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The 3-point Steiner problem on a cylinder

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The 3-point Steiner problem in the Euclidean plane is to find the least length path network connecting three points. In this paper we will demonstrate an algorithm for solving the 3-point Steiner problem on the cylinder.

1. Introduction

Say we have three points on a cylinder. What would be the shortest possible path network connecting our three points? Our goal is to develop an algorithm to find the minimal path network connecting three points on a cylinder. Finding the least length path network connecting a given set of fixed points in a surface is called the Steiner problem. We will first show that the Steiner problem on the cylinder is related to the Steiner problem on the plane. We then will work with a covering map from the plane to the cylinder so that the correspondence between the Steiner problem on the plane and on the cylinder is clarified. We will follow this with a few results culminating in the cutting theorem. The cutting theorem, Theorem 5.3, guarantees that for any configuration of three points on a cylinder there exists a straight line in the cylinder through which we can make a "cut," then flatten the cut surface out in the plane, and finally construct the minimal path network connecting the three points within the flattened surface. The cutting theorem is an important result that leads us to the cutting algorithm. The cutting algorithm determines the minimal path network connecting the three points on the cylinder. The algorithm requires two cuts in order to compare the principal minimal path network candidates obtained when flattening the cut surface of the cylinder out in the plane.

Only within the last 40 years has the Steiner problem really begun to be studied on nonplanar surfaces. Local properties of minimal path networks on smooth surfaces were investigated in [Weng 2001]. Cockayne [1972] and Brazil et al. [1998] provided analytic methods to solve the 3-point Steiner problem in the sphere. Analytic methods for finding the solution to Steiner problems on the hyperbolic plane and surfaces of revolution were given in [Halverson and March 2005] and

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[Caffarelli et al. 2012], respectively. Geometric methods for solving the two- and 3-point Steiner problems on the regular tetrahedron were provided in [Brune and Sipe 2009; Moon et al. 2011]. A cutting algorithm to find the solution to 3-point Steiner problems on the cone, similar to the one in this paper, is given in [Lee et al. 2011]. Results providing for reductions in solving the 3-point Steiner problem on the torus are found in [Halverson and Penrod 2007; Ivanov and Tuzhilin 1994; May and Mitchell 2007]. Furthermore, Ivanov and Tuzhilin [1994] classify all the closed local minimal networks on closed surfaces of constant nonnegative curvature (spheres, projective planes, flat tori, and Klein bottles) and present similar results for the regular tetrahedron. Helmandollar and Penrod [2007] used a generalization of the method of paired calibrations to solve Steiner problems in the hyperbolic plane for four fixed points that are the vertices of a square. Hwang et al. [1992] offer a detailed discussion on various strategies, extensions, and modifications of the Steiner problem.

The importance of this paper is that it provides an algorithm that does not just give a reduction to the list of possible solutions or refer to a set of analytic equations which must be solved, but finds an actual geometric solution to any 3-point Steiner problem on the cylinder.

2. The Steiner problem on the plane

In this section we will give a brief background of the Steiner problem in the plane. For a more extensive study on the Steiner problem in the Euclidean plane see [Hwang et al. 1992; Ivanov and Tuzhilin 1994]. First we will begin with a few definitions and a basic result concerning the Steiner problem. Then we will give a brief history of the development of solutions to this problem. Finally, we will finish with an algorithm for finding a minimal path network connecting three points in the plane.

Definition 2.1. Let A, B, and C be points in $\mathbb{R}^2$. A *Steiner minimal tree*, denoted $\mathrm{SMT}(A, B, C)$, is the set of minimal length path networks contained in $\mathbb{R}^2$ that connect A, B, and C.

It is a classical result that, for three points A, B, and C in the plane, $\mathrm{SMT}(A, B, C)$ contains precisely one element (see [Hwang et al. 1992]). It is a common practice to denote this unique path network itself as $\mathrm{SMT}(A, B, C)$. We will also apply this practice in our paper when considering the 3-point Steiner problem on the plane. It is also a classical result that, if $\triangle ABC$ has no interior angle with measure $\geq 120°$, then $\mathrm{SMT}(A, B, C) = \overline{AS} \cup \overline{BS} \cup \overline{CS}$ for some point S, called the *Steiner point* (see [Courant and Robbins 1979]). In this case we say that $\mathrm{SMT}(A, B, C)$ is *full*. If $\triangle ABC$ has an interior angle with measure $\geq 120°$, say $m\angle ABC \geq 120°$, then $\mathrm{SMT}(A, B, C) = \overline{AB} \cup \overline{BC}$. In this case we say $\mathrm{SMT}(A, B, C)$ is *degenerate*.

