now, we take the Laplace transform of both sides to get

$$\int_0^\infty \left(\frac{u_t^2}{2} - fu\right)' e^{-st}\,dt = \int_0^\infty Nu_t e^{-st}\,dt. \tag{3-2}$$

Here $'$ means the derivative with respect to time t. Integrating by parts we get

$$\begin{aligned} 0 &= \int_0^\infty \left(\frac{u_t^2}{2} - fu\right)' e^{-st}\,dt \\ &= \left[\left(\tfrac{1}{2}u_t^2 - fu\right)e^{-st}\right]_0^\infty + s\int_0^\infty \left(\frac{u_t^2}{2} - fu\right) e^{-st}\,dt. \end{aligned} \tag{3-3}$$

Since $E = o(e^t)$ as $t \uparrow \infty$, there is a constant $M > 0$ such that

$$|E(t)| = \left|\frac{u_t^2}{2} - fu\right| \le Me^t \quad \text{for some large } t > 0.$$

Since we require the convergence of the improper integral on the right side of (3-3), we need to impose the condition that $1 - s < 0$. Thus it follows from (3-3) that, for $s > 1$,

$$(\mathscr{L}E(t))(s) = (1/2s)\big(u_t^2(0) - 2fu(0)\big).$$

We can also see that $(\mathscr{L}E(0))(s) = (1/2s)(u_t^2(0) - 2fu(0))$ for $s > 0$. Thus, we note that the Laplace transform requires one-to-one mapping for only $s > 1$. Since $(\mathscr{L}E(t))(s) = (\mathscr{L}E(0))(s)$ for $s > 1$, $E(0) = E(t)$ for almost all $t \in (0, \infty)$, as required. □

Remarks 2. In Lemma 1, the displacement u may be semismooth (see its definition in [Facchinei and Pang 2003b, Section 7.4]), since we have the condition $u(t) = \varphi$ for all $t \in [t_* - \epsilon, t_* + \epsilon]$ with the fixed $t_* \in (0, \infty)$.

Unfortunately, the technique used in Lemma 1 does not work for the viscoelastic or elastic cases, since it is relatively more difficult to handle the elastic energy included in the energy function.

4. Numerical formulations and their convergence

In this section, we set up three numerical equations based on the continuous formulations (2-1)–(2-2), with (4-2) being an extra equation where we set the change in the displacement equal to the average velocity between the time steps. First, we partition the time interval $[0,\ T]$ such that

$$0 = t_0 < t_1 < t_2 < \cdots < t_l < \cdots < t_{n-1} < t_n = T,$$

where n is the number of time steps. The uniform time step $h = T/n$ is used and thus the size of the time step is $h = t_{l+1} - t_l$ and each discretized time is $t_l = lh$ for any integers $l \geq 0$. Then, the numerical approximations $u(t_l)$, $v(t_l)$ and $N(t_l)$ are denoted by u^l, v^l and N^l, respectively. Assume that there is no change of body force f. Using the implicit Euler method (sometimes referred to as the backwards Euler method) for the CCs, we are led to the following numerical formulations:

$$\frac{v^{l+1} - v^l}{h} = N^l + f, \tag{4-1}$$

$$\frac{u^{l+1} - u^l}{h} = \frac{v^{l+1} + v^l}{2}, \tag{4-2}$$

$$0 \leq u^{l+1} - \varphi \perp N^l \geq 0. \tag{4-3}$$

The solutions (u, v, N) of our contact problem (2-1)–(2-4) will be approximated by the numerical trajectories (u_h, v_h, N_h), which satisfy the numerical formulations (4-1)–(4-3); let $u_h(t)$ be a piecewise linear interpolant satisfying $u(t_l) = u^l$ and $u(t_{l+1}) = u^{l+1}$, and let $v_h(t)$ be a piecewise constant interpolant satisfying $v(t) = v^{l+1}$ for $t \in (t_l, t_{l+1}]$. We also set up the numerical approximation $N_h(t)$ of the contact forces, which is the step function; that is, $N(t) = N^l$ for $t \in [t_l, t_{l+1})$ and thus the approximation N_h has to be defined in the distributional sense to be

$$N_h(t) = h \sum_{l=0}^{n-1} \delta(t - (l+1)h) N^l, \tag{4-4}$$

where δ is the Dirac delta function. We also define the energy function for the discrete case to be

$$E(t_l) := E^l = \tfrac{1}{2}(v^l)^2 - f u^l, \tag{4-5}$$

which plays a very important role in showing the boundedness of numerical solutions from the theoretical perspective and addressing the stability in the numerical perspective.

Thanks to our numerical scheme, the numerical formulations (4-1)–(4-3) confirm the regularity of numerical solutions (u_h, v_h, N_h) for any $h > 0$. Lemma 3 demonstrates a possibility of energy conservation and supports the regularity of solutions.

Lemma 3. *Suppose that our numerical solutions satisfy the numerical formulations* (4-1)–(4-3) *for any time step $h > 0$ and the body force f is given as a constant function. If the initial data u^0, v^0 are finite*, *we have the following estimates*:

$$\max_{0\le l\le n} |v^l| \le \sqrt{\big(2E^0 + 2|f|\,|u^0| + |f|T\big)\big(1 + |f|T e^{|f|T}\big)} < \infty,$$

$$\max_{0\le l\le n} |u^l| \le |u^0| + \frac{T}{2}\sqrt{\big(2E^0 + 2|f|\,|u^0| + |f|T\big)\big(1 + |f|T e^{|f|T}\big)} < \infty.$$

Proof. Using (4-1) and (4-2), for any $h > 0$ we have

$$\frac{(v^{l+1})^2 - (v^l)^2}{2h} = \frac{N^l(u^{l+1} - u^l)}{h} + \frac{f(u^{l+1} - u^l)}{h}.$$

It follows from the numerical CCs that

$$\begin{aligned}\frac{(v^{l+1})^2 - (v^l)^2}{2} - f(u^{l+1} - u^l) &= N^l(u^{l+1} - u^l)\\ &= N^l[u^{l+1} - \varphi - (u^l - \varphi)]\\ &= -N^l(u^l - \varphi) \le 0.\end{aligned} \tag{4-6}$$

