involve

a journal of mathematics

The surgery unknotting number of Legendrian links

Bianca Boranda, Lisa Traynor and Shuning Yan

msp

2013 vol. 6, no. 3

INVOLVE 6:3 (2013)
dx.doi.org/10.2140/involve.2013.6.273

The surgery unknotting number of Legendrian links

Bianca Boranda, Lisa Traynor and Shuning Yan

(Communicated by Kenneth S. Berenhaut)

The surgery unknotting number of a Legendrian link is defined as the minimal number of particular oriented surgeries that are required to convert the link into a Legendrian unknot. Lower bounds for the surgery unknotting number are given in terms of classical invariants of the Legendrian link. The surgery unknotting number is calculated for every Legendrian link that is topologically a twist knot or a torus link and for every positive, Legendrian rational link. In addition, the surgery unknotting number is calculated for every Legendrian knot in the Legendrian knot atlas of Chongchitmate and Ng whose underlying smooth knot has crossing number 7 or less. In all these calculations, as long as the Legendrian link of j components is not topologically a slice knot, its surgery unknotting number is equal to the sum of $j-1$ and twice the smooth 4-ball genus of the underlying smooth link.

1. Introduction

A classical invariant for smooth knots is the unknotting number: the unknotting number of a diagram of a knot K is the minimum number of crossing changes required to change the diagram into a diagram of the unknot; the unknotting number of K is the minimum of the unknotting numbers of all diagrams of K. In the following, we will define a surgery unknotting number for Legendrian knots and links.

Legendrian links are smooth links that satisfy an additional geometric condition imposed by a contact structure. We will focus on Legendrian links in $\mathbb{R}^3$ with its standard contact structure. The notion of Legendrian equivalence is more refined than smooth equivalence: there is only one smooth unknot, but there are an infinite number of Legendrian unknots. Figure 1 shows the front projections of three

MSC2010: primary 53D35, 57R17; secondary 57M25.
Keywords: Legendrian links, unknotting number, genus, Lagrangian cobordism.
Lisa Traynor is partially supported by NSF grant DMS-0909021. Bianca Boranda and Shuning Yan completed this work as Bryn Mawr College undergraduates; they received support from NSF grant DMS-0909021 to pursue this research.

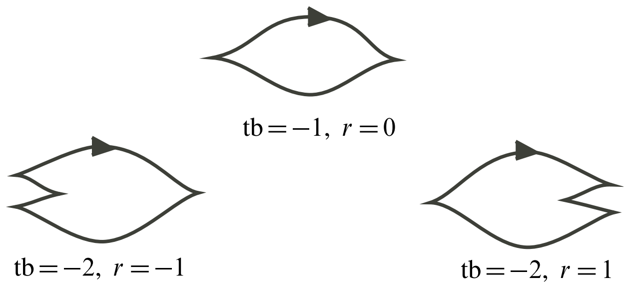


Figure 1. Three different Legendrian knots that are topologically the unknot.

different Legendrian unknots; the infinite structure representing all Legendrian unknots is depicted in Figure 7 on page 281.

The act of changing a crossing (smoothly passing a knot through itself) is not a natural operation in a contact manifold. Instead, given a Legendrian link, we will attempt to arrive at a Legendrian unknot through a Legendrian "surgery" operation in which two oppositely oriented strands in a Legendrian 0-tangle are replaced by an oriented, Legendrian ∞-tangle as illustrated in Figure 2. It is shown in Proposition 3.5 that every Legendrian link can become a Legendrian unknot after a finite number of surgeries. The surgery unknotting number of a Legendrian link Λ, $\sigma_0(\Lambda)$, measures the minimal number of these surgeries that are required to convert Λ to a Legendrian unknot; see Definitions 3.1 and 3.6. In the following, our goal is to study and calculate this Legendrian invariant $\sigma_0(\Lambda)$.

Main results. Lower bounds on $\sigma_0(\Lambda)$ exist in terms of the classical invariants of Λ. These invariants include invariants of the underlying smooth link type L_Λ and the classical Legendrian invariants of Λ: the Thurston–Bennequin, $\operatorname{tb}(\Lambda)$, and rotation number, $r(\Lambda)$, as defined in Section 2.

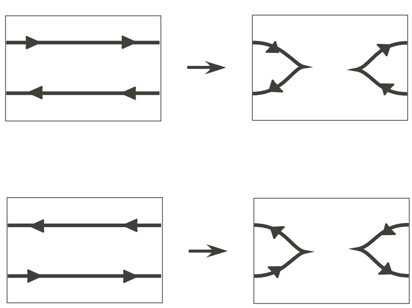


Figure 2. Oriented Legendrian surgeries: a basic, compatibly oriented 0-tangle is replaced by a basic, compatibly oriented ∞-tangle.

Theorem 1.1. *Let Λ be a Legendrian link. Then*:

(1) $\mathrm{tb}(\Lambda) + |r(\Lambda)| + 1 \leq \sigma_0(\Lambda)$.

(2) *If Λ has j components, L_Λ denotes the underlying smooth link type of Λ, and $g_4(L_\Lambda)$ denotes the smooth 4-ball genus of L_Λ,*[1] *then*

$$2g_4(L_\Lambda) + (j-1) \leq \sigma_0(\Lambda).$$

Remark 1.2. In parallel to Theorem 1.1 (1), when Λ is a Legendrian knot with underlying smooth knot type K_Λ, the well known slice-Bennequin inequality says that

$$\mathrm{tb}(\Lambda) + |r(\Lambda)| + 1 \leq 2g_4(K_\Lambda). \tag{1-1}$$

There are now a number of proofs of this result, but all use deep theory. Lisca and Matić [1998] prove this using their adjunction inequality obtained by Seiberg–Witten theory. See also [Akbulut and Matveyev 1997; Rudolph 1995]. In contrast, the proof of Theorem 1.1 is elementary and is given in Lemmas 3.8 and 3.9.

When Λ is a knot, combining Theorem 1.1(2) and the slice-Bennequin inequality (1-1), we find:

Corollary 1.3. *For any Legendrian knot Λ, if K_Λ denotes the smooth knot type of Λ then*

$$\mathrm{tb}(\Lambda) + |r(\Lambda)| + 1 \leq 2g_4(K_\Lambda) \leq \sigma_0(\Lambda).$$

Thus $\sigma_0(\Lambda) = 2g_4(K_\Lambda)$ when $\sigma_0(\Lambda) = \mathrm{tb}(\Lambda) + |r(\Lambda)| + 1$.

As we will see below, this corollary sometimes allows us to calculate the smooth 4-ball genus of a knot.

Using the established lower bounds, we can calculate $\sigma_0(\Lambda)$ when the underlying smooth link type of Λ falls within some important families.

Theorem 1.4. (1) *If Λ is a Legendrian knot that is topologically a nontrivial twist knot, then $\sigma_0(\Lambda) = 2$.*

(2) *If Λ is a j-component Legendrian link that is topologically a (jp, jq)-torus link, $|p| > q > 1$ and $\gcd(p, q) = 1$, then*

$$\sigma_0(\Lambda) = (|jp| - 1)(jq - 1).$$

Theorem 1.4 is proved in Section 4 as Theorems 4.1 and 4.2. The proof of this theorem relies heavily on the classification of Legendrian twist knots given by Etnyre, Ng and Vértesi [Etnyre et al. 2013], and the classification of Legendrian torus knots by Etnyre and Honda [2001], which was extended to a classification of Legendrian torus links by Dalton [2008]. When Λ is topologically a positive torus

[1] That is, $g_4(L_\Lambda)$ denotes the minimal genus of a smooth, compact, connected, oriented surface $\Sigma \subset B^4$ with $\partial\Sigma = L_\Lambda \subset \mathbb{R}^3 \subset S^3 = \partial B^4$.

link, $p > 0$, of maximal Thurston–Bennequin invariant, the calculation of $\sigma_0(\Lambda)$ is obtained realizing the lower bound given in Theorem 1.1 by the Legendrian invariants of Λ. Thus by Corollary 1.3, which employs the deep slice-Bennequin inequality in (1-1), we are able to deduce the Milnor conjecture about torus knots, originally proved by Kronheimer and Mrowka:

Corollary 1.5 [Kronheimer and Mrowka 1993]. *If $T(p,q)$ is a (p,q)-torus knot, $|p| > q > 1$, then*

$$2g_4(T(p,q)) = (|p|-1)(q-1).$$

By comparing σ_0 of the Legendrian and g_4 of the underlying smooth link type, we can rephrase the conclusions of Theorem 1.4 as:

Corollary 1.6. *If Λ is a Legendrian link that is topologically a nonslice twist knot*[2] *or a j-component torus link, L_Λ, then*

$$\sigma_0(\Lambda) = 2g_4(L_\Lambda) + (j-1).$$

As an additional family of Legendrian links, we consider *positive, Legendrian rational links*. These links are defined as Legendrian numerator closures of the Legendrian rational tangles studied, for example, in [Traynor 1998] and [Schneider 2011]. These links are positive in the sense that an orientation is chosen on the components so that all the crossings have a positive sign. Such Legendrian links are specified by a vector $(c_n, \dots, c_1)$ of positive integers; see Definition 4.4 and Figure 18. Lemma 4.5 gives conditions on the c_i that guarantee that the link is positive.

Theorem 1.7. *If $\Lambda(c_n, \dots, c_2, c_1)$ is a positive, Legendrian rational link, then*

$$\sigma_0(\Lambda(c_n, \dots, c_2, c_1)) = \sum_{i \text{ odd}} c_i - p(n),$$

where $p(n)$ equals 1 when n is odd and equals 0 when n is even.

This is proved in Section 4; see Theorem 4.6.

