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Hyperbolic construction of Cantor sets

Zair Ibragimov and John Simanyi

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In this paper we present a new construction of the ternary Cantor set within the context of Gromov hyperbolic geometry. Unlike the standard construction, where one proceeds by removing middle-third intervals, our construction uses the collection of the removed intervals. More precisely, we first hyperbolize (in the sense of Gromov) the collection of the removed middle-third open intervals, then we define a visual metric on its boundary at infinity and then we show that the resulting metric space is isometric to the Cantor set.

1. The ternary Cantor set

The ternary Cantor set $\mathscr{C}$ is one of the most familiar fractals in mathematics. Recall its standard construction, which is based on the Euclidean notion of length. Begin with the closed unit interval $C_0 = [0, 1] \subseteq \mathbb{R}$, then remove the open middle-third interval, constructing $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$. We then remove the middle-third of each resulting closed interval again, finding

$$C_2 = \left[0, \tfrac{1}{9}\right] \cup \left[\tfrac{2}{9}, \tfrac{1}{3}\right] \cup \left[\tfrac{2}{3}, \tfrac{7}{9}\right] \cup \left[\tfrac{8}{9}, 1\right].$$

Continuing in this manner, we construct $\mathscr{C}$ by taking the intersection of all C_k's,

$$\mathscr{C} = \bigcap_{k=1}^{\infty} C_k.$$

Graphically, C_0 through C_6 are shown in Figure 1. The ternary Cantor set has many interesting properties. As the intersection of closed intervals in $(\mathbb{R}, |\,.\,|)$, it is compact. It is also perfect (i.e., it contains no isolated points), uncountable and totally disconnected. The complement of the ternary Cantor set in $[0, 1]$, $\mathscr{C}\mathscr{S}$, is called the *Cantor string*. It consists of the countable union of the removed open middle-third intervals. Cantor strings are subjects of study in fractal geometry [Lapidus and van Frankenhuijsen 2006].

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Figure 1. Sets C_0 through C_6.

2. Hyperbolic construction

We begin with a brief discussion of Gromov hyperbolic spaces. A metric space (X, d) is called *Gromov δ-hyperbolic* (or *δ-hyperbolic*) if there exists a $\delta \geq 0$ such that for all $x, y, z, w \in X$,

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta. \tag{2-1}$$

For $x, y, z \in X$, the *Gromov product* of x and y with respect to z is defined by

$$(x|y)_z = \tfrac{1}{2}[d(x, z) + d(y, z) - d(x, y)]. \tag{2-2}$$

Alternatively, the space (X, d) is δ-hyperbolic if

$$(x|y)_v \geq \min\{(x|z)_v, (z|y)_v\} - \delta,$$

for all $x, y, z, v \in X$ (see, for example, [Väisälä 2005]). A bounded metric space X is always δ-hyperbolic with $\delta \leq \operatorname{diam} X$, so only unbounded metric spaces may have more interesting characteristics.

To each Gromov hyperbolic space X, we associate a boundary at infinity, ∂X (also called the *Gromov boundary*). Fix a base point $v \in X$. A sequence $\{a_i\}$ in X is said to *converge at infinity* if $(a_i|a_j)_v \to \infty$ as $i, j \to \infty$. Two such sequences $\{a_i\}$ and $\{b_i\}$ are *equivalent* if $(a_i|b_i)_v \to \infty$ as $i \to \infty$. The boundary at infinity is defined to be the equivalence classes of sequences converging at infinity. The boundary at infinity supports a family of so-called *visual metrics*. A metric d on ∂X is called a visual metric if there exists a $v \in X$, $C \geq 1$ and $\epsilon > 0$ such that for all $x, y \in \partial X$,

$$\frac{1}{C}\rho_{\epsilon,v}(x, y) \leq d(x, y) \leq C\rho_{\epsilon,v}(x, y), \quad \text{where } \rho_{\epsilon,v}(x, y) = e^{-\epsilon(x|y)_v}.$$

Here $(x|y)_v$ is the Gromov product on ∂X, defined by

$$(x|y)_v = \inf\{\liminf_{i\to\infty}(a_i|b_i)_v : a_i \in x, b_i \in y\}$$

and we set $e^{-\infty} = 0$. The boundary at infinity of any Gromov hyperbolic space endowed with a visual metric is bounded and complete [Bonk and Schramm 2000].