Therefore, from (4-6),

$$E^{l+1} = \tfrac{1}{2}(v^{l+1})^2 - f u^{l+1} \le \tfrac{1}{2}(v^l)^2 - f u^l = E^l.$$

So repeating the inequality at each time step $t = t_l$, we can get $E^l \le E^0$ for any $l \ge 1$. Thus,

$$\tfrac{1}{2}(v^l)^2 \le E^0 + f u^l \le E^0 + |f|\,|u^l| \le E^0 + |f|\bigg(|u^0| + \frac{1}{2}\int_0^{t_l} |v_h(\tau)|\,d\tau\bigg).$$

Note that

$$|u^l| \le |u^0| + \frac{1}{2}\int_0^{t_l} |v_h(\tau)|\,d\tau. \tag{4-7}$$

Since v_h is a constant interpolant, by Cauchy's inequality, we can set up

$$|v_h(t_l)|^2 \leq 2E^0 + 2|f|\left(|u^0| + \frac{1}{2}T + \frac{1}{2}\int_0^{t_l} |v_h(\tau)|^2\,d\tau\right).$$

Using Gronwall's inequality, we have

$$(v^l)^2 = |v_h(t_l)|^2 \leq \big(2E^0 + 2|f||u^0| + |f|T\big)\big(1 + |f|Te^{|f|T}\big) \quad \text{for any } l \geq 0. \tag{4-8}$$

It also follows from (4-7)–(4-8) that

$$|u^l| \leq |u^0| + \frac{T}{2}\sqrt{\big(2E^0 + 2|f||u^0| + |f|T\big)\big(1 + |f|Te^{|f|T}\big)},$$

as desired. □

We note that the estimates in Lemma 3 can be obtained even if the body force f is not a constant function. Now, we introduce notations to see how to show the existence of solutions. If $u : [0, T] \to \mathbb{R}$ is continuous, then the p-th Hölder norm of u is defined by

$$\|u\|_{C^p[0,T]} = \sup_{t\in[0,T]} |u(t)| + \sup_{s\neq t\in[0,T]} \frac{|u(t) - u(s)|}{|t - s|^p}.$$

Considering Hölder spaces would be useful to show the compactness of continuous solutions for PDEs. Applying Lemma 3 to the construction of numerical solutions, we can see that $u_h \in C[0, T]$ and $v_h \in L^\infty[0, T]$ for any time step size $h > 0$. However, showing the boundedness of solutions is not enough to prove the existence of solutions. Thus, we need compactness to show that u_h converges strongly in $C[0, T]$ as $h \downarrow 0$. Now, we choose any s_1, s_2 such that $0 \leq s_1 < s_2 \leq T$, $|s_1 - s_2| < h$, $s_1 \in (t_{l-1}, t_l]$, and $s_2 \in (t_l, t_{l+1}]$. We can use Lemma 3 again to have

$$\begin{aligned} |u_h(s_2) - u_h(s_1)| &= |u_h(s_2) - u_h(s_1)|^p |u_h(s_2) - u_h(s_1)|^{1-p} \\ &\leq \tfrac{1}{2}\left(\int_{s_1}^{s_2} |v_h(t-h) + v_h(t)|\,dt\right)^p \Big(|u_h(s_2)| + |u_h(s_1)|\Big)^{1-p} \\ &\leq C|s_2 - s_1|^p. \end{aligned}$$

Consequently, we can see easily that the interpolant $u_h \in C^p[0, T]$ with exponent $0 < p \leq 1$. By the Arzelà–Ascoli theorem, $C^p[0, T]$ is compactly embedded in $C[0, T]$. Therefore, there is a subsequence of u_h (denoting this sequence by u_h), such that u_h converges strongly to u, that is, $u_h \to u$ in $C[0, T]$, as $h \downarrow 0$.

We regard the numerical contact force N_h as the Borel measure on the time interval $[0, T]$:

$$N_h([0, T]) = \int_{[0,T]} N_h(t)\,dt.$$

Using (4-4), we can show the boundedness of N_h easily. Recalling the numerical formulation (4-1), we have

$$\int_{[0,T]} N_h(t)\,dt = h\sum_{l=0}^{n-1} N^l = v^n - v^0. \tag{4-9}$$

Equation (4-9) does make sense from a physical point view, since the velocity v moves down initially, and thus $v^0 < 0$ and the particles bounce off, and thus their velocity $v^n > 0$. Therefore, for any $h > 0$ we have

$$\int_{[0,T]} N_h(t)\,dt \le \sqrt{\bigl(2E^0 + 2|f||u^0| + |f|T\bigr)\bigl(1 + |f|Te^{|f|T}\bigr)} - v^0 < \infty.$$

Applying the Riesz representation theorem [Renardy and Rogers 1993, p. 199] and Alaoglu's theorem [ibid., p. 209], N_h has a subsequence that is weakly* convergent to N in the sense of measures as $h \downarrow 0$. We denote the subsequence by N_h. Thus, $N_h \rightharpoonup^* N$. Finally, we check if our solutions, which converged by numerical trajectories, satisfy the CCs (2-2). Since $u_h - \varphi \ge 0$ and $u_h \to u$ as $h \downarrow 0$, we have $u - \varphi \ge 0$. Since $N_h \ge 0$ and $N_h \rightharpoonup^* N$ as $h \downarrow 0$, we have $N \ge 0$. We claim that $N(u-\varphi) = 0$ in the weak sense. Taking the integral of $N_h(u_h - \varphi)$, we have

$$\begin{aligned}\int_0^T N_h(t)(u_h(t)-\varphi)\,dt &= h\int_0^T \sum_{l=0}^{n-1} \delta(t-(l+1)h)N^l(u_h(t)-\varphi)\,dt\\ &= h\int_0^T \sum_{l=0}^{n-1} N^l(u^{l+1}-\varphi)\,dt = 0.\end{aligned} \tag{4-10}$$