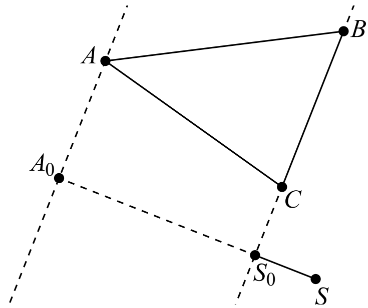

Figure 1. Demonstrating that τ_0 is shorter than τ in the proof of Propositon 2.3.

Note that in this case $\mathrm{SMT}(A, B, C) = \overline{AB} \cup \overline{BB} \cup \overline{BC}$, so in some sense B takes on a similar role as the Steiner point in the full case.

Definition 2.2. Let A, B, and C be points in $\mathbb{R}^2$. We call the point S a *generalized Steiner point* if $\overline{AS} \cup \overline{BS} \cup \overline{CS} \in \mathrm{SMT}(A, B, C)$.

Another result of the Steiner problem in the plane is that the minimal path network connecting three points in a plane is contained in the convex hull of the triangle whose vertices lie on those three points. Since we use this result in proving future theorems in this paper, we will demonstrate a proof here in this section.

Propositon 2.3. *If A, B, and C are points in the plane, then* $\mathrm{SMT}(A, B, C)$ *is contained in the convex hull of* $\triangle ABC$.

Proof. Let $\tau \in \mathrm{SMT}(A, B, C)$ and let $S \in \mathbb{R}^2$ be the generalized Steiner point of τ.

Suppose τ is not contained in the convex hull of $\triangle ABC$. Then S lies outside of the convex hull of $\triangle ABC$. Hence S is opposite one of the points A, B, or C of the lines $\overleftrightarrow{BC}$, $\overleftrightarrow{AC}$, or $\overleftrightarrow{AB}$, respectively. Suppose without loss of generality S is on the side of the line $\overleftrightarrow{BC}$ opposite point A (see Figure 1). Then there is a line perpendicular to $\overleftrightarrow{BC}$ that passes through S. Let S_0 be the point of intersection of the two lines. Let $\tau_0 = \overline{AS_0} \cup \overline{BS_0} \cup \overline{CS_0}$. Note that $SS_0 > 0$ because S is not on $\overleftrightarrow{BC}$. Since $BS = \sqrt{(BS_0)^2 + (SS_0)^2}$ and $CS = \sqrt{(CS_0)^2 + (SS_0)^2}$, then $BS_0 < BS$ and $CS_0 < CS$. Let l be the line parallel to BC passing through A and let A_0 be the point of intersection of l and $\overleftrightarrow{SS_0}$. Since $A_0S_0 < A_0S$,

$$AS = \sqrt{(AA_0)^2 + (A_0S)^2} > \sqrt{(AA_0)^2 + (A_0S_0)^2} = AS_0.$$

Thus τ_0 is shorter than τ, which yields a contradiction.

Therefore τ is contained in the convex hull of $\triangle ABC$. $\square$

Other interesting results of the Steiner problem on the plane are found in [Cieslik 1998; Hwang et al. 1992; Ivanov and Tuzhilin 1994; Jarník and Kössler 1934; Lee et al. 2011; Roussos 2012].

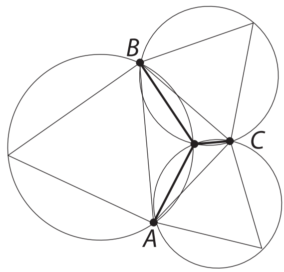

Figure 2. Torricelli's solution.

Brief history. The history of the Steiner problem is briefly described in [Cieslik 1998; Courant and Robbins 1979; Kuhn 1974; Roussos 2012]. We give a summary here.

Fermat posed the following problem in the early 17th century: "Given three points in the plane, find a fourth point such that the sum of its distances to the three given points is minimal." Around 1640 Torricelli presented a geometric solution to Fermat's problem. He showed in the full case that the three circles circumscribing the equilateral triangles constructed on the sides of and outside the triangle intersect at the desired point which is often referred to as the Fermat–Torricelli point [Cieslik 1998] (see Figure 2). SMT(A, B, C) is the configuration of the bold lines in Figure 2. In 1836 Gauss considered the Fermat problem for $n > 3$ points, sometimes referred to as Gauss's problem.