Remark 1.8. When Λ is a positive, Legendrian rational link, the calculation of $\sigma_0(\Lambda)$ is obtained realizing the lower bound given in Theorem 1.1 given by the classical Legendrian invariants of Λ. Thus by Corollary 1.3, when $\Lambda(c_n, \dots, c_1)$ is a positive, Legendrian rational knot, Theorem 1.7 gives a formula for twice the smooth 4-ball genus of the underlying smooth knot. This can be used to get

[2]Casson and Gordon [1986] proved that the only twist knots that are slice are the unknot, 6_1, and $m(6_1)$.

formulas for the smooth 4-ball genus of a knot in terms of its rational notation. In particular,

$$\begin{aligned} g_4(5_2) &= g_4(N(3,2)) = \tfrac{1}{2}\sigma_0(\Lambda(3,2)) = \tfrac{1}{2}(2) = 1, \\ g_4(7_5) &= g_4(N(3,2,2)) = \tfrac{1}{2}\sigma_0(\Lambda(3,2,2)) = \tfrac{1}{2}(2+3-1) = 2, \\ g_4(N(5,244,4,16,3,104,2,12,1)) &= \tfrac{1}{2}(1+2+3+4+5-1). \end{aligned}$$

This is an alternate to formulas for calculating the smooth 4-ball genus in terms of crossings and Seifert circles as given by Nakamura in [2000]. In turn, using Nakamura's formula, we see that when the underlying link type of $\Lambda(c_n, \dots, c_2, c_1)$ is a 2-component link L_Λ,

$$\sigma_0(\Lambda(c_n, \dots, c_2, c_1)) = 2g_4(L_\Lambda) + 1;$$

see Remark 4.7.

Given the above calculations, it is natural to ask:

Question 1.9. If Λ is a Legendrian knot that is topologically a nonslice knot K_Λ, is $\sigma_0(\Lambda) = 2g_4(K_\Lambda)$? More generally, if Λ is a Legendrian link of $j \geq 2$ components that is topologically the link L_Λ, is $\sigma_0(\Lambda) = 2g_4(L_\Lambda) + (j-1)$?

To investigate the knot portion of this question, we examined Legendrian representatives of knots with crossing number 7 or less. There is not yet a Legendrian classification of all these knot types, but a conjectured classification is given by Chongchitmate and Ng [2013].

Proposition 1.10. *Assuming the conjectured classification of Legendrian knots in* [Chongchitmate and Ng 2013],[3] *if Λ is a Legendrian knot that is topologically a nonslice knot K_Λ with crossing number* 7 *or less*, $\sigma_0(\Lambda) = 2g_4(K_\Lambda)$.

The only non-torus and non-twist knots with crossing number at most 7 are $6_2, m(6_2)$, $6_3 = m(6_3)$, 7_3, $m(7_3)$, 7_4, $m(7_4)$, 7_5, $m(7_5)$, 7_6, $m(7_6)$, 7_7, and $m(7_7)$. While doing the calculations for Legendrians with these knot types, in general we found that for a Legendrian Λ whose underlying smooth knot type K_Λ satisfies $g_3(K_\Lambda) = g_4(K_\Lambda)$, where $g_3(K_\Lambda)$ denotes the (3-dimensional) genus of the knot, it is fairly straight forward to show that $\sigma_0(\Lambda) = 2g_4(K_\Lambda)$. Legendrians that are topologically $7_3, m(7_3), 7_4, m(7_4), 7_5$, and $m(7_5)$ fall into this category. For the remaining knot types under consideration, the calculation of the smooth 4-ball genus follows from the fact that the topological unknotting number of these knots is equal to 1. We show that in a front projection of a Legendrian knot, it is possible to locally change any negative crossing to a positive one by 2 surgeries; see Lemma 5.2. This allowed us to prove Proposition 1.10 in the cases where Λ is

[3] Potential duplications in their atlas will not affect the statement.

topologically $6_2, 6_3 = m(6_3), 7_6$, or 7_7. For the remaining cases of $m(6_2)$, $m(7_6)$, and $m(7_7)$, results of [Soteros et al. 2011] show that it is not possible to find a front projection that can be unknotted at a negative crossing. However, we found front projections that could be unknotted at a positive crossing in a special "S" or "hooked-X" form: a positive crossing in one of these special forms can be locally changed to a negative crossing by 2 surgeries; see Lemma 5.5.

The Lagrangian motivation and discussion. All of our calculations indicate that $\sigma_0(\Lambda)$ is measuring an invariant of the underlying smooth link type and that this invariant will be the same for Λ and Λ' when they represent smooth knots that differ by the topological mirror operation. Below is an explanation for why this may be true.

Although the definition of the surgery unknotting number has been formulated above combinatorially, the motivation comes from trying to understand the flexibility and rigidity of Lagrangian submanifolds of a symplectic manifold. From [Bourgeois et al. $\geq$ 2013] (see also [Ekholm et al. 2012]) the existence of an unknotting surgery string $(\Lambda_n, \ldots, \Lambda_0)$, as defined in Definition 3.1, implies the existence of an oriented Lagrangian cobordism Σ in $(\mathbb{R}\times\mathbb{R}^3 = \{(s, x, y, z)\}) \cap \{0 \leq s \leq n\}$ so that $(\Sigma \cap \{s = i\}) = \Lambda_i$, for $i = n, \ldots, 0$. Furthermore, if Λ_0 is the Legendrian unknot with maximal Thurston–Bennequin invariant, this cobordism can be "filled in" with a Lagrangian $\bar{\Sigma} \subset \{s \leq n\}$ so that $\partial\bar{\Sigma} = \Lambda_n$. In fact, it is shown in [Chantraine 2010] that if Λ_0 is not the Legendrian unknot with maximal Thurston–Bennequin invariant, then the cobordism Σ cannot be filled in to $\bar{\Sigma}$; moreover, when there does exist the filling to $\bar{\Sigma}$ and the smooth underlying knot type of Λ_n is K_n, then the genus of $\bar{\Sigma}$ agrees with the smooth 4-ball genus of K_n.

From this Lagrangian perspective, it is a bit more natural to consider surgery strings $(\Lambda_n, \ldots, \Lambda_0)$ where Λ_0 is a Legendrian unlink (a trivial link of Legendrian unknots), and define a corresponding "surgery unlinking number"; this is a project that the second author has begun to pursue with other undergraduates. A Lagrangian analogue of Question 1.9 is:

Question 1.11. If Λ is a Legendrian knot with underlying smooth knot type K_Λ, does there exist a Lagrangian cobordism constructed from Legendrian isotopy and oriented Legendrian surgeries between Λ and Λ_0, a Legendrian that is a smooth unlink, that realizes $g_4(K_\Lambda)$?

Any Lagrangian constructed from Legendrian isotopy and oriented Legendrian surgeries would be in ribbon form; this means that the restriction of the height function, given by the s coordinate, to the cobordism would not have any local maxima in the interior of the cobordism. So a positive answer to Question 1.11 would imply that the slice genus agrees with the ribbon genus; for some background on this and related problems, see, for example, [Livingston 2005].

2. Background information on Legendrian links

Below is some basic background on Legendrian links. More information can be found, for example, in [Etnyre 2005].

The *standard contact structure* on $\mathbb{R}^3$ is the field of hyperplanes ξ where $\xi_p = \ker(dz - y\,dx)_p$. A *Legendrian link* is a submanifold, L, of $\mathbb{R}^3$ diffeomorphic to a disjoint union of circles so that for all $p \in L$, $T_pL \subset \xi_p$. It is common to examine Legendrian links from their xz-projections, known as their *front projections*. A Legendrian link will generically have an immersed front projection with semicubical cusps and no vertical tangents; conversely, any such projection can be uniquely lifted to a Legendrian link using $y = dz/dx$. Figure 3 shows Legendrian versions of the trefoils 3_1 and $m(3_1)$.

Λ_0 and Λ_1 are *equivalent Legendrian links* if there exists a 1-parameter family of Legendrian links Λ_t joining Λ_0 and Λ_1. In fact, Legendrian links Λ_0, Λ_1 are equivalent if and only if their front projections are equivalent by planar isotopies that do not introduce vertical tangents and the *Legendrian Reidemeister moves* as shown in Figure 4.

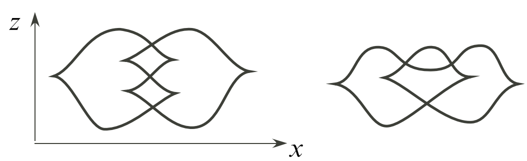


Figure 3. Left: front projection of a Legendrian knot that is topologically the (negative/left) trefoil 3_1. Right: front projection of a Legendrian knot that is topologically the mirror trefoil $m(3_1)$. At crossings, it is not necessary to specify which strand is the overstrand: the strand with lesser slope will always be on top.

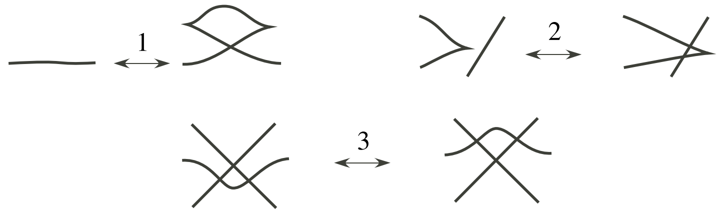


Figure 4. The three Legendrian Reidemeister moves. There is another type 1 move obtained by flipping the planar figure about a horizontal line, and there are three additional type 2 moves obtained by flipping the planar figure about a vertical, a horizontal, and both a vertical and horizontal line.

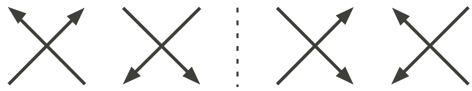


Figure 5. Negative (left) and positive (right) crossings.