Our goal is to produce the ternary Cantor set within the framework of Gromov hyperbolic spaces. As mentioned above, we do this by hyperbolizing the collection of the removed middle-third intervals. Let $\boldsymbol{X}$ be the collection of all such intervals. Hence, $\boldsymbol{X}$ contains intervals such as $\left(\frac{1}{3}, \frac{2}{3}\right)$, $\left(\frac{1}{9}, \frac{2}{9}\right)$, $\left(\frac{7}{9}, \frac{8}{9}\right)$ and so on. Note that

$$\mathscr{C} = [0, 1] \setminus \bigcup_{I \in \boldsymbol{X}} I.$$

We now proceed to construct a metric h on $\boldsymbol{X}$ so that the space $(\boldsymbol{X}, h)$ is Gromov hyperbolic. Let u_H be a distance function defined on the set of all nonempty subsets of $[0, 1]$, defined by

$$u_H(A, B) = \sup\{|x - y| : x \in A \text{ and } y \in B\}.$$

This distance function is called the *upper Hausdorff* distance (see, for instance, [Hausdorff 1957; Ibragimov 2011a]). If $I, J \in \boldsymbol{X}$ with $I = (a, b)$, $J = (c, d)$ and $b < c$, then $u_H = |a - d|$. Note also that for each $I, J \in \boldsymbol{X}$, we have

$$u_H(I, J) \geq l(I) \vee l(J) \geq \sqrt{l(I) \cdot l(J)}, \tag{2-3}$$

where the first equality holds only if $I = J$ and the second equality holds only if $l(I) = l(J)$. Here, and in what follows, $l(I)$ denotes the Euclidean length of $I \in \boldsymbol{X}$ and $a \vee b = \max\{a, b\}$ for positive numbers $a, b \in \mathbb{R}$.

Observe that since $\boldsymbol{X}$ consists of a disjoint collection of open intervals, it has a natural order $\preceq$ induced by the usual order $\leq$ on $\mathbb{R}$. Namely, we say that $I \preceq J$ if I is to the left of J or if $I = J$. Observe also that if $I \preceq J \preceq K$, then

$$u_H(I, K) \geq u_H(I, J). \tag{2-4}$$

Now we define a distance function h on $\boldsymbol{X}$. Given $I, J \in \boldsymbol{X}$, let

$$h(I, J) = 2 \log \frac{u_H(I, J)}{\sqrt{l(I) \cdot l(J)}}.$$

It is an immediate consequence of (2-3) that h is nonnegative, symmetric and $h(I, J) = 0$ if and only if $I = J$. To show that h also satisfies the triangle inequality, let I, J and K be arbitrary elements of $\boldsymbol{X}$. Then the triangle inequality $h(I, J) \leq h(I, K) + h(K, J)$ is equivalent to

$$\begin{aligned} 2 \log \frac{u_H(I, J)}{\sqrt{l(I) \cdot l(J)}} &\leq 2 \log \frac{u_H(I, K)}{\sqrt{l(I) \cdot l(K)}} + 2 \log \frac{u_H(K, J)}{\sqrt{l(K) \cdot l(J)}} \\ &= 2 \log \frac{u_H(I, K) \cdot u_H(K, J)}{l(K)\sqrt{l(I) \cdot l(J)}}. \end{aligned}$$

This is true if and only if

$$l(K) \cdot u_H(I, J) \le u_H(I, K) \cdot u_H(K, J). \tag{2-5}$$

It is also a consequence of (2-3) and (2-4) that inequality (2-5) holds if either $K \preceq I$ and $K \preceq J$ or $I \preceq K$ and $J \preceq K$. Therefore, due to symmetry, it is enough to verify the validity of (2-5) when $I \preceq K \preceq J$. In this case, since

$$u_H(I, J) = u_H(I, K) + u_H(J, K) - l(K),$$

inequality (2-5) is equivalent to $(u_H(I, K) - l(K))(u_H(J, K) - l(K)) \ge 0$, whose validity follows from (2-3). Thus, h is a metric on $\boldsymbol{X}$.

Next, we will show that h satisfies the Gromov hyperbolicity condition (2-1) with $\delta = \log 2$. We will need the following lemma.