Steiner gave a geometric construction of the Fermat–Torricelli point in the early 19th century and used it in the construction of distance-minimizing trees and graphs [Roussos 2012]. Courant and Robbins [1979] popularized the minimizing of path networks for n points and (mis)labeled it the *Steiner problem*; see [Cieslik 1998] for discussion.

Note that Torricelli's solution only holds when all angles in $\triangle ABC$ are less than or equal to 120°. If we were to perform Torricelli's algorithm of the solution on a triangle with an interior angle greater than 120 degrees we would get a point outside of the convex hull of that triangle which contradicts Propositon 2.3; hence the distinction between full and degenerate minimal path networks.

Solution to the 3-point Steiner problem in the plane. We will now present a useful algorithm [Melzak 1961] for finding SMT(A, B, C) and its length.

First draw the triangle connecting the three points. If one of the angles of $\triangle ABC$ has measure $\geq 120°$, remove the opposite side. The union of the remaining two sides is SMT(A, B, C) and its length is the sum of the lengths of the two sides. In this case SMT(A, B, C) is degenerate.

Otherwise choose one of the sides of the triangle (for example in Figure 3 we chose side $\overline{BC}$) and draw an equilateral triangle, $\triangle BCE$, where E is on the side of

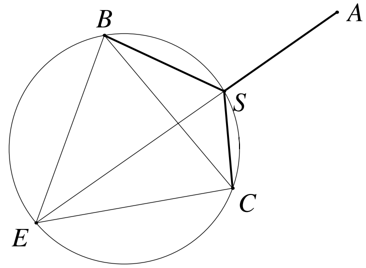

Figure 3. Constructing a full minimal length path network in the plane.

the line $\overline{BC}$ opposite point A. Draw a circle circumscribing $\triangle BCE$ and draw a line from point E to point A. The intersection of the line and the circle will give us the point S, the Steiner point. Then EA will be the length of SMT(A, B, C), and SMT(A, B, C) $= \overline{AS} \cup \overline{BS} \cup \overline{CS}$. In this case SMT($A$, B, C) is full.

3. The cylinder

We will now introduce the cylinder and the covering map we will be using in this paper. (Refer to Figure 4.)

Let $\mathscr{C} \subseteq \mathbb{R}^3$ be the cylinder defined by $\mathscr{C} : x^2 + y^2 = 1$. Then $\mathbb{R}^2$ is a covering for $\mathscr{C}$, where $p : \mathbb{R}^2 \to \mathscr{C}$ is the covering map such that $p(u, v) = (\cos u, \sin u, v)$. Let x denote an arbitrary point of $\mathscr{C}$. Let X_i be the point of $p^{-1}(x)$ contained in $[-\pi + 2i\pi, \pi + 2i\pi)$. We denote by (u_X, v_X) the coordinates of an arbitrary point X in $\mathbb{R}^2$.

Definition 3.1. For points $A, B \in \mathbb{R}^2$ where $A = (u_A, v_A)$ and $B = (u_B, v_B)$, the *strip* Σ_{AB} is the set $\Sigma_{AB} = \{(u, v) \in \mathbb{R}^2 \mid u_A \leq u \leq u_B\}$.

In this paper we will order without loss of generality the three fixed points a, b, and c in such a way that $u_{A_0} \leq u_{B_0} \leq u_{C_0}$.

For a 3-point Steiner problem on a cylinder with fixed points a, b, and c, it will be convenient to distinguish the three regions partitioned by the vertical lines for

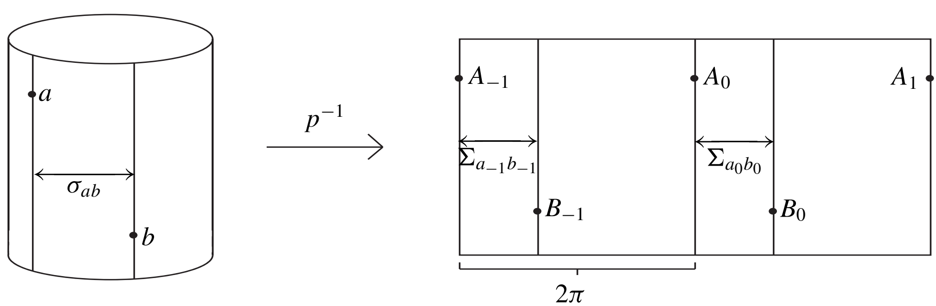

Figure 4. The covering map p.

each of the fixed points. In particular, let $\sigma_{ab} = p(\Sigma_{A_0B_0})$, $\sigma_{bc} = p(\Sigma_{B_0C_0})$, and $\sigma_{ca} = p(\Sigma_{C_0A_0})$.