Every Legendrian knot and link has a Legendrian representative. In fact, every Legendrian knot and link has an infinite number of different Legendrian representatives. For example, Figure 1 shows three different Legendrians that are all topologically the unknot. These unknots can be distinguished by classical Legendrian invariants, the Thurston–Bennequin and rotation number. These invariants can easily be computed from a front projection of the Legendrian link once we understand how to assign a $\pm$ sign to each crossing and an up/down direction to each cusp.

A *positive* (*negative*) *crossing* of a front projection of an oriented Legendrian link is a crossing where the strands point to the same side (opposite sides) of a vertical line passing through the crossing point; see Figure 5. Each cusp can also be assigned an *up* or *down* direction; see Figure 6. Then for an oriented Legendrian link Λ, we have the following formulas for the *Thurston–Bennequin*, $\mathrm{tb}(\Lambda)$, and *rotation number*, $r(\Lambda)$, invariants:

$$\mathrm{tb}(\Lambda) = P - N - R, \qquad r(\Lambda) = \tfrac{1}{2}(D - U), \tag{2-1}$$

where P is the number of positive crossings, N is the number of negative crossings, R is the number of right cusps, D is the number of down cusps, and U is the number of up cusps in a front projection of Λ. Given that two front projections of equivalent Legendrian links differ by the Legendrian Reidemeister moves described in Figure 4, it is easy to verify that $\mathrm{tb}(\Lambda)$ and $r(\Lambda)$ are Legendrian link invariants.

The two unknots in the second line of Figure 1 are obtained from the one at the top by adding an up or down *zig-zag* (also known as a $\mp$ *stabilization*). In general, this stabilization procedure will not change the underlying smooth knot type but will decrease the Thurston–Bennequin number by 1; adding an up (down) zig-zag will decrease (increase) the rotation number by 1. If Λ is a Legendrian knot, we will use the notation $S_{\pm}(\Lambda)$ to denote the *double stabilization* of Λ, the Legendrian knot obtained by adding both a positive and negative zig-zag.

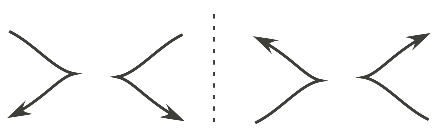


Figure 6. Right and left down cusps (left) and right and left up cusps (right).

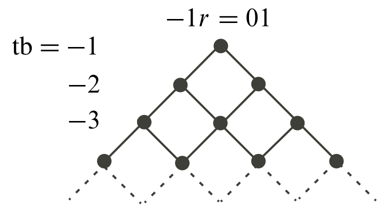


Figure 7. The single-peaked mountain of all Legendrian unknots.

In fact, as discovered by Eliashberg and Fraser, all Legendrian unknots are classified by their Thurston–Bennequin and rotation numbers:

Theorem 2.1 [Eliashberg and Fraser 2009; Etnyre and Honda 2001]. *Suppose* Λ_0 *and* Λ'_0 *are oriented Legendrian knots that are both topologically the unknot. Then* Λ_0 *is equivalent to* Λ'_0 *if and only if* $\mathrm{tb}(\Lambda_0) = \mathrm{tb}(\Lambda'_0)$ *and* $r(\Lambda_0) = r(\Lambda'_0)$.

Figure 7 describes all the Legendrian unknots. Notice that any Legendrian unknot is equivalent to one that is obtained by adding up and/or down zig-zags to the unknot with Thurston–Bennequin number equal to -1 and rotation number equal to 0 shown in Figure 1.

In general, it is an important question to understand the "geography" of other knot types. By [Etnyre and Honda 2001; Etnyre et al. 2013], we understand the mountain ranges for all torus and twist knots. The Legendrian knot atlas [Chongchitmate and Ng 2013] gives the known and conjectured mountain ranges for all Legendrian knots with arc index at most 9; this includes all knot types with crossing number at most 7 and all non-alternating knots with crossing number at most 9.

3. The surgery unknotting number

In this section, we define the surgery operation, show that every Legendrian link can be unknotted by surgeries, define the surgery unknotting number, and give some basic properties of the surgery unknotting number.

The surgery operation can be viewed as a *tangle surgery*: the replacement of one Legendrian tangle by another. A *basic, compatibly oriented Legendrian* 0-*tangle* is a Legendrian tangle that is topologically the 0-tangle where the strands are oppositely oriented and each strand has neither crossings nor cusps; the two basic, compatibly oriented Legendrian 0-tangles can be seen on the left side of Figure 2. A *basic, compatibly oriented Legendrian* ∞*-tangle* is Legendrian tangle that is topologically the ∞-tangle where the strands are oppositely oriented and each strand has precisely one cusp and no crossings; the two basic, compatibly oriented Legendrian ∞-tangles can be seen on the right side of Figure 2.

Definition 3.1. An *oriented, Legendrian surgery* of an oriented, Legendrian link is the Legendrian link obtained by replacing a basic, compatibly oriented Legendrian

0-tangle with a basic, compatibly oriented Legendrian ∞-tangle; see Figure 2. An *oriented surgery string* consists of a vector of oriented, Legendrian links

$$(\Lambda_n, \Lambda_{n-1}, \dots, \Lambda_0),$$

where, for all $j \in \{n-1, \dots, 0\}$, Λ_j is obtained from Λ_{j+1} by Legendrian isotopy and an oriented, Legendrian surgery. An *oriented, unknotting surgery string of length n for* Λ consists of an oriented surgery string $(\Lambda_n, \Lambda_{n-1}, \dots, \Lambda_0)$ where $\Lambda_n = \Lambda$ and Λ_0 is topologically an unknot.

To start, we have the following relationships between the classic invariants of two Legendrian links related by surgery:

Lemma 3.2. *If* Λ *is an oriented, Legendrian link and* Λ' *is obtained from* Λ *by an oriented, Legendrian surgery, then:*

(1) *the parity of the number of components of* Λ *and* Λ' *differ;*

(2) $\operatorname{tb}(\Lambda') = \operatorname{tb}(\Lambda) - 1$, *and* $r(\Lambda') = r(\Lambda)$.

Proof. The statements about the Thurston–Bennequin and rotation numbers are easily verified using Equation (2-1). Regarding the parity, one surgery to a knot will always produce a link of two components, while doing a surgery to a link will increase or decrease the number of components by 1 depending on whether or not the strands in the 0-tangle belong to the same component of the link. □

Recall that for any Legendrian knot Λ, the Legendrian knot $\Lambda' = S_\pm(\Lambda)$ obtained as the double $\pm$ stabilization of Λ will have $r(\Lambda') = r(\Lambda)$ and $\operatorname{tb}(\Lambda') = \operatorname{tb}(\Lambda) - 2$. Thus it is potentially possible that Λ' can be obtained from Λ by two oriented Legendrian surgeries. In fact, it is possible.

Lemma 3.3. *For any oriented, Legendrian knot* Λ *there exists an oriented surgery string* $(\Lambda_2, \Lambda_1, \Lambda_0)$ *with* $\Lambda_2 = \Lambda$ *and* $\Lambda_0 = S_\pm(\Lambda)$.

Proof. These surgeries are illustrated in Figure 8. Every Legendrian link Λ must have a right cusp. By a Legendrian isotopy, we can pull a right cusp far to the right and perform one surgery near this right cusp. This produces a link consisting of the original link and a Legendrian unknot. After a Legendrian isotopy, a second surgery can be done using one strand near the same cusp of the original link and a strand from the unknot. The result is $S_\pm(\Lambda)$. □

In the chart of Legendrian unknots given in Figure 7, we see that any two unknots with the same rotation number are related by a sequence of double $\pm$ stabilizations. Thus we get:

Corollary 3.4. *If* Λ *and* Λ' *are oriented, Legendrian unknots with* $r(\Lambda) = r(\Lambda')$ *and* $\operatorname{tb}(\Lambda) = \operatorname{tb}(\Lambda') + 2m$, *for* $m \geq 0$, *then there exists an oriented surgery string* $(\Lambda_{2m}, \Lambda_{2m-1}, \dots, \Lambda_0)$, *where* $\Lambda_{2m} = \Lambda$, *and* $\Lambda_0 = \Lambda'$.

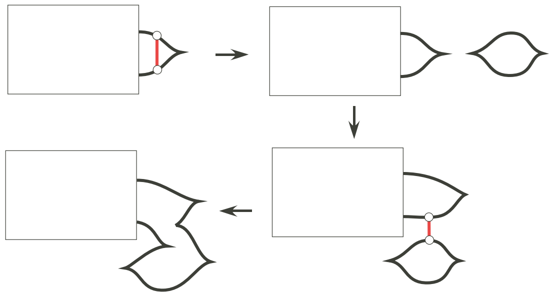


Figure 8. Two oriented, Legendrian surgeries produce $S_{\pm}(\Lambda)$ from Λ.

Thus if we can reach a Legendrian unknot by surgeries, then we can reach an infinite number of Legendrian unknots by surgery. The basis for our new invariant is the fact that every Legendrian link can be "unknotted" by a string of surgeries:

Proposition 3.5. *For any oriented, Legendrian link* Λ, *there exists an oriented, unknotting surgery string* $(\Lambda = \Lambda_u, \Lambda_{u-1}, \dots, \Lambda_0)$. *Moreover, if* Λ *has* j *components and there exists a front projection of* Λ *with* m *crossings, then* $u \leq 2m + j - 1$.