Lemma 2.6. For all $I, J, K, L \in \boldsymbol{X}$, we have

$$u_H(I, J) \cdot u_H(K, L) \le u_H(I, K) \cdot u_H(J, L) + u_H(I, L) \cdot u_H(J, K).$$

Proof. Without loss of generality we can assume that $I \preceq J \preceq K \preceq L$. Then inequality (2-4) implies that

$$u_H(I, K) \cdot u_H(J, L) \ge u_H(I, J) \cdot u_H(K, L).$$

It also implies that

$$(u_H(I, K) - u_H(J, K))(u_H(J, L) - u_H(J, K)) \ge 0,$$

which is equivalent to

$$u_H(I, K) \cdot u_H(J, L) \ge u_H(J, K)((u_H(I, K) + u_H(J, L) - u_H(J, K)).$$

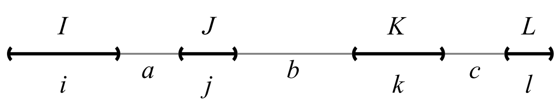
Since $u_H(I, L) = u_H(I, K) + u_H(J, L) - u_H(J, K)$, we obtain that

$$u_H(I, K) \cdot u_H(J, L) \ge u_H(J, K) \cdot u_H(I, L).$$

Therefore, to prove the lemma it is enough to show that

$$u_H(I, K) \cdot u_H(J, L) \le u_H(I, L) \cdot u_H(J, K) + u_H(I, K) \cdot u_H(J, L).$$

Let i, j, k, l denote the lengths of I, J, K, L and let a, b, c denote the distances between I and J, J and K, K and L, respectively:



Then

$$\begin{aligned}u_H(I,K)\cdot u_H(J,L)&\\ &=(i+a+j+b+k)(j+b+k+c+l)\\ &=(i+a+j+b+k)(j+b+k)+(i+a+j+b+k)(c+l)\\ &=(i+a+j+b+k)(j+b+k)+(c+l)(j+b+k)+(i+a)(c+l)\\ &<(i+a+k+b+j+c+l)(j+b+k)+(i+a+j)(k+c+l)\\ &=u_H(I,L)\cdot u_H(J,K)+u_H(I,J)\cdot u_H(K,L),\end{aligned}$$

completing the proof. □

Theorem 2.7. The metric space $(\boldsymbol{X}, h)$ is Gromov δ-hyperbolic with $\delta \le \log 2$.

Proof. Let $I, J, K, L \in \boldsymbol{X}$ be arbitrary. Lemma 2.6 implies that

$$u_H(I,J)\cdot u_H(K,L) \le 2\big[u_H(I,K)\cdot u_H(J,L) \vee u_H(I,L)\cdot u_H(J,K)\big].$$

Hence

$$\begin{aligned}h(I,J)+h(K,L) &= 2\log\frac{u_H(I,J)}{\sqrt{l(I)\cdot l(J)}}+2\log\frac{u_H(K,L)}{\sqrt{l(K)\cdot l(L)}}\\ &=2\log\frac{u_H(I,J)\cdot u_H(K,L)}{\sqrt{l(I)\cdot l(J)\cdot l(K)\cdot l(L)}}\\ &\le 2\log\frac{2\big([u_H(I,K)\cdot u_H(J,L)]\vee[u_H(I,L)\cdot u_H(J,K)]\big)}{\sqrt{l(I)\cdot l(J)\cdot l(K)\cdot l(L)}}\\ &=[h(I,K)+h(J,L)]\vee[h(I,L)+h(J,K)]+2\log 2,\end{aligned}$$

as required. □

3. The boundary at infinity

We now discuss the boundary at infinity $\partial \boldsymbol{X}$ of the Gromov hyperbolic space $(\boldsymbol{X}, h)$. Our goal is to construct a visual metric d on $\partial \boldsymbol{X}$ so that the space $(\partial \boldsymbol{X}, d)$ is isometric to the Cantor set $\mathscr{C}$ equipped with the standard Euclidean metric of the real line. Denote the distance between real numbers x and y by $|x-y|$. Recall that $\partial \boldsymbol{X}$ is the collection of equivalence classes of sequences in $\boldsymbol{X}$ converging at infinity. Fix $V=\left(\frac{1}{3},\frac{2}{3}\right)\in \boldsymbol{X}$ to be the base point. Observe that if the sequence $\{I_n\}$ converges at infinity, then $\lim_{j,k\to\infty}(I_j|I_k)_V=\infty$.

Lemma 3.1. Given $a\in\partial\boldsymbol{X}$, there exists unique $x_a\in\mathscr{C}$ with the property that

$$\lim_{n\to\infty} u_H(I_n,\{x_a\})=0 \quad \text{for each } I_n\in a.$$

Conversely, for each $x \in \mathscr{C}$ there exists $a \in \partial \boldsymbol{X}$ such that

$$\lim_{n\to\infty} u_H(J_n, \{x\}) = 0 \quad \text{for each } J_n \in a.$$

Proof. Given $\{I_n\} \in a$, we have

$$\begin{aligned}(I_j|I_k)_V &= \tfrac{1}{2}\big(h(I_j, V) + h(I_k, V) - h(I_j, I_k)\big) = \log \frac{u_H(I_j, V)\cdot u_H(I_k, V)}{l(V)\cdot u_H(I_j, I_k)}\\ &\le \log \frac{\frac{2}{3}\cdot\frac{2}{3}}{\frac{1}{3}\cdot u_H(I_j, I_k)} = \log \frac{\frac{4}{3}}{u_H(I_j, I_k)}.\end{aligned}$$

Since $\lim_{j,k\to\infty}(I_j|I_k)_V = \infty$, we obtain $\lim_{j,k\to\infty} u_H(I_j, I_k) = 0$. For each n choose some point $x_n \in I_n$.