Definition 3.2. Let $\mathscr{Z}$ be a subset of $\mathscr{C}$. A map $f : \mathscr{Z} \to \mathbb{R}^2$ is said to be a *lift of the inclusion map* $\mathscr{Z} \hookrightarrow \mathscr{C}$ provided, for all $z \in \mathscr{Z}$, $z = p \circ f(z)$. We also say the set $f(\mathscr{Z})$ is a *lift* of $\mathscr{Z}$.

4. Regarding the 3-point Steiner problem on a smooth surface

The Steiner problem on any smooth surface is similar to, but more complicated than, the Steiner problem in the plane [Weng 2001]. In this section we provide definitions and notations for a minimal length path network and a generalized Steiner point on a smooth surface. On a cylinder, and in other smooth surfaces, the minimal length path network need not be unique. Hence we have the following definitions.

Definition 4.1. Let a, b, and c be points in a smooth surface $\mathscr{X}$. Then $\mathrm{SMT}(a, b, c)$ is the set of minimal path networks contained in $\mathscr{X}$ that connect a, b, and c.

Definition 4.2. Let a, b, and c be points in a smooth surface $\mathscr{X}$. If $\tau = \overline{as} \cup \overline{bs} \cup \overline{cs} \in \mathrm{SMT}(a, b, c)$, then we say that s is a *generalized Steiner point* for τ.

There have been many studies of the Steiner problem on general curved surfaces. We cannot address all results and studies in this paper, but refer the interested reader to [Brazil et al. 1998; Cockayne 1972; Dolan et al. 1991; Ivanov and Tuzhilin 1994; Weng 2001] for more details.

5. The cutting theorem

Our purpose in this paper is to present an algorithm for finding a minimal path network on a cylinder. We first need to prove the cutting theorem that we will use in the cutting algorithm; this result, in short, informs us that any minimal path network on a cylinder will be contained in the union of two of the strips σ_{ab}, σ_{bc}, and σ_{ca}. In preparation for the proof of the cutting theorem we need the following proposition.

Propositon 5.1. *Let T be a minimal path network for three fixed points in the plane such that $p(T) \in \mathrm{SMT}(a, b, c)$, S be the generalized Steiner point of T, and $X \in T$ be a fixed point of T such that $p(X) \in \{a, b, c\}$. Then $|u_X - u_S| \leq \pi$.*

Proof. Suppose $|u_X - u_S| > \pi$. By properties of the covering map p there is a point $X_i \in p^{-1}(p(X))$ so that $|u_{X_i} - u_S| \leq \pi$. Then T' obtained by replacing $\overline{XS}$ in T with $\overline{X_i S}$ is a shorter path network where $p(T')$ connects a, b, c. Hence $p(T) \notin \mathrm{SMT}(a, b, c)$. This is a contradiction, so $|u_X - u_S| \leq \pi$. □

Corollary 5.2. *Let T be a minimal path network for three fixed points in the plane such that $p(T) \in \mathrm{SMT}(a, b, c)$ and S be the generalized Steiner point of T. Then $T \subseteq \Gamma = \{(u, v) \in \mathbb{R}^2 : |u - u_S| \leq \pi\}$.*

Proof. Let $T = \overline{A_l S} \cup \overline{B_m S} \cup \overline{C_n S}$. Since $|u_{A_l} - u_S| \le \pi$, then $\overline{A_l S} \subset \Gamma$. Likewise $\overline{B_m S}$, $\overline{C_n S} \subset \Gamma$. Thus $T \subseteq \Gamma$. □

The following theorem demonstrates that the lift of a minimal path network connecting three points a, b, and c on a cylinder is contained in one of the following:

$$\Sigma_{B_{k-1}A_k} = \Sigma_{B_{k-1}C_{k-1}} \cup \Sigma_{C_{k-1}A_k},$$
$$\Sigma_{C_{k-1}B_k} = \Sigma_{C_{k-1}A_k} \cup \Sigma_{A_k B_k},$$
$$\Sigma_{A_k C_k} = \Sigma_{A_k B_k} \cup \Sigma_{B_k C_k}.$$

A proof of a similar result regarding the flat torus can be found in [Halverson and Penrod 2007].