Proof. Assume that there is a front projection of Λ with m crossings. We will first show that there is an oriented surgery string $(\tilde{\Lambda}_m, \tilde{\Lambda}_{m-1}, \dots, \tilde{\Lambda}_0)$, where $\tilde{\Lambda}_m = \Lambda$ and $\tilde{\Lambda}_0$ is a trivial link of Legendrian unknots. If $\tilde{\Lambda}_0$ has c components, we will then show that it is possible to do an additional $c - 1$ surgeries to get this into a single component unknot.

Given the initial Legendrian link Λ having a projection with m crossings, assume that n of these crossings are negative. It is then possible to construct a surgery string $(\tilde{\Lambda}_m, \tilde{\Lambda}_{m-1}, \dots, \tilde{\Lambda}_{m-n})$ where $\tilde{\Lambda}_m = \Lambda$ and $\tilde{\Lambda}_{m-n}$ has a front projection with $m - n$ crossings, all of which are positive. This surgery string is obtained by doing a surgery to the right of each negative crossing and then doing a Legendrian isotopy to remove the positive crossing introduced by the surgery, as shown in Figure 9. Next, by applying a planar Legendrian isotopy, it is possible to assume that all the crossings of $\tilde{\Lambda}_{m-n}$ have distinct x-coordinates. The left cusps associated to the leftmost positive crossing are either nested or stacked and fall into one of the 6 cases listed in Figure 10; by an additional Legendrian planar isotopy, we can assume that all other left cusps occur to the right of this crossing. For each case, it is possible

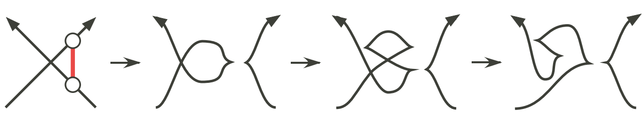


Figure 9. A negative crossing can be removed by an oriented Legendrian surgery and then Legendrian isotopy.

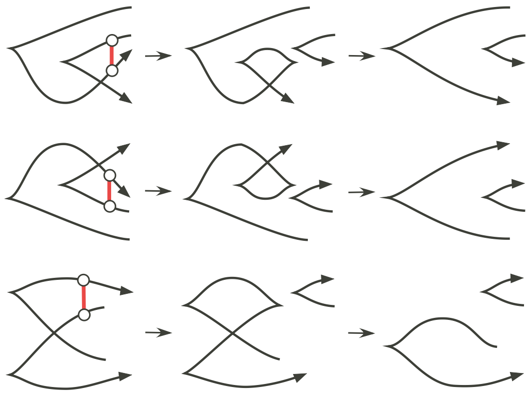


Figure 10. Three cases for the leftmost positive crossing and their associated left cusps; three additional cases are obtained by reversing the orientations on both strands.

to do a surgery immediately to the right of this leftmost crossing. After Legendrian Reidemeister moves, the crossing is eliminated and the number of crossings of the projection of the resulting link has decreased by 1; see Figure 10. What was the second leftmost positive crossing is now the leftmost positive crossing and the procedure can be repeated. In this way, we obtain a surgery string of Legendrian links $(\tilde{\Lambda}_m, \dots, \tilde{\Lambda}_{m-n}, \tilde{\Lambda}_{m-n-1}, \dots, \tilde{\Lambda}_0)$ where $\tilde{\Lambda}_0$ has a front projection with no crossings. It follows that $\tilde{\Lambda}_0$ is topologically a trivial link of unknots. By applying a Legendrian isotopy, we can assume that $\tilde{\Lambda}_0$ consists of c Legendrian unknots which are vertically stacked and where each unknot is oriented "clockwise"; an example of this is shown in Figure 11. It is then easy to see that after applying $c-1$ additional surgeries, we can obtain a Legendrian unknot. Thus there is a length $u = m + c - 1$ unknotting surgery sequence for Λ. By Lemma 3.2, if $\Lambda = \tilde{\Lambda}_m$ has j components, $\tilde{\Lambda}_0$ has at most $c = j + m$ components. Thus we see that $u \leq 2m + j - 1$, as claimed. □

Definition 3.6. Given a Legendrian link Λ, the (oriented) *Legendrian surgery unknotting number of* Λ, $\sigma_0(\Lambda)$, is defined as the minimal length of an oriented, unknotting surgery string for Λ.

Remark 3.7. Here are some basic properties of $\sigma_0(\Lambda)$:

(1) By Lemma 3.2, for any Legendrian link Λ, the parity of $\sigma_0(\Lambda)$ is opposite the parity of the number of components of Λ;

(2) For any oriented, Legendrian link Λ with j components, $j - 1 \leq \sigma_0(\Lambda) < \infty$, with $0 = \sigma_0(\Lambda)$ if and only if Λ is topologically an unknot.

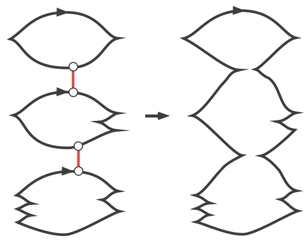


Figure 11. After all crossings are eliminated, a Legendrian isotopy can be applied so that Λ_0 is a stack of c Legendrian unknots oriented clockwise. After $c-1$ additional surgeries, a Legendrian unknot is obtained.

(3) If Λ is a topologically nontrivial Legendrian knot and there exists an oriented unknotting surgery string for Λ of length 2, then $\sigma_0(\Lambda) = 2$.

(4) If Λ' is obtained from Λ by stabilization(s), then $\sigma_0(\Lambda') \leq \sigma_0(\Lambda)$.

Proposition 3.5 and, more importantly, explicit calculations will give upper bounds for $\sigma_0(\Lambda)$. Now we turn to examining some lower bounds for $\sigma_0(\Lambda)$.

First, by Theorem 2.1, if Λ' is a Legendrian unknot, then

$$\mathrm{tb}(\Lambda') + |r(\Lambda')| \leq -1.$$

Thus if Λ is a Legendrian link with a "large" Thurston–Bennequin and/or rotation number, one is forced to do a certain number of Legendrian surgeries. More precisely, Lemma 3.2 implies:

Lemma 3.8. *For any Legendrian link* Λ,

$$\mathrm{tb}(\Lambda) + |r(\Lambda)| + 1 \leq \sigma_0(\Lambda).$$

Lemma 3.8 gives us improved lower bounds over those given in Remark 3.7 when $2 \leq \mathrm{tb}(\Lambda) + |r(\Lambda)|$.[4] For example, there exists a Legendrian whose underlying smooth knot type is $m(5_1)$ and whose classical invariants satisfy

$$\mathrm{tb}(\Lambda) + |r(\Lambda)| = 3;$$

see, for example, [Chongchitmate and Ng 2013]. Thus Lemma 3.8 implies that $4 \leq \sigma_0(\Lambda)$. However for many links, $\mathrm{tb}(\Lambda) + |r(\Lambda)| \leq 2$. For example, for any Legendrian Λ that is topologically the 5_1 knot, $\mathrm{tb}(\Lambda) + |r(\Lambda)| \leq -5$. Although Lemma 3.8 will not help us, in this case we can make use of another result:

[4]The parity of $\mathrm{tb}(\Lambda) + |r(\Lambda)|$ agrees with the parity of the number of components of Λ, so for knots, we get interesting new bounds when $3 \leq \mathrm{tb}(\Lambda) + |r(\Lambda)|$.

Lemma 3.9. *For a Legendrian link Λ with j components, let L_Λ denote the underlying smooth link type of Λ, and let $g_4(L_\Lambda)$ denote the smooth 4-ball genus of L_Λ. Then*

$$2g_4(L_\Lambda) + (j-1) \le \sigma_0(\Lambda).$$

Proof. From a Legendrian surgery string of length n that ends at an unknot, one can construct a smooth, orientable, compact, and connected 2-dimensional surface in B^4 with boundary equal to L_Λ and Euler characteristic equal to $1-n$; the genus, g, of this surface satisfies $1-n = 2-2g-j$. Thus, by definition of the smooth 4-ball genus,

$$(j-1) + 2g_4(L_\Lambda) \le (j-1) + 2g = n.$$

Since $\sigma_0(\Lambda)$ is the minimum length of a surgery unknotting string, the claim follows. □

A convenient table of smooth 4-ball genera of knots can be found at KnotInfo [Cha and Livingston 2012].

4. The surgery unknotting number for families of knots

In this section we will calculate the surgery unknotting numbers for Legendrian twist knots, Legendrian torus links, and positive, Legendrian rational links. The fact that we can precisely calculate these numbers for the first two families rests upon classification results of [Etnyre et al. 2013; Etnyre and Honda 2001; Dalton 2008].

Legendrian twist knots. A twist knot is a knot that is smoothly equivalent to a knot K_m in the form of Figure 12. In other words, a twist knot is a twisted Whitehead double of the unknot.

Theorem 4.1. *If Λ is a Legendrian knot that is topologically a nontrivial twist knot then $\sigma_0(\Lambda) = 2$.*

Proof. Etnyre, Ng and Vértesi [Etnyre et al. 2013] have classified all Legendrian twist knots. In particular, every Legendrian knot Λ with maximal Thurston–Bennequin invariant that is topologically K_m, for some $m \le -2$, is Legendrian isotopic to one

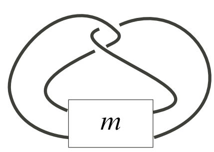


Figure 12. The twist knot K_m; the box contains m right-handed half twists if $m \ge 0$, and $|m|$ left-handed twists if $m < 0$. Notice that K_0 and K_{-1} are unknots.

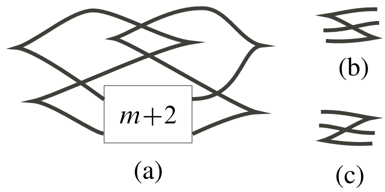


Figure 13. Any Legendrian knot that is topologically a negative twist knot, K_m with $m \leq -2$, and has maximal Thurston–Bennequin invariant is Legendrian isotopic to one of the form in (a) where the box contains $|m+2|$ half twists, each of form S as shown in (b) or of form Z as shown in (c).