We notice that (4-10) is identified by the numerical CCs (4-3). Finally, we claim that $\int_0^T N_h(t)(u_h(t)-\varphi)\,dt \to \int_0^T N(t)(u(t)-\varphi)\,dt$. Since $u_h \to u$ and $N \rightharpoonup^* N$ as $h \downarrow 0$,

$$\begin{aligned}&\left|\int_0^T N_h(t)(u_h(t)-\varphi)\,dt - \int_0^T N(t)(u(t)-\varphi)\,dt\right|\\ &\quad\le \int_0^T \bigl|N_h(t)(u_h(t)-\varphi) - N(t)(u(t)-\varphi)\bigr|\,dt\\ &\quad= \int_0^T \bigl|N_h(t)(u_h(t)-\varphi) - N_h(t)(u(t)-\varphi) + N_h(t)(u(t)-\varphi) - N(t)(u(t)-\varphi)\bigr|\,dt\\ &\quad\le \int_0^T \bigl|N_h(t)(u_h(t)-u)\bigr|\,dt + \int_0^T \bigl|(N_h(t)-N(t))(u(t)-\varphi)\bigr|\,dt \to 0.\end{aligned}$$

Therefore, by the squeeze theorem, we can obtain

$$0 = \int_0^T N_h(t)(u_h(t)-\varphi)\,dt \to \int_0^T N(t)(u(t)-\varphi)\,dt \quad \text{as } h \downarrow 0.$$

Thus, we conclude that there exist solutions $u \in C[0, T] \cap C^p[0, T] \cap W^{1,\infty}[0, T]$ with $0 < p \leq 1$ satisfying (2-1)–(2-4), where the space $W^{1,\infty}[0, T]$ is defined by $W^{1,\infty}[0, T] = \{u \mid \sup_{0 \leq t \leq T}(|u(t)| + |u_t(t)|) < \infty\}$. We notice that the derivative u_t has to be considered in the weak sense.

Lemma 4 requires an additional condition that the solutions are absolutely continuous. We denote by $\mathrm{COR}(u)$ the coefficient of restitution for the particle which is defined by $\mathrm{COR}(u) = -v_a/v_b$, where v_a is the velocity after contact and v_b is the velocity before contact. Therefore, solutions that we seek have to be considered in the stronger sense in order to prove their uniqueness. We note that showing the uniqueness is trivial unless we take contact forces into consideration.

Lemma 4. *Suppose that there exist two solutions* (u, N_1) *and* (w, N_2) *satisfying* (2-1)–(2-4). *If either the initial velocity* $u_t^0 = 0$ *and* $u_t(t) = w_t(t) = 0$ *for some* $t \in [0, T]$ *or* $\mathrm{COR}(u) = \mathrm{COR}(w) = 1$, *then the two solutions are the same*; *that is*, $u(t) = w(t)$ *for all* $t \in [0, T]$ *and* $N_1(t) = N_2(t)$ *for almost all* $t \in [0, T]$.

Proof. We assume that there exist two solutions (u, N_1) and (w, N_2) such that

$$u_{tt} = N_1(t) + f(t) \quad \text{and} \quad w_{tt} = N_2(t) + f(t). \tag{4-11}$$

Letting $z(t) = u(t) - w(t)$, it is easy to see that $z_{tt} = N_1(t) - N_2(t)$. Multiplying by z_t and taking the integral over $[0, t] \subset [0, T]$, we can obtain

$$\begin{aligned}\int_0^t z_{\tau\tau} z_\tau \, d\tau &= \int_0^t (N_1(\tau) - N_2(\tau))(u_\tau(\tau) - w_\tau(\tau)) \, d\tau \\ &= \int_0^t N_1(\tau)u_\tau(\tau) - N_1(\tau)w_\tau(\tau) - N_2(\tau)u_\tau(\tau) + N_2(\tau)w_\tau(\tau) \, d\tau.\end{aligned}$$

In Lemma 1, it has been shown from the CCs (2-2) that

$$N_1(\tau)u_\tau(\tau) = N_2(\tau)w_\tau(\tau) = 0.$$

Using the two equations in (4-11) and applying integration by parts, we have

$$\begin{aligned}&\frac{1}{2}\int_0^t \frac{d}{d\tau}(z_\tau^2(\tau)) \, d\tau \\ &\quad = -\int_0^t N_1(\tau)w_\tau(\tau) + N_2(\tau)u_\tau(\tau) \, d\tau \\ &\quad = -\int_0^t u_{\tau\tau}(\tau)w_\tau(\tau) + w_{\tau\tau}(\tau)u_\tau(\tau) \, d\tau \\ &\quad = -\left(u_t(t)w_t(t) - u_t(0)w_t(0) - \int_0^t u_\tau(\tau)w_{\tau\tau}(\tau) \, d\tau + \int_0^t w_{\tau\tau}(\tau)u_\tau(\tau) \, d\tau\right) \\ &\quad = -u_t(t)w_t(t) + u_t(0)w_t(0).\end{aligned} \tag{4-12}$$

If the initial two velocities $u_t(0) = w_t(0) = 0$ and $u_t(t) = w_t(t) = 0$ for some $t \in [0, T]$, it is easy to see from (4-12) that

$$\frac{1}{2}\int_0^t \frac{d}{d\tau}(z_\tau^2(\tau))\,d\tau = z_t^2(t) - z_t^2(0) = 0. \tag{4-13}$$

If $\mathrm{COR}(u) = \mathrm{COR}(w) = 1$, the identities (4-13) also hold. Since the two solutions satisfy the initial data (2-4), from both cases we have $z_t^2(0) = 0$, which gives us $u_t(t) = v_t(t)$ for almost all $t \in [0, T]$. Therefore, $u(t) = w(t)$ for all $t \in [0, T]$ and the corresponding contact forces $N_1(t) = N_2(t)$ for almost all $t \in [0, T]$, as required. □

When we consider Equation (4-12), we could impose the more general condition that $\mathrm{COR}(u), \mathrm{COR}(w) \geq 1$. However, the condition requires that the obstacle is deformed. Therefore, the uniqueness is shown under the assumption that particles collide with the rigid foundation elastically.