Theorem 5.3 (cutting theorem). *Let T be a minimal path network for three fixed points in the plane such that $p(T) \in \mathrm{SMT}(a, b, c)$. Then T is contained in one of $\Sigma_{B_{k-1}A_k}$, $\Sigma_{C_{k-1}B_k}$, and $\Sigma_{A_k C_k}$ for some $k \in \mathbb{Z}^+$.*

Proof. Let $T = \overline{A_l S} \cup \overline{B_m S} \cup \overline{C_n S}$. Let $t = \min\{u_{A_l}, u_{B_m}, u_{C_n}\}$. Without loss of generality let $t = u_{A_l}$. By Corollary 5.2, $|u_{A_l} - u_S| \le \pi$ and $|u_{B_m} - u_S| \le \pi$. Using $u_{A_l} \le u_{B_m}$ and the triangle inequality, we have

$$u_{B_m} - u_{A_l} = |u_{B_m} - u_{A_l}| \le |u_{B_m} - u_S| + |u_{A_l} - u_S| \le 2\pi.$$

Note that $m \ge l$. Let $m = l + j$ for some $j \in \mathbb{Z}^+$. Then $u_{B_m} = u_{B_l} + 2\pi j \le 2\pi + u_{A_l}$. Thus

$$0 \le u_{B_l} - u_{A_l} \le 2\pi - 2\pi j.$$

This is only possible if j is either 0 or 1. Furthermore, when $j = 1$ equality must occur. In particular, if $j = 1$, then $u_{B_l} = u_{A_l}$ and hence $u_{B_m} = u_{A_{l+1}}$. So either $m = l$ or $m = l + 1$, and in the case $m = l + 1$ necessarily $u_{B_m} = u_{A_{l+1}}$. Similar considerations of C_n yield either $n = l$ or $n = l + 1$, and in the case $n = l + 1$ necessarily $u_{C_n} = u_{A_{l+1}}$.

Case 1. Suppose $m = l$ and $n = l$. Then $T = \mathrm{SMT}(A_l, B_l, C_l)$. By Propositon 2.3, T is in the convex hull of $\triangle A_l B_l C_l$. Thus $T \subset \Sigma_{A_l C_l}$. Letting $k = l$ gives the desired result.

Case 2. Suppose $m = l + 1$ and $n = l$. Then $u_{B_m} = u_{A_{l+1}}$ and hence $u_{B_{m-1}} = u_{A_l}$. Thus T is in the convex hull of $\triangle A_l B_{l+1} C_l$. It follows that $T \subset \Sigma_{A_l B_{l+1}} = \Sigma_{B_l A_{l+1}}$. Letting $k = l + 1$ gives the desired result.

Case 3. Suppose $n = l + 1$. Then $u_{C_n} = u_{A_{l+1}}$. Thus $u_{A_l} = u_{C_l}$. Since $u_{A_l} \le u_{B_l} \le u_{C_l}$, then $u_{A_l} = u_{B_l} = u_{C_l}$. Since $v_{X_i} = v_{X_j}$ for any $i, j \in \mathbb{Z}$, the length of T is

$$\begin{aligned}\sqrt{(u_S-u_{A_{l+1}})^2+(v_S-v_{A_{l+1}})^2}+\sqrt{(u_S-u_{B_m})^2+(v_S-v_{B_m})^2}+\sqrt{(u_S-u_{C_n})^2+(v_S-v_{C_n})^2}&\\ \geq |v_S-v_{A_{l+1}}|+|v_S-v_{B_m}|+|v_S-v_{C_{l+1}}|&\\ \geq |v_S-v_{A_l}|+|v_S-v_{B_l}|+|v_S-v_{C_l}|&\\ \geq \max\{|v_{A_l}-v_{C_l}|, |v_{A_l}-v_{B_l}|, |v_{B_l}-v_{C_l}|\}&\\ = \max\{A_lC_l, A_lB_l, B_lC_l\}.&\end{aligned}$$

Let T' be the minimal path connecting A_l, B_l, and C_l. Note that, since A_l, B_l and C_l are collinear, T' is one of $\overline{A_lC_l}$, $\overline{A_lB_l}$, and $\overline{B_lC_l}$. Then the length of T' is $\max\{A_lC_l, A_lB_l, B_lC_l\}$ which is less than or equal to the length of T. Also note that equality can only hold when $u_S=u_{A_l}=u_{B_l}=u_{C_l}$, implying $T=T'$ which is a contradiction. Therefore this case is not possible.