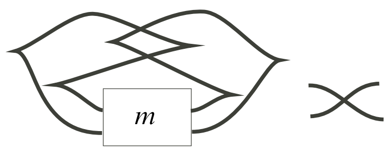
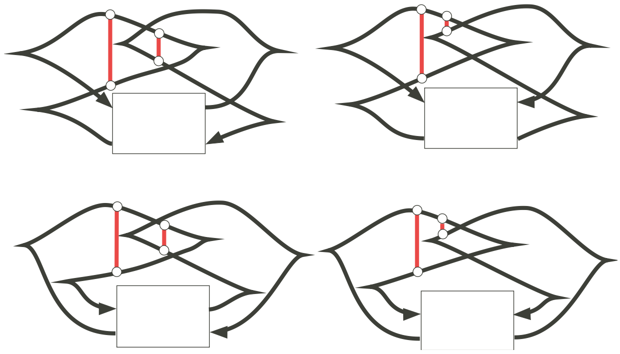


Figure 14. Any Legendrian knot that is topologically a positive twist knot, K_m with $m \geq 0$, and has maximal Thurston–Bennequin invariant is Legendrian isotopic to one of the form on the left. The box contains m half twists, each of form X as shown on the right.

of the form in Figure 13, and every Legendrian knot Λ with maximal Thurston–Bennequin invariant that is topologically K_m, for $m \geq 1$ with maximal Thurston–Bennequin invariant is Legendrian isotopic to one of the form in Figure 14.[5] Every Legendrian knot Λ that is topologically a nontrivial twist knot is obtained by stabilization of one of these with maximal Thurston–Bennequin invariant. By Remark 3.7, it suffices to show for any Legendrian knot Λ^+ that is topologically a nontrivial twist knot and has maximal Thurston–Bennequin invariant, $\sigma_0(\Lambda^+) = 2$. For Λ^+, we can do the two unknotting surgeries near the "clasp". The sign of the crossings in the clasp will depend on whether m is even or odd: Figure 15 shows the positions of two surgeries that result in an unknot. □

Legendrian torus links. A torus link is a link that can be smoothly isotoped so that it lies on the surface of an unknotted torus in $\mathbb{R}^3$. Every torus knot can be specified by a pair (p, q) of coprime integers: we will use the convention that the (p, q)-torus knot, $T(p, q)$, winds p times around a meridional curve of the torus and q times in the longitudinal direction. See, for example, [Adams 2004]. In fact, $T(p, q)$ is equivalent to $T(q, p)$ and to $T(-p, -q)$. So we will always assume that $|p| > q > 0$; in addition we will assume $q > 1$ since we are interested in nontrivial

[5] We omit $m = 0, -1$ since those correspond to the unknot.



(a) (b)

Figure 15. For a Legendrian knot with maximal Thurston–Bennequin invariant that is topologically K_m, (a) gives the surgery points when m is even, and (b) gives the surgery points when m is odd.

torus knots. For $j \geq 2$, $T(jp, jq)$, with $|p| > q > 1$ and $\gcd(p,q) = 1$, will be a j-component link where each component is a $T(p,q)$ torus link. We will only consider torus links of nontrivial components.

Theorem 4.2. *If Λ is a j-component Legendrian link that is topologically the (jp, jq)-torus link, $|p| > q > 1$, then $\sigma_0(\Lambda) = (|jp|-1)(jq-1)$.*

Proof. First consider the case where Λ is topologically a positive torus knot, $T(p,q)$ with $p > 0$. As shown by Etnyre and Honda [2001], the list of different Legendrian representations of a positive torus knot can be represented as a "single-peaked mountain" in parallel to the mountain of unknots shown in Figure 7. Namely, for fixed $p > q > 1$, there is a unique Legendrian knot Λ^+ that is topologically $T(p,q)$ with maximal Thurston–Bennequin invariant $\mathrm{tb}(\Lambda^+) = pq - p - q$ and $r(\Lambda^+) = 0$; any Legendrian knot Λ that is topologically $T(p,q)$ is obtained by stabilizations of Λ^+. By Remark 3.7, it suffices to show that if Λ^+ is a Legendrian knot that is topologically $T(p,q)$ and has maximal Thurston–Bennequin invariant, then $\sigma_0(\Lambda^+) = (p-1)(q-1)$. By Lemma 3.8,

$$\mathrm{tb}(\Lambda^+) + |r(\Lambda^+)| + 1 = (p-1)(q-1) \leq \sigma_0(\Lambda).$$

In fact, it is possible to unknot with $(p-1)(q-1)$ surgeries. Starting from the left most string of crossings, do $(q-1)$ successive surgeries as illustrated for the $(5,3)$-torus knot in Figure 16; in this sequence of surgeries, one begins with the surgery on the innermost strands, and then performs a Legendrian isotopy so that it is possible to do a surgery on the next set of innermost strands. In general, this

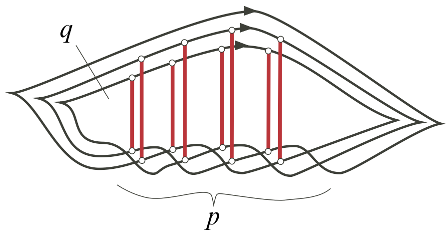


Figure 16. The Legendrian $(5, 3)$-torus knot with maximal tb invariant. The general, positive Legendrian (p, q)-torus knot with maximal Thurston–Bennequin invariant is constructed using q strands and a length p string of crossings. Shown are the $(p-1)(q-1)$ oriented Legendrian surgeries that unknot the Legendrian positive (p, q)-torus knot with maximal tb.

takes the (p, q)-torus knot to the $(p-1, q)$-torus link. Repeating this $p-1$ times results in the $(1, q)$-torus knot, which is an unknot.[6]

The above proof easily generalizes to positive torus links of nontrivial components. Dalton [2008] showed that there is a unique Legendrian link Λ^+ that is topologically $T(jp, jq)$ with maximal Thurston–Bennequin invariant $\mathrm{tb}(\Lambda^+) = jpjq - jp - jq$. The construction of this one exactly parallels the construction in Figure 16, and so the same pattern of $(jp-1)(jq-1)$ surgeries will produce a Legendrian unknot.

Next consider the case where Λ is topologically a negative torus knot, $T(p, q)$ with $p < 0$. In this case, Etnyre and Honda have shown that the list of different Legendrian representations of a negative torus knots, $T(p, q)$ for $p < 0$ and $|p| > q > 1$, can be represented as a many-peaked "mountain range" where the number of representatives with maximal Thurston–Bennequin invariant depends on the divisibility of p by q. Namely, if $|p| = mq + e$, $0 < e < q$, then there will be $2m$ Legendrian representatives of $T(p, q)$ with maximal Thurston–Bennequin invariant of $pq < 0$. Half of these different representatives with maximal Thurston–Bennequin invariant are obtained by writing $m = 1 + n_1 + n_2$, where $n_1, n_2 \geq 0$, and then $\Lambda^+_{(n_1,n_2)}$ is constructed using the form shown in Figure 17 with n_1 and n_2 copies of the tangle B inserted as indicated:

$$r\bigl(\Lambda^+_{(n_1,n_2)}\bigr) = q(n_2 - n_1) + e.$$

The other m Legendrian versions of $T(p, q)$ with maximal Thurston–Bennequin invariant are obtained by reversing the orientation. For negative torus knots, Lemma 3.8 will not be a useful lower bound. However, since the calculation of the 4-ball genus is the same for both the knot and its mirror, the calculations in

[6]By Corollary 1.3, we can now deduce the Milnor conjecture as mentioned in Corollary 1.5.

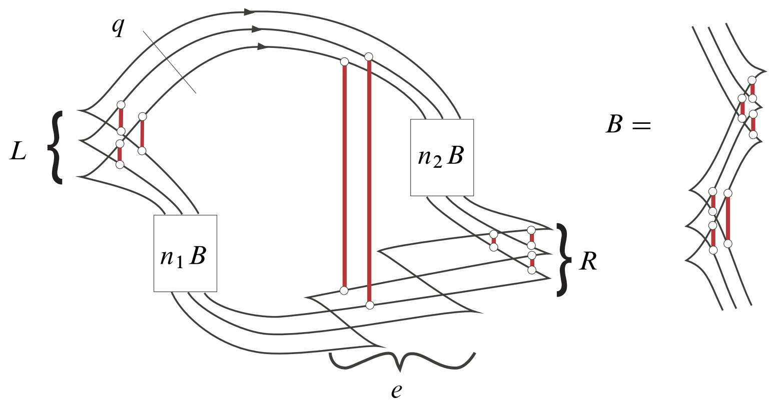


Figure 17. The $(|p|-1)(q-1)$ oriented Legendrian surgeries that unknot a Legendrian negative (p,q)-torus knot with maximal Thurston–Bennequin invariant.