5. Numerical results and discussion

In this section, numerical results are presented implementing several methods. For the sake of simplicity, we assume that $\varphi = 0$ throughout this section. Even though we use different methods with $f = 0$ and without considering the coefficient of restitution, we obtain almost equivalent numerical results (simulations) which are displayed in Figures 2–3. These results may enable us to demonstrate some evidence for the numerical stability. Lemma 1 is proven by the main idea that our numerical schemes guarantee that energy does not increase. We shall observe numerical results later on that show the numerical evidence for energy conservation. This means that the numerical solutions are stable, because they satisfy the criterion that solutions never show increasing energy. The first method that we describe is an infinite-dimensional approach that has a completely different perspective from the other two numerical schemes. Indeed, the infinite-dimensional approach is motivated by the normal compliance (see [Klarbring et al. 1988]). If the contact conditions are rather changed to the normal compliance condition, the contact forces N will be replaced by

$$N(t) = p(u(t) - \varphi),$$

where a prescribed function p can be defined by $p(r) = c_N \max(r, 0)$ for $c_N \geq 0$ and c_N is called the normal compliance stiffness coefficient. As we shall see in Lemma 5, contact forces satisfying Signorini contact conditions (or CCs) can be approximated as $c_N \uparrow \infty$. Instead of using Signorini contact conditions, the normal compliance condition enables us to consider well-conditioned dynamic contact problems and more realistic physical situations. In Section 5.1, we shall use the normal compliance to see how to construct approximations, depending on the parameter of penetration, $\epsilon > 0$. Mathematically speaking, the normal compliance plays a fundamental role in showing better regularity of solutions and the uniqueness of solutions for dynamic contact problems. In Section 5.2, we shall discuss two numerical methods based on time discretization; one is directly implemented from the numerical CCs (4-3) and another is carried out with the nonsmooth Newton's method. There is a classification for dynamic contact problems on $\mathbb{R}^d$ with $d \geq 1$; one class is a class of thick obstacle problems and the other is a class of boundary thin obstacle problems. The meaning of "thick" is that obstacles (or constraints) are applied over a subset of the whole domain, while the meaning of "thin" is that the obstacles are applied on a subset of only the boundary of the domain. Readers who are interested in this classification may refer to [Ahn and Stewart 2006]. Concerning the corresponding numerical schemes for the two classes, the nonsmooth Newton's method will be very useful and efficient for thick obstacle problems and it is not necessary for the boundary thin obstacle problems in the case that $d = 1$.

5.1. ***Numerical results via the infinite-dimensional approach.*** Our physical interpretation is that particles touch and penetrate a rigid obstacle over the short contact time period $(t_* - \epsilon, t_* + \epsilon)$ with $\epsilon > 0$. We assume that the solutions to the ODE (2-1) are smooth enough. Then, assuming that $f(t) = 0$, we can construct the natural cubic splines to interpolate the solutions:

$$\begin{cases} S_1(t) = -\dfrac{3}{2}(t - t_* + \epsilon) + \dfrac{1}{2\epsilon^2}(t - t_* + \epsilon)^3 & \text{on } (t_* - \epsilon, t_*], \\ S_2(t) = -\epsilon + \dfrac{3}{2\epsilon}(t - t_*)^2 - \dfrac{1}{2\epsilon^2}(t - t_*)^3 & \text{on } [t_*, t_* + \epsilon), \end{cases}$$

where ϵ is called an approximate parameter of penetration. The piecewise linear functions below are included in the entire solution for the displacement:

$$\begin{cases} S_3(t) = -\frac{3}{2}(t - t_*) - \frac{3}{2}\epsilon & \text{on } [0, t_* - \epsilon], \\ S_4(t) = \frac{3}{2}(t - t_*) - \frac{3}{2}\epsilon & \text{on } [t_* + \epsilon, \infty). \end{cases}$$

So $S_3(t)$ and $S_4(t)$ are the outer functions for the piecewise solution for the displacement of the particle. Let $S_\epsilon = S_1 \cup S_2 \cup S_3 \cup S_4$ be an approximation of the solutions u. Then, this approximation S is a smooth function, but it does not satisfy Signorini contact conditions. In Lemma 5, we will see that the approximation of the contact forces N_ϵ satisfying the normal compliance condition converges to δ in the distributional sense, as $\epsilon \downarrow 0$. Since we expect contact forces over the interval $(t_* - \epsilon, t_* + \epsilon)$, we consider the translated Dirac delta function $\delta(t - t_*)$ in Lemma 5. Let Ω be a nonempty open set in $\mathbb{R}$. Then, the set of all test functions on Ω is denoted by $\mathcal{D}(\Omega)$.

Lemma 5. *Let $N_\epsilon = S_1'' \cup S_2''$ be an approximation of contact forces over the interval $(t_* - \epsilon, t_* + \epsilon)$ for $t_* > 0$ with small $\epsilon > 0$. Then, $N_\epsilon \to \delta$ in the sense of distributions.*

Proof. We consider the sequence of contact forces as follows:

$$N_\epsilon(t) := \frac{1}{6}\begin{cases} S_1''(t) = \dfrac{3}{\epsilon^2}(t - t_* + \epsilon) & \text{if } t \in (t_* - \epsilon, t_*], \\ S_2''(t) = \dfrac{3}{\epsilon} - \dfrac{3}{\epsilon^2}(t - t_*) & \text{if } t \in [t_*, t_* + \epsilon), \\ \qquad 0 & \text{if } t \in (0, t_* - \epsilon] \cup [t_* + \epsilon, \infty). \end{cases}$$

Then, we claim that, for any test function $\psi \in \mathscr{D}(\mathbb{R}^+)$,

$$\int_0^\infty N_\epsilon(t)\psi(t)\,dt \to \int_0^\infty \delta(t - t_*)\psi(t)\,dt \quad \text{as } \epsilon \downarrow 0.$$

For any fixed $t_* > 0$ and $\epsilon > 0$ we define the integral functions F_1 and F_2 to be

$$F_1(\tau) = \int_{t_*}^{\tau} (t - t_* + \epsilon)\psi(t)\,dt,$$
$$F_2(\tau) = \int_{t_*}^{\tau} \Big(1 - \frac{1}{\epsilon}(t - t_*)\Big)\psi(t)\,dt \quad \text{for } \tau > 0.$$