Similar arguments apply when $t=u_{B_m}$ and $t=u_{C_n}$. □

6. The cutting algorithm

Justification. Let a, b, and c be points on the cylinder $\mathscr{C}$ and let T be a lift of $\tau\in\mathrm{SMT}(a,b,c)$ contained in $\Sigma_{B_{-1}C_0}$. This is possible from the cutting theorem since we know that there is a lift of τ contained in one of $\Sigma_{B_{-1}A_0}$, $\Sigma_{C_{-1}B_0}$, and $\Sigma_{A_0C_0}$. Notice that if we cut along the vertical line containing a and lay it out in a plane we get copies of $\Sigma_{B_{-1}A_0}$ and $\Sigma_{A_0C_0}$, contained in the cut surface. If we cut along the vertical line containing b and lay it out in a plane we get copies of $\Sigma_{B_{-1}A_0}$ and $\Sigma_{C_{-1}B_0}$, contained in the cut surface. If we cut along the vertical line containing c and lay it out in a plane we get copies of $\Sigma_{A_0C_0}$ and $\Sigma_{C_{-1}B_0}$, contained in the cut surface. With all three cuts together we get copies of each of $\Sigma_{B_{-1}A_0}$, $\Sigma_{C_{-1}B_0}$, and $\Sigma_{A_0C_0}$ twice. One way to determine the $\mathrm{SMT}(a,b,c)$ is comparing the minimal path networks in each strip. However the following algorithm demonstrates how to do this more efficiently with just two cuts.

The cutting algorithm. **Step 1.** Cut along the vertical line containing a of our cylinder. Then there are two possible minimal path networks, one in $\Sigma_{A_0C_0}$ and one in $\Sigma_{B_{-1}A_0}$. Let T_1 be $\mathrm{SMT}(A_0,B_0,C_0)$, and T_2 be $\mathrm{SMT}(B_{-1},C_{-1},A_0)$. Since T_1 and T_2 are both in the plane, perform the algorithm presented in Section 2 to compare the two minimal path networks and find which one is shorter.

Step 2. If T_1 is at least as short as T_2, then cut vertically up the cylinder at the point c and unwrap it as before, laying it out on the plane contained in $\Sigma_{C_{-1}C_0}$. Then there are two possible minimal path networks, one in $\Sigma_{C_{-1}B_0}$ and the other in $\Sigma_{A_0C_0}$. Let T_3 be $\mathrm{SMT}(C_{-1},A_0,B_0)$. Note that T_1 is contained in $\Sigma_{A_0C_0}$. Since T_3 is in the plane, use the algorithm for finding minimal path networks in the plane presented in Section 2 and compare T_3 to T_1. Let i be any index where T_i is at least as short as T_j for all $j \neq i$. Then $p(T_i) \in \mathrm{SMT}(a, b, c)$.

Otherwise, cut vertically up the cylinder at the point b and unwrap it, laying it out on the plane contained in $\Sigma_{B_{-1}B_0}$. Then there are two possible minimal path networks, one in $\Sigma_{B_{-1}A_0}$ and the other in $\Sigma_{C_{-1}B_0}$. Let T_3 be $\mathrm{SMT}(C_{-1}, A_0, B_0)$ as in the first case. Note that T_2 is contained in $\Sigma_{B_{-1}A_0}$. Since T_3 is in the plane use the algorithm for finding minimal path networks in the plane presented in Section 2 and compare T_3 to T_2. Let i be any index where T_i is at least as short as T_j where $j \neq i$. Then $p(T_i) \in \mathrm{SMT}(a, b, c)$.

That's all there is to it.

7. Conclusion

Further problems that can be investigated include:

(1) The n-point Steiner problem on the cylinder. Jarník and Kössler [1934] have developed an algorithm for solving any n-point Steiner problem in the plane. How could the results in this paper be generalized to solve any n-point problem on the cylinder?

(2) The 3-point Steiner problem on the flat torus in 4-space. The cylinder is a covering space for the flat torus in 4-space. How can the results produced in this paper be applied to solve the 3-point Steiner problem on the flat torus in 4-space?

(3) The n-point Steiner problem on the flat torus in 4-space. Could results of (1) and (2) be combined to solve any n-point Steiner problem on the flat torus in 4-space?

We hope that the results found in this paper can serve as a basis in many future findings.

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