the positive torus knot case and Corollary 1.3, (or [Kronheimer and Mrowka 1993]), show that for a negative torus knot $T(p,q)$, $2g_4(T(p,q)) = (|p|-1)(q-1)$. Thus, by Lemma 3.9

$$(|p|-1)(q-1) \leq \sigma_0(\Lambda).$$

In fact, it is possible to arrive at an unknot with $(|p|-1)(q-1)$ surgeries. Figure 17 shows the claimed surgeries: a surgery is done to the right of all crossings in the L, R, and B regions (contributing $\frac{1}{2}q(q-1)+\frac{1}{2}q(q-1)+(n_1+n_2)q(q-1)$ surgeries), and between each Z in the e string one does $q-1$ successive surgeries (contributing $(e-1)(q-1)$ surgeries). Thus the total number of surgeries is

$$(1+n_1+n_2)q(q-1)+(e-1)(q-1) = (mq+e-1)(q-1) = (|p|-1)(q-1).$$

The proof easily generalizes to negative torus links. It follows from [Nakamura 2000] that $g_4(T(jp, jq)) + (j-1) = (j|p|-1)(jq-1)$; see Remark 4.3. It was shown in [Dalton 2008] that there are $2m$ Legendrian links Λ^+ that are topologically $T(jp, jq)$ with maximal Thurston–Bennequin invariant, and all Legendrians that are topologically $T(jp, jq)$ are obtained by stabilizations of one of these. Each of these with maximal Thurston–Bennequin invariant can be constructed as in Figure 17, and so the same pattern of $(j|p|-1)(jq-1)$ surgeries will produce a Legendrian unknot. □

Remark 4.3. Nakamura's formula [2000] for the smooth 4-ball genus of a j-component positive link L is that

$$2g_4(L) = 2 - j - s(D) + c(D),$$

where $s(D)$ is the number of Seifert circles and $c(D)$ is the number of crossings in a non-split positive diagram D for L. It is straightforward to see that when L is the positive torus link $T(jp, jq)$, using the diagram D corresponding to Figure 16, $s(D) = jq$ and $c(D) = jp(jq-1)$. So,

$$2g_4(T(jp, jq)) = 2 - j - jq + jp(jq-1) = (1-j) + (jp-1)(jq-1).$$

Thus for any Legendrian link Λ that is topologically $T(jp, jq)$, for either p positive or negative,

$$2g_4(T(jp, jq)) + (j-1) = \sigma_0(\Lambda).$$

Positive, Legendrian rational links.

Definition 4.4. Given a vector of integers $(c_n, \dots, c_2, c_1)$, where $c_n \geq 2$, and $n \geq 2$ implies $c_i \geq 1$ for $i = 1, \dots, n-1$, we construct the *rational Legendrian link* $\Lambda(c_n, \dots, c_2, c_1)$ to be the Legendrian numerator closure of the Legendrian tangle $(c_n, \dots, c_2, c_1)$ as demonstrated in Figure 18; see also [Adams 2004; Traynor 1998; Schneider 2011]. The rational Legendrian link $\Lambda(c_n, \dots, c_2, c_1)$ is *positive* if all crossings are positive.

This Legendrian link $\Lambda(c_n, \dots, c_1)$ is topologically the numerator closure of the rational tangle associated to the rational number q with continued fraction expansion $q = c_1 + 1/(c_2 + 1/(c_3 + \dots))$; see [Conway 1970].

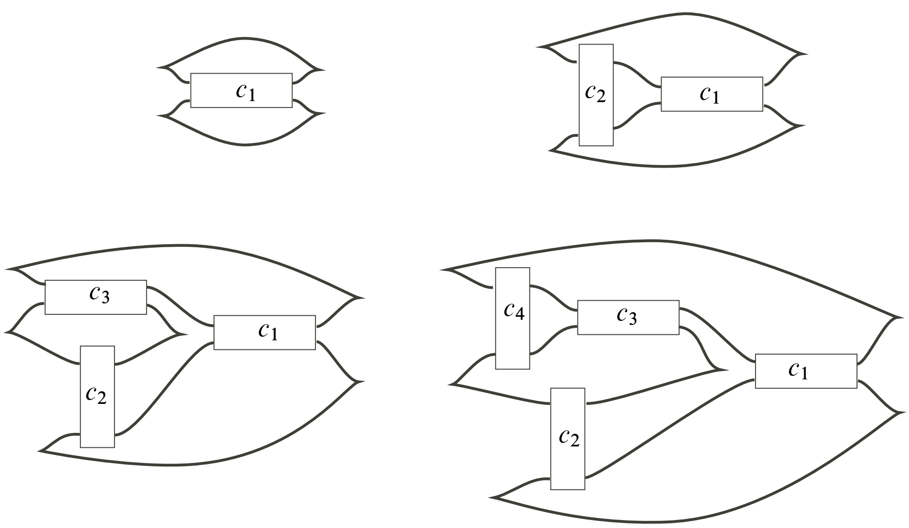


Figure 18. The general form of $\Lambda(c_1)$, $\Lambda(c_2, c_1)$, $\Lambda(c_3, c_2, c_1)$, and $\Lambda(c_4, c_3, c_2, c_1)$.

The "even" entries $c_2, c_4, \ldots$ of the vector $(c_n, \ldots, c_2, c_1)$ denote the strings of vertical crossings. It is straightforward to verify that the parity of these vertical entries determine when $\Lambda(c_n, \ldots, c_1)$ is a positive link:

Lemma 4.5. (1) *When n is odd, there exists an orientation on the components of $\Lambda(c_n, \ldots, c_1)$ so that it is a positive link if and only if c_i is even, for all i even. Moreover, $\Lambda(c_n, \ldots, c_1)$ is a knot when $\sum_{i \text{ odd}} c_i$ is odd.*

(2) *When n is even, there exists an orientation on the components of $\Lambda(c_n, \ldots, c_1)$ so it is a positive link if and only if c_n is odd and $c_{n-2}, c_{n-4}, \ldots, c_2$ are all even. Moreover, $\Lambda(c_n, \ldots, c_1)$ is a knot when $\sum_{i \text{ odd}} c_i$ is even.*

The Legendrian surgery unknotting number of a positive link has a convenient formula in terms of the "odd" entries, which correspond to the strings of horizontal crossings. There will be some differences in following formulas depending on whether Λ is constructed from an odd or an even length vector. Define

$$p(n) = \begin{cases} 1, & n \text{ odd}, \\ 0, & n \text{ even}; \end{cases}$$

$p(n)$ measures the parity of the "length" of the vector $(c_n, \ldots, c_1)$.

Theorem 4.6. *If $\Lambda(c_n, \ldots, c_2, c_1)$ is a positive, Legendrian rational link, then*

$$\sigma_0(\Lambda(c_n, \ldots, c_2, c_1)) = \sum_{i \text{ odd}} c_i - p(n).$$

Proof. This will be proved using the lower bound on $\sigma_0(\Lambda)$ provided by Lemma 3.8, and explicit calculations.

We will first show that

$$r(\Lambda(c_n, \ldots, c_2, c_1)) = 0 \quad \text{and} \quad \operatorname{tb}(\Lambda(c_n, \ldots, c_2, c_1)) = \sum_{i \text{ odd}} c_i - p(n) - 1.$$

It is easy to verify that when all the crossings are positive, the up and down cusps cancel in pairs and thus the rotation number vanishes. To calculate $\operatorname{tb}(\Lambda(c_n, \ldots, c_1))$, notice that when n is odd the number of right cusps is 2 more than the number of vertical crossings, $\sum_{i \text{ even}} c_i$, while when n is even, the number of rights cusps is 1 more than the number of vertical crossings. Thus:

$$\operatorname{tb}(\Lambda(c_n, \ldots, c_2, c_1)) = \sum_{i=1}^{n} c_i - \left(\sum_{i \text{ even}} c_i + 1 + p(n) \right) = \sum_{i \text{ odd}} c_i - 1 - p(n).$$

Thus, by Lemma 3.8,

$$\sum_{i \text{ odd}} c_i - p(n) \leq \sigma_0(\Lambda(c_n, \ldots, c_2, c_1)).$$

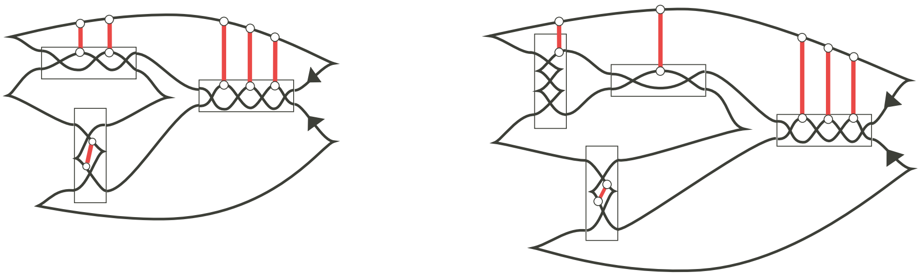


Figure 19. Two positive, Legendrian rational knots of odd and even lengths. In both cases, it is possible to unknot by doing $c_i - 1$ surgeries in each horizontal segment (i odd) and 1 surgery in each vertical segment.