Thus, it follows that

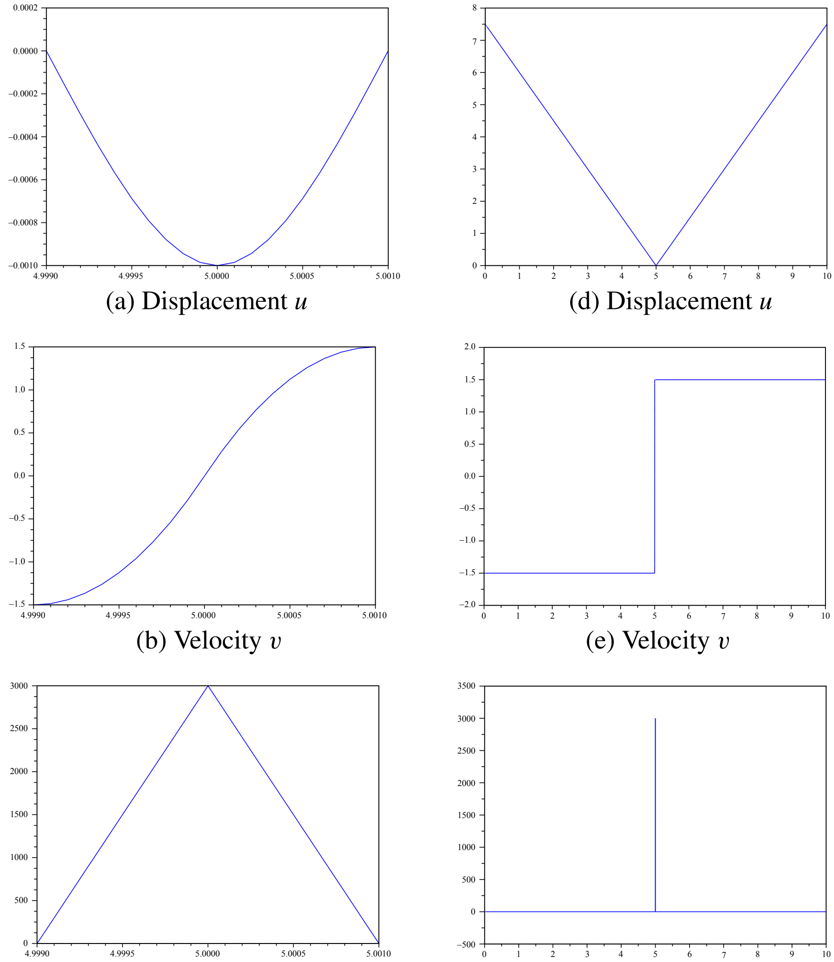
$$\begin{aligned}\int_0^\infty & N_\epsilon(t)\psi(t)\,dt \\ &= \frac{1}{2\epsilon^2}\int_{t_*-\epsilon}^{t_*} (t - t_* + \epsilon)\psi(t)\,dt + \frac{1}{2\epsilon}\int_{t_*}^{t_*+\epsilon} \Big(1 - \frac{1}{\epsilon}(t - t_*)\Big)\psi(t)\,dt \\ &= \frac{1}{2\epsilon}\frac{F_1(t_* - \epsilon) - F_1(t_*)}{-\epsilon} + \frac{1}{2}\frac{F_2(t_* + \epsilon) - F_2(t_*)}{\epsilon} = \frac{1}{2\epsilon}\frac{dF_1(t_*)}{dt} + \frac{1}{2}\frac{dF_2(t_*)}{dt}.\end{aligned}$$

By the fundamental theorem of calculus, part 2, we can obtain

$$\int_0^\infty N_\epsilon(t)\psi(t)\,dt = \tfrac{1}{2}\psi(t_*) + \tfrac{1}{2}\psi(t_*) = \psi(t_*) = \int_0^\infty \delta(t - t_*)\psi(t)\,dt \quad \text{as } \epsilon \downarrow 0,$$

which implies that $N_\epsilon \to \delta$ in the distributional sense as $\epsilon \downarrow 0$, as desired. □

The approximation S_ϵ computed with the small parameter $\epsilon = 10^{-3}$ is presented in Figure 2. The top of Figure 2 is a visual representation of the natural cubic splines ($S_1(t)$ and $S_2(t)$ and their applicable derivatives) for the displacement, velocity, and contact forces. We can observe a little penetration of a particle due to the parameter $\epsilon = 10^{-3}$. In addition, we can guess that the less the penetration depth is, the larger the magnitude of the contact forces is. While the cubic splines only consider the time period when the particle is in contact with the rigid obstacle, the piecewise linear functions $S_3(t)$ and $S_4(t)$ and their applicable derivatives are



(a) Displacement u (d) Displacement u

(b) Velocity v (e) Velocity v

(c) Contact force N (f) Contact force N

Figure 2. For $\epsilon = 0.001$ and $t_* = 5$, the graphs on the left, (a)–(c), represent the natural cubic splines for u, v, and N over the short time period $[t_* - \epsilon, t_* + \epsilon]$; the graphs on the right, (d)–(f), represent the entire piecewise functions for u, v, and N.

added to the ends of the splines to get the total picture of what is really happening throughout the particle's motion. This can be seen in the graphs in the right-hand column of Figure 2. Unfortunately, this infinite approach does not work in the dynamic adhesive contact model; see [Wolf 2012].

5.2. ***Numerical results via the finite-dimensional approach.*** In this subsection, two different numerical schemes are introduced and it is assumed that the body force f is a constant.

First, we explain our numerical scheme where we can directly compute the next step solution from the numerical CCs (4-3). The numerical equations (4-1)–(4-3) can be manipulated so that we obtain the solutions (u^{l+1}, N^l) at the next time step $t = t_{l+1}$. Using (4-2), from (4-1) we can solve for the next step solution u^{l+1}:

$$u^{l+1} = h\left(\frac{h(N^l + f)}{2} + v^l\right) + u^l. \tag{5-1}$$

The next step solution u^{l+1} needs to satisfy the CCs (4-3). So, if $u^{l+1} > \varphi$, we accept the solution (u^{l+1}, N^l) with $N^l = 0$. If $u^{l+1} = \varphi$, then we need to compute the previous contact force N^l:

$$N^l = \frac{2}{h}\left(\frac{\varphi - u^l}{h} - v^l\right) - f.$$

Once the next step solution u^{l+1} is obtained, we can compute the next step velocity v^{l+1} from the extra equation (4-2):

$$v^{l+1} = \frac{2}{h}(u^{l+1} - u^l) - v^l.$$

Secondly, we apply the nonsmooth Newton's method to compute u^{l+1}. Basically, solutions of dynamic contact problems are not smooth, because of the nature of the CCs. However, we can reformulate the approach by substituting a smooth function; see [Facchinei and Pang 2003a, p. 73 ff.]. One of the functions commonly used for this purpose is the Fischer–Burmeister function F, given by

$$F(a, b) = (a + b) - \sqrt{a^2 + b^2}. \tag{5-2}$$

It is not hard to see that $0 \le a \perp b \ge 0$ is equivalent to the equation $F(a, b) = 0$. This function is not still applied practically. In order to avoid the singularity happening, we set up the approximate function

$$F_\varepsilon(a, b) = (a + b) - \sqrt{a^2 + b^2 + \varepsilon}$$

for sufficiently small $\varepsilon > 0$, where ε is called a smoothing parameter. As $\varepsilon \downarrow 0$, $F_\varepsilon(a, b) \to F(a, b)$ in the strong sense.