In fact, it is possible to unknot $\Lambda(c_n, \ldots, c_2, c_1)$ by doing $c_i - 1$ surgeries in each horizontal component and 1 surgery in each vertical segment; Figure 19 illustrates some examples of this. When $n = 1$, there are no vertical segments; for other odd n, the number of vertical components is one less than the number of horizontal components, and when n is even, the number of vertical components agrees with the number of horizontal components. Thus

$$\sigma_0(\Lambda(c_n, \ldots, c_1)) \leq \sum_{i \text{ odd}} c_i - p(n),$$

and the desired calculation of $\sigma_0(\Lambda(c_n, \ldots, c_1))$ follows. □

Remark 4.7. In the above proof, $\sigma_0(\Lambda(c_n, \ldots, c_1))$ is obtained by realizing the lower bound given by the classical Legendrian invariants. Thus, by Corollary 1.3, we see that when $\Lambda(c_n, \ldots, c_1)$ has an underlying topological type of the knot K_Λ, $\sigma_0(\Lambda(c_n, \ldots, c_1)) = 2g_4(K_\Lambda)$. Moreover, when $\Lambda(c_n, \ldots, c_1)$ has an underlying topological type of a 2-component link L_Λ, we can compare $\sigma_0(\Lambda(c_n, \ldots, c_1))$ to the smooth 4-ball genus of L_Λ using Nakamura's formula (see Remark 4.3) for the smooth 4-ball genus of a positive link. When n is odd, the number of Seifert circles is $s(D) = 2 + \sum_{i \text{ even}} c_i$, while when n is even, $s(D) = 1 + \sum_{i \text{ even}} c_i$. Thus we find that for a 2-component, positive, Legendrian rational link $\Lambda(c_n, \ldots, c_1)$,

$$2g_4(L_\Lambda) + 1 = c(D) - s(D) + 1 = \sum_{i \text{ odd}} c_i - p(n) = \sigma_0(\Lambda(c_n, \ldots, c_1)).$$

5. The surgery unknotting number for small crossing knots

Given the calculations of the previous section, it is natural to ask Question 1.9 in the Introduction. To investigate the knot portion of this question, we examined Legendrian representatives of low-crossing knots. There is not a Legendrian

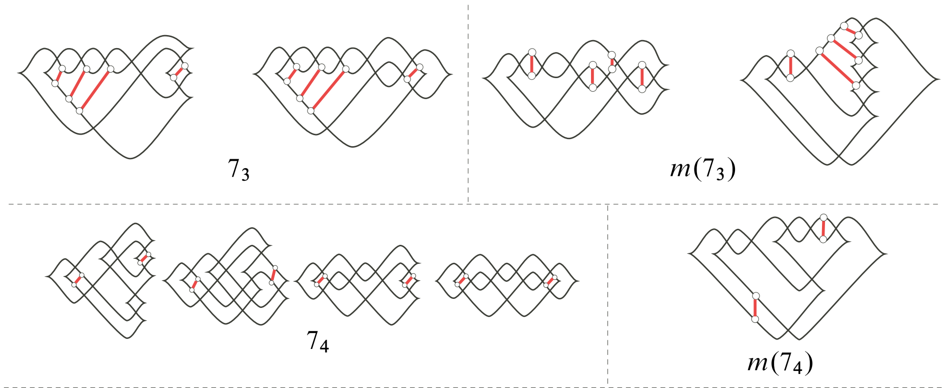


Figure 20. Front projections representing all conjectured Legendrian representatives of 7_3, $m(7_3)$, 7_4, and $m(7_4)$ with maximal Thurston–Bennequin invariant. For all of these knot types, $g_3(K_\Lambda) = g_4(K_\Lambda)$; the indicated surgery points realize $\sigma_0(\Lambda) = 2g_4(K_\Lambda)$.

classification of all these knot types, but a conjectured classification of these knot types can be found in [Chongchitmate and Ng 2013]. In the following, we prove Proposition 1.10, which says that the surgery unknotting number of the Legendrian agrees with twice the smooth 4-ball genus of the underlying smooth knot for all Legendrians that are topologically a nonslice knot with crossing number at most 7.

In Section 4, Proposition 1.10 is verified for all torus and twist knots. The only non-torus and non-twist knots with 7 or fewer crossings are $6_2, m(6_2)$, $6_3 = m(6_3)$, 7_3, $m(7_3)$, 7_4, $m(7_4)$, 7_5, $m(7_5)$, 7_6, $m(7_6)$, 7_7, and $m(7_7)$. The needed calculations fall into three categories as described below.

Example 5.1. For the smooth knots $7_3, m(7_3), 7_4, m(7_4), 7_5$, and $m(7_5)$, the genus, g_3, agrees with the smooth 4-ball genus g_4.[7] In general, we find that for a Legendrian Λ whose underlying knot type K_Λ satisfied $g_3(K_\Lambda) = g_4(K_\Lambda)$, it is fairly straightforward to show that $\sigma_0(\Lambda) = 2g_4(L_\Lambda)$. For example, Figure 20 shows all conjectured representatives of 7_3, $m(7_3)$, 7_4, $m(7_4)$, 7_5, and $m(7_5)$ with maximal Thurston–Bennequin invariant (after perhaps selecting alternate orientations and/or performing the Legendrian mirror operation, which consists of rotating the diagram 180°). For each of these with maximal Thurston–Bennequin invariant, it is possible to unknot with $2g_4(K_\Lambda)$ surgeries as indicated.

In general, we found that for a Legendrian Λ whose underlying knot type K_Λ satisfied $g_4(K_\Lambda) < g_3(K_\Lambda)$, it is more difficult to calculate $\sigma_0(\Lambda)$. To do calculations for our remaining cases, we made use of the well known fact that the

[7] This is also the situation for the torus and nonslice twist knots studied in Section 4.

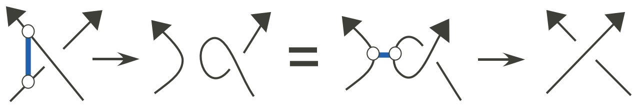


Figure 21. A sequence of two topological surgeries in a neighborhood of a negative crossing that topologically change the crossing. An analogous picture shows that a positive crossing can be changed into a negative crossing by two topological surgeries.

unknotting number of a knot, $u(K)$, gives an upper bound to the smooth 4-ball genus:

$$g_4(K) \leq u(K). \tag{5-1}$$

Figure 21 demonstrates two topological surgeries that produce a crossing change; an argument as in the proof of Lemma 3.9 then proves inequality (5-1). Notice that the topological Reidemeister moves used in the equivalence are not Legendrian Reidemeister moves. However, near a negative crossing, it is possible to "Legendrify" this construction:

Lemma 5.2. *If the Legendrian knot* Λ *has a front projection that can be topologically unknotted by changing a negative crossing*, *then*

$$\sigma_0(\Lambda) \leq 2.$$

Proof. Figure 22 demonstrates how two surgeries can locally produce a topological crossing change. □

Example 5.3. Using Lemma 5.2, it is possible to show that for any conjectured Legendrian representative Λ of 6_2, 6_3, 7_6, or 7_7, $\sigma_0(\Lambda) = 2g_4(K_\Lambda)$. Figure 23 shows the conjectured Legendrian representatives of these knot types with maximal Thurston–Bennequin invariant (after perhaps selecting alternate orientations and/or performing a mirror operation, which corresponds to a rotation of the front diagram

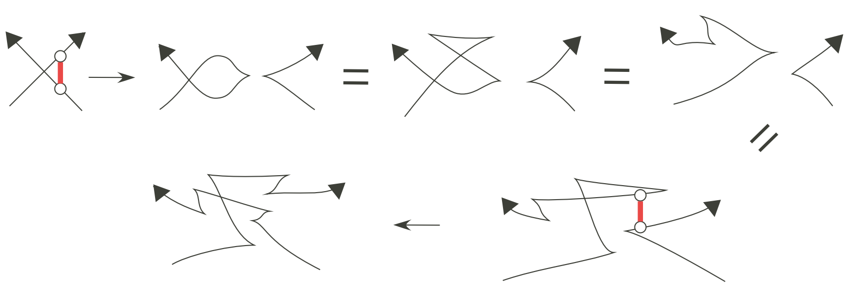


Figure 22. A sequence of two oriented surgeries in a neighborhood of a negative crossing that topologically change the crossing.

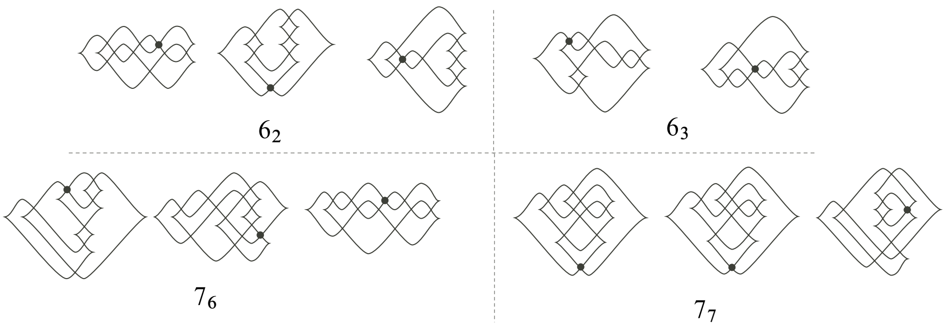


Figure 23. Front projections representing all conjectured Legendrian representatives of 6_2, 6_3, 7_6, and 7_7 with maximal Thurston–Bennequin invariant. These projections can be topologically unknotted at the indicated negative crossing.

by 180°), and the negative crossing that when topologically changed produces an unknot.

We were not able to find front projections of the conjectured maximal Thurston–Bennequin representatives of $m(6_2)$, $m(7_6)$, or $m(7_7)$ that could be topologically unknotted by changing a negative crossing; in fact, by [Soteros et al. 2011], it is *not* possible to do this even in the smooth setting. Luckily, sometimes we can topologically change a positive crossing when it has a special form.

Definition 5.4. A positive crossing is of *S form*, *Z form*, or *hooked-X form* if it takes the form as shown in Figure 24.

Lemma 5.5. *If* Λ *is a nontrivial Legendrian knot that has a projection that can be topologically unknotted by changing a positive crossing in* S, Z, *or hooked-X form, then*

$$\sigma_0(\Lambda) \leq 2.$$

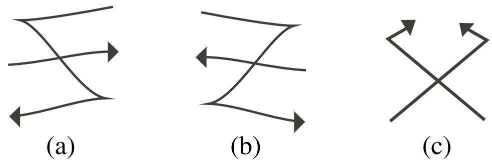


Figure 24. A positive crossing of (a) S form, (b) Z form, and (c) hooked-X form. Reversing the orientations on both strands keeps the respective forms. Also reflecting the planar figure in (c) about a horizontal line produces another hooked-X form.

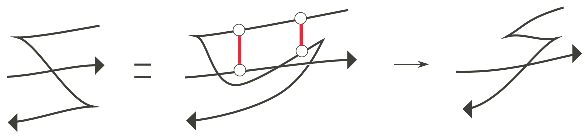


Figure 25. A positive crossing of S form can be transformed into a negative crossing with 2 surgeries. Similarly, a positive crossing of Z form can be transformed into a negative crossing with 2 surgeries.

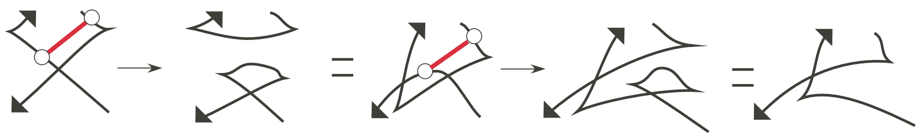


Figure 26. A positive crossing of hooked-X form can be transformed into a negative crossing with 2 Legendrian surgeries.

Proof. Figures 25 and 26 show how a positive crossing in S, Z, or hooked-X form can be transformed into a negative crossing using two surgeries and Legendrian isotopies. □

Example 5.6. Using Lemma 5.5, it is possible to show that for any conjectured Legendrian representative Λ of $m(6_2)$, $m(7_6)$, or $m(7_7)$, $\sigma_0(\Lambda) = 2g_4(K_\Lambda)$. Figure 27 shows the conjectured Legendrian representatives of these knot types with maximal Thurston–Bennequin invariant (after perhaps selecting alternate orientations and/or

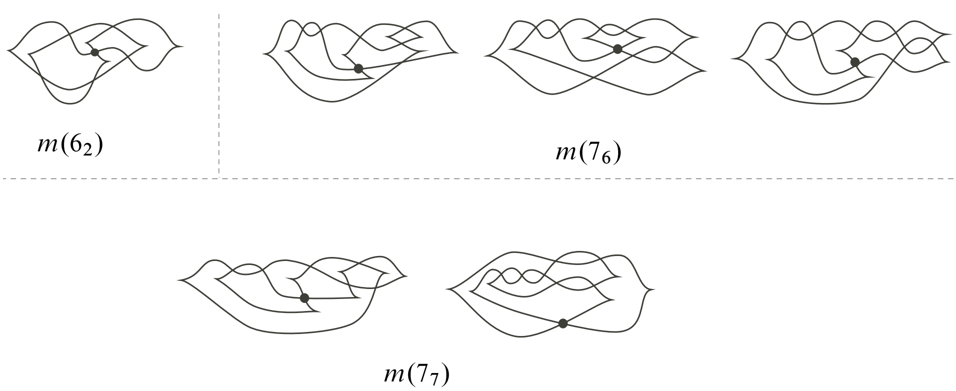


Figure 27. Front projections representing all conjectured Legendrian representatives of $m(6_2)$, $m(7_6)$ and $m(7_7)$ with maximal Thurston–Bennequin invariant. Each of these can be topologically unknotted by changing the indicated positive crossing in S form or hooked-X form.

performing a mirror operation). These projections differ from those in [Chongchitmate and Ng 2013] by Legendrian Reidemeister moves of type II and III. The black dot indicates a positive crossing that when topologically changed produces an unknot.

The proofs of Lemmas 5.2 and 5.5 in fact show that if the Legendrian knot Λ has a front projection that can be topologically unknotted by changing ν negative crossings and ρ crossings in S, Z, or hooked-X form, then $\sigma_0(\Lambda) \leq 2\nu + 2\rho$. However, for our calculations we did not need this more general form.

Acknowledgements

The ideas of this project were inspired by joint work of Josh Sabloff and the second author. We are extremely grateful for the many fruitful discussions we have had with Sabloff throughout this project. We also gained much by discussions with Chuck Livingston about the smooth 4-ball genus; we are very thankful for his clear explanations. We also thank Paul Melvin and other members of our PACT (Philadelphia Area Contact/Topology) seminar for useful comments during a series of presentations on this work.

References

[Adams 2004] C. C. Adams, *The knot book: an elementary introduction to the mathematical theory of knots*, American Mathematical Society, Providence, RI, 2004. MR 2005b:57009 Zbl 1065.57003

[Akbulut and Matveyev 1997] S. Akbulut and R. Matveyev, "Exotic structures and adjunction inequality", *Turkish J. Math.* **21**:1 (1997), 47–53. MR 98d:57053 Zbl 0885.57011 arXiv math/0011055

[Bourgeois et al. $\geq$ 2013] F. Bourgeois, J. Sabloff, and L. Traynor, "Lagrangian cobordisms via generating families: constructions and geography". In preparation.

[Casson and Gordon 1986] A. J. Casson and C. M. Gordon, "Cobordism of classical knots", pp. 181–199 in *À la recherche de la topologie perdue*, edited by L. Guillou and A. Marin, Progr. Math. **62**, Birkhäuser, Boston, 1986. MR 900252

[Cha and Livingston 2012] J. C. Cha and C. Livingston, "Knotinfo: table of knot invariants", web site, 2012, http://www.indiana.edu/~knotinfo.

[Chantraine 2010] B. Chantraine, "Lagrangian concordance of Legendrian knots", *Algebr. Geom. Topol.* **10**:1 (2010), 63–85. MR 2011f:57049 Zbl 1203.57010

[Chongchitmate and Ng 2013] W. Chongchitmate and L. Ng, "An atlas of Legendrian knots", *Exp. Math.* **22**:1 (2013), 26–37. MR 3038780 Zbl 06180370

[Conway 1970] J. H. Conway, "An enumeration of knots and links, and some of their algebraic properties", pp. 329–358 in *Computational problems in abstract algebra* (Oxford, 1967), edited by J. Leech, Pergamon, Oxford, 1970. MR 41 #2661 Zbl 0202.54703

[Dalton 2008] J. Dalton, *Legendrian torus links*, Ph.D. thesis, Bryn Mawr College, Bryn Mawr, PA, 2008, http://repository.brynmawr.edu/dissertations/6. MR 2712646

[Ekholm et al. 2012] T. Ekholm, K. Honda, and T. Kálmán, "Legendrian knots and exact Lagrangian cobordisms", preprint, 2012. arXiv 1212.1519

[Eliashberg and Fraser 2009] Y. Eliashberg and M. Fraser, "Topologically trivial Legendrian knots", *J. Symplectic Geom.* **7**:2 (2009), 77–127. MR 2010d:57024 Zbl 1179.57040

[Etnyre 2005] J. B. Etnyre, "Legendrian and transversal knots", pp. 105–185 in *Handbook of knot theory*, edited by W. Menasco and M. Thistlethwaite, Elsevier, Amsterdam, 2005. MR 2006j:57050 Zbl 1095.57006

[Etnyre and Honda 2001] J. B. Etnyre and K. Honda, "Knots and contact geometry, I: Torus knots and the figure eight knot", *J. Symplectic Geom.* **1**:1 (2001), 63–120. MR 2004d:57032 Zbl 1037.57021

[Etnyre et al. 2013] J. B. Etnyre, L. Ng, and V. Vértesi., "Legendrian and transverse twist knots", *J. Eur. Math. Soc.* **15**:3 (2013), 969–995. arXiv 1002.2400

[Kronheimer and Mrowka 1993] P. B. Kronheimer and T. S. Mrowka, "Gauge theory for embedded surfaces, I", *Topology* **32**:4 (1993), 773–826. MR 94k:57048 Zbl 0799.57007

[Lisca and Matić 1998] P. Lisca and G. Matić, "Stein 4-manifolds with boundary and contact structures", *Topology Appl.* **88**:1-2 (1998), 55–66. MR 99f:57037 Zbl 0978.53122

[Livingston 2005] C. Livingston, "A survey of classical knot concordance", pp. 319–347 in *Handbook of knot theory*, edited by W. Menasco and M. Thistlethwaite, Elsevier, Amsterdam, 2005. MR 2006k:57020 Zbl 1098.57006 arXiv math/0307077

[Nakamura 2000] T. Nakamura, "Four-genus and unknotting number of positive knots and links", *Osaka J. Math.* **37**:2 (2000), 441–451. MR 2001e:57005 Zbl 0968.57008

[Rudolph 1995] L. Rudolph, "An obstruction to sliceness via contact geometry and "classical" gauge theory", *Invent. Math.* **119**:1 (1995), 155–163. MR 95k:57013 Zbl 0843.57011

[Schneider 2011] G. R. Schneider, *On the classification of Legendrian rational tangles via characteristic foliations of compressing discs*, Ph.D. thesis, State University of New York, Buffalo, NY, 2011, http://tinyurl.com/schneider-thesis. MR 2898627

[Soteros et al. 2011] C. Soteros, K. Ishihara, K. Shimokawa, M. Szafron, and M. Vazquez, "Signed unknotting number and knot chirality discrimination via strand passage", *Prog. Theor. Phys. Suppl.* **191** (2011), 78–95.

[Traynor 1998] L. Traynor, "A Legendrian stratification of rational tangles", *J. Knot Theory Ramifications* **7**:5 (1998), 659–700. MR 2000c:57020 Zbl 0912.57003

Received: 2012-05-07 Revised: 2013-05-20 Accepted: 2013-05-25

aboranda@brynmawr.edu *Bryn Mawr College, Bryn Mawr, PA 19010, United States*

ltraynor@brynmawr.edu *Department of Mathematics, Bryn Mawr College, Bryn Mawr, PA 19010, United States*

syan@brynmawr.edu *Bryn Mawr College, Bryn Mawr, PA 19010, United States*

mathematical sciences publishers msp

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2013 is US $105/year for the electronic version, and $145/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2013 vol. 6 no. 3

1944-4176(2013)6:3;1-8