Thanks to the equations (4-1)–(4-2), we can express the previous contact force N^l in terms of the next step solution u^{l+1}:

$$N^l = \frac{2}{h}\left(\frac{u^{l+1} - u^l}{h} - v^l\right) - f.$$

Thus, finding the next step solution u^{l+1} satisfying the CCs (4-3) is equivalent to finding the solution u^{l+1} satisfying the following nonlinear equation:

$$\left(u^{l+1} + \left[\frac{2}{h}\left(\frac{u^{l+1}}{h} - \frac{u^l}{h} - v^l\right) - f\right]\right)$$
$$= \sqrt{(u^{l+1})^2 + \left[\frac{2}{h}\left(\frac{u^{l+1}}{h} - \frac{u^l}{h} - v^l\right) - f\right]^2 + \varepsilon}. \quad (5\text{-}3)$$

Now, we move the right side of (5-3) and replace the left side by the nonlinear function $S_\varepsilon(u^{l+1})$. So the next step solution u^{l+1} can be found for nonlinear equation $S_\varepsilon(u^{l+1}) = 0$. In order to compute the next step solution u^{l+1}, we can set up Newton's iterative formula:

$$u^{l+1}_{m+1} = u^{l+1}_m - \frac{S_\varepsilon(u^{l+1}_m)}{S'_\varepsilon(u^{l+1}_m)},$$

where u^{l+1}_{m+1} is the next solution and u^{l+1}_m is the previous solution for Newton's iteration. We note that S'_ε does not contain any singularity.

Based on the numerical equations (4-1)–(4-3), we tested the two numerical schemes. The results, which are almost indistinguishable, are shown in Figures 3 and 4, using an initial displacement of $u^0 = 5$, an initial velocity of $v^0 = -1$, an end time $T = 10$, and the step size $h = 0.001$. The body force f is not applied in this numerical experiment. When we implement the nonsmooth Newton's method, the smoothing parameter $\varepsilon = 10^{-15}$ is used and 10^{-15} is used for the stop criterion.

As can be seen in the left column of graphs in Figure 3, with no coefficient of restitution, the particle's motion reflects that of an absolute value function. Also note that we see a very similar graph as our natural cubic spline for the particle's displacement (as was displayed in Figure 2). Its velocity resembles the Heaviside function, as expected from our continuous result for the velocity of the particle. The impulse function δ can be seen in the graph of the contact force. The bottom left picture in Figure 3 supports conservation of energy numerically.

Numerically, we would also like to consider the particle's motion with a given coefficient of restitution since this would be more realistic. To change our numerical code to take the COR into account, we must alter the velocity at the instant that the particle is in contact with the surface where

$$0 \le \text{COR} = \frac{-v_a}{v_b} \le 1.$$

As expected, when a coefficient of restitution is introduced into our numerical formulations (in this inelastic case COR $= 0.75$), both the displacement and the velocity of the particle are dampened after impact; see Figure 3, right column. The implementation of a coefficient of restitution has no effect on the results of the contact force, but does have a rather large effect on the graph of the energy function. With a coefficient of restitution, we see that energy is lost after the particle's impact,

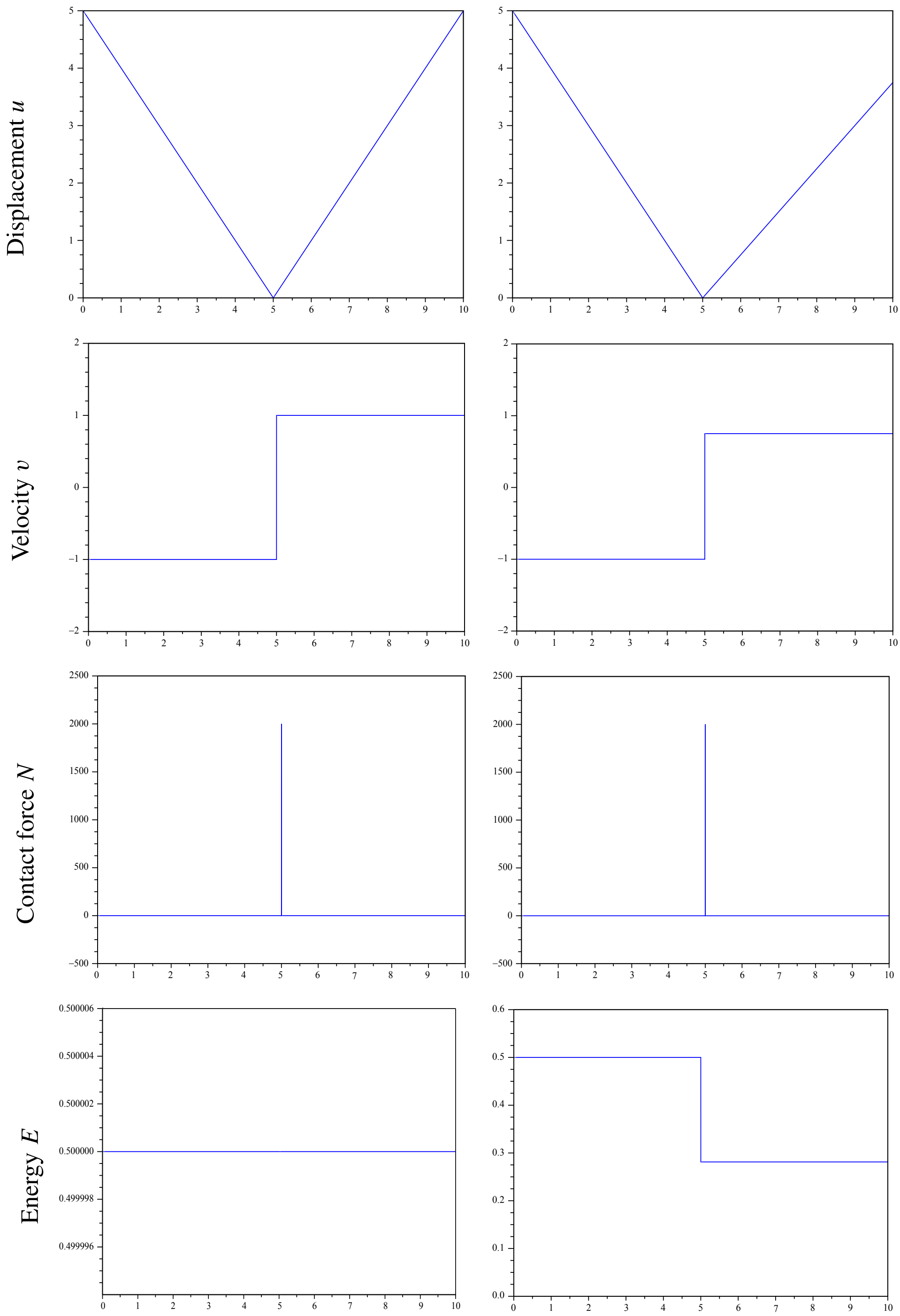


Figure 3. Numerical results without considering the coefficient of restitution (left) and with COR $= 0.75$ (right).

which can be shown theoretically. Also, the nonsmooth Newton's method with the Fischer–Burmeister equation still works very well when a coefficient of restitution is thrown into the mix.

Going back to our original numerical equation of motion (4-1), we note that we still need to incorporate a body force into the system. In a real-world sense of the situation, there is no better choice for a body force to impose on the particle than one that resembles Earth's gravitational force.

With this gravity-like body force, $f(t) = -9.80665$, we see some interesting graphs in Figure 4. The left column shows the simulations without a coefficient of resitution. The top graph, for the displacement, shows that the body force causes the particle to repeatedly bounce off the rigid obstacle until coming to a stop at around $t = 7.5$ for the selected initial conditions. However, we note that the height of the bounces does not decrease at a constant rate when only a body force is applied. The velocity shows a continual "zig-zag" centered about a velocity of zero. Conceptually, we can agree that the body force would continually pull the particle down, causing an increasingly negative velocity before bouncing back up, causing a jump of the function to a positive value, before falling again. Graphs of the contact forces each show multiple Dirac deltas, whose magnitude decreases over time until the particle comes to rest. With the energy function, like the contact and displacement graphs, energy decreases in steps with just a body force applied. Without considering a coefficient of restitution one might expect that energy conserves. Indeed, as the time step size h_t gets smaller and smaller, numerical simulations show that the energy function becomes flatter than the graph in the left column of Figure 4.

The application of both a coefficient of restitution and a body force combines to give us the most realistic solutions possible when thinking of a real-world situation. As seen in the right-hand column of Figure 4, the coefficient of restitution, in addition to the body force, gives us solutions for the displacement, velocity, and energy function that trend more steadily in comparison to those on the left column.

6. Conclusion

In this paper, we consider a second-order ODE with constraints. The existence of solutions is proved by using time discretization and passing to the limit as the time step size h decreases to zero. Although conservation of energy and uniqueness are proven in this paper under some restrictive assumptions, they are still open questions in general. Several numerical methods are introduced to present simulations which support conservation of energy. The two numerical methods provide almost identical results when we use the same input data. Therefore, our numerical schemes seem to be reasonably stable. In our future work, we will investigate a possibility of doing

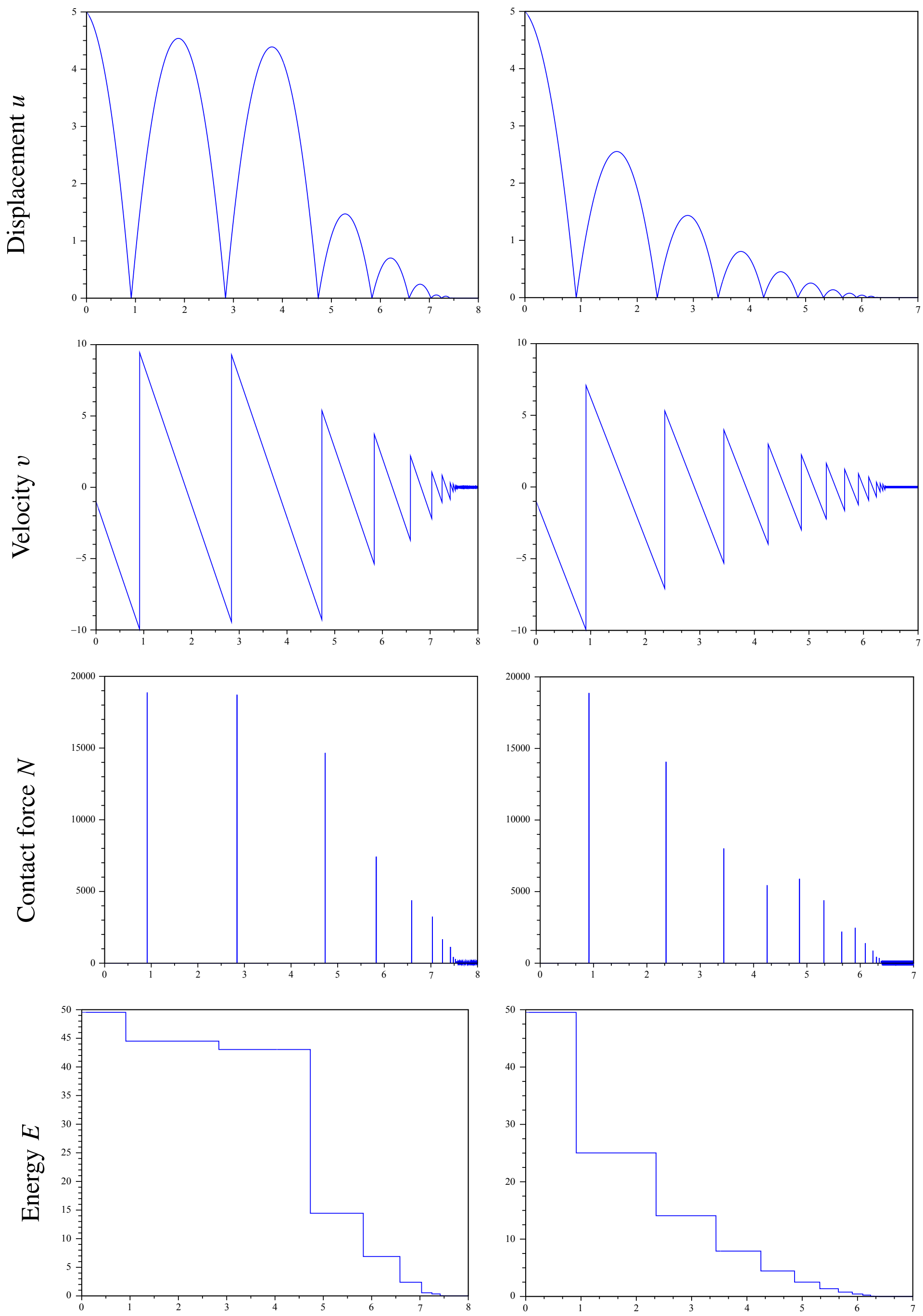


Figure 4. Numerical results with a body force of $f(t) = -9.80665$, representing gravity: without considering a coefficient of restitution (left column) and with COR $= 0.75$ (right column).

error analysis and study a more realistic contact model (see [Wolf 2012]) where we add the effect of a bonding field.

Acknowledgments

The authors thank Dr. Debra Ingram, department chair, who has improved the presentation of the introduction in this paper. The authors also thank the anonymous referee for helpful comments.

References

[Ahn 2007] J. Ahn, "A vibrating string with dynamic frictionless impact", *Appl. Numer. Math.* **57**:8 (2007), 861–884. MR 2008c:35166 Zbl 1114.74021

[Ahn 2008] J. Ahn, "Thick obstacle problems with dynamic adhesive contact", *M2AN Math. Model. Numer. Anal.* **42**:6 (2008), 1021–1045. MR 2009j:74072 Zbl 1149.74043

[Ahn 2012] J. Ahn, "A viscoelastic Timoshenko beam with Coulomb law of friction", *Appl. Math. Comput.* **218**:13 (2012), 7078–7099. MR 2880294 Zbl 06056829

[Ahn and Stewart 2006] J. Ahn and D. E. Stewart, "Existence of solutions for a class of impact problems without viscosity", *SIAM J. Math. Anal.* **38**:1 (2006), 37–63. MR 2007g:35158 Zbl 1116.35096

[Ahn and Stewart 2009] J. Ahn and D. E. Stewart, "Dynamic frictionless contact in linear viscoelasticity", *IMA J. Numer. Anal.* **29**:1 (2009), 43–71. MR 2010b:74018 Zbl 1155.74029

[Ahn et al. 2012] J. Ahn, K. L. Kuttler, and M. Shillor, "Dynamic contact of two Gao beams", *Electron. J. Diff. Equ.* **2012**:194 (2012), 1–42. MR 3001680

[Facchinei and Pang 2003a] F. Facchinei and J.-S. Pang, *Finite-dimensional variational inequalities and complementarity problems, I*, Springer, New York, 2003. MR 2004g:90003a Zbl 1062.90001

[Facchinei and Pang 2003b] F. Facchinei and J.-S. Pang, *Finite-dimensional variational inequalities and complementarity problems, II*, Springer, New York, 2003. MR 2004g:90003b Zbl 1062.90002

[Gao 1996] D. Y. Gao, "Nonlinear elastic beam theory with application in contact problems and variational approaches", *Mech. Res. Comm.* **23**:1 (1996), 11–17. MR 96m:73026 Zbl 0843.73042

[Hertz 1881] H. Hertz, "Über die Berührung fester elastischer Körper", *Journal für die reine und angewandte Mathematik* **92** (1881), 156–171. Zbl 14.0807.01

[Kikuchi and Oden 1988] N. Kikuchi and J. T. Oden, *Contact problems in elasticity: a study of variational inequalities and finite element methods*, SIAM Studies in Applied Mathematics **8**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1988. MR 89j:73097 Zbl 0685.73002

[Klarbring et al. 1988] A. Klarbring, A. Mikelić, and M. Shillor, "Frictional contact problems with normal compliance", *Internat. J. Engrg. Sci.* **26**:8 (1988), 811–832. MR 89j:73098 Zbl 0662.73079

[Moreau et al. 1988] J.-J. Moreau, P. D. Panagiotopoulos, and G. Strang (editors), *Topics in nonsmooth mechanics*, Birkhäuser, Basel, 1988. MR 89c:00054 Zbl 0646.00014

[Renardy and Rogers 1993] M. Renardy and R. C. Rogers, *An introduction to partial differential equations*, Texts in Applied Mathematics **13**, Springer, New York, 1993. MR 94c:35001 Zbl 0917.35001

[Signorini 1933] A. Signorini, "Sopra alcune questioni di elastostatica", *Atti della Societa Italiana per il Progresso delle Scienze* (1933).

[Stewart 2000] D. E. Stewart, "Rigid-body dynamics with friction and impact", *SIAM Rev.* **42**:1 (2000), 3–39. MR 2001c:70017 Zbl 0962.70010

[Widder 1941] D. V. Widder, *The Laplace transform*, Princeton Mathematical Series **6**, Princeton University Press, 1941. MR 3,232d Zbl 0063.08245

[Wolf 2012] J. R. Wolf, *Dynamic contact of a particle: Mathematical theories and numerical approaches*, honors thesis, Arkansas State University, 2012, http://dbellis.library.astate.edu/vwebv/holdingsInfo?bibId=1541047.

Received: 2011-10-04 Revised: 2012-04-24 Accepted: 2012-05-06

jahn@astate.edu
Department of Mathematics and Statistics, Arkansas State University, P.O. Box 70, State University, AR 72467, United States

jared.wolf@smail.astate.edu
Department of Mathematics and Statistics, Arkansas State University, P.O. Box 70, State University, AR 72467, United States

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2013 is US $105/year for the electronic version, and $145/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2013 vol. 6 no. 2

1944-4176(2013)6:2;1-9