Next, given $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that

$$|x_j - x_k| \le u_H(I_j, I_k) < \epsilon \quad \text{whenever } j, k \ge n_0.$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence in [0, 1]. Since [0, 1] is complete, it converges to some point in [0, 1], call it x_a. Now if we choose a different sequence $\{y_n\}$, where $y_n \in I_n$, then

$$|y_n - x_a| \le |y_n - x_n| + |x_n - x_a| \le u_H(I_n, I_n) + |x_n - x_a|,$$

which implies that $\{y_n\}$ also converges to x_a. Therefore, the point x_a is well defined. Finally, since

$$u_H(I_n, \{x_a\}) \le u_H(I_n, \{x_n\}) + u_H(\{x_n\}, \{x_a\}) \le u_H(I_n, I_n) + |x_n - x_a|,$$

we obtain that $\lim_{n\to\infty} u_H(I_n, \{x_a\}) = 0$, as required.

Now let $\{K_n\}$ be another sequence converging at infinity and equivalent to $\{I_n\}$, i.e.,$\{K_n\} \in a$. Then we need to show that $\lim_{n\to\infty} u_H(K_n, \{x_a\}) = 0$. Recall that the equivalence of the two sequences $\{I_n\}$ and $\{K_n\}$ means that $\lim_{n\to\infty}(I_n|K_n)_V = \infty$. The latter implies, by the same argument as above, that $\lim_{n\to\infty} u_H(I_n, K_n) = 0$. Since

$$u_H(K_n, \{x_a\}) \le u_H(K_n, I_n) + u_H(I_n, \{x_a\}),$$

we obtain that $\lim_{n\to\infty} u_H(K_n, \{x_a\}) = 0$. Thus, we have shown the existence and uniqueness of x_a.

It remains to show that $x_a \in \mathscr{C}$. Assume by contrary that $x \in [0, 1] \setminus \mathscr{C}$. Then $x_a \in I$ for some $I \in \boldsymbol{X}$. Since

$$0 < \frac{l(I)}{2} \le u_H(I_n, \{x_a\}) \quad \text{and} \quad \lim_{n\to\infty} u_H(I_n, \{x_a\}) = 0,$$

we obtain the required contradiction. Thus, $x_a \in \mathscr{C}$, completing the proof of the first part.

To prove the second part, we first show that there exists a sequence $\{J_n\}$ in $\boldsymbol{X}$ converging at infinity and such that $\lim_{n\to\infty} u_H(J_n, \{x\}) = 0$. To construct such a sequence, index $\boldsymbol{X}$ as follows: let $J_{i,j} \in \boldsymbol{X}$, where 3^{-i} is the length of the interval $J_{i,j}$ and j represents each interval in $\boldsymbol{X}$ of length 3^{-i}. Here

$$i = 1, 2, 3, \dots \quad \text{and} \quad j = 1, 2, \dots 2^{i-1}.$$

Note that $J_{1,1} = V$ and that $u_H(\{x\}, J_{1,1}) \le 2/3$ while $l(J_{1,1}) = 1/3$. We can then find J_{2,j_2} such that $u_H(\{x\}, J_{2,j_2}) \le \frac{2}{9}$ and $l(J_{2,j_2}) = \frac{1}{9}$. Continuing in this manner, for each $n \in \mathbb{N}$, there exists j_n such that

$$u_H(\{x\}, J_{n,j_n}) \le \frac{2}{3^n} \quad \text{and} \quad l(J_{n,j_n}) = 3^{-n}.$$

Put $J_n = J_{n,j_n}$. Then $\lim_{n\to\infty} u_H(J_n, \{x\}) = 0$, as required. Observe that since $u_H(J_j, J_k) \le u_H(J_j, \{x\}) + u_H(J_k, \{x\})$, we have $\lim_{j,k\to\infty} u_H(J_j, J_k) = 0$. Also, since

$$(J_j|J_k)_V = \log \frac{u_H(J_j, V) \cdot u_H(J_k, V)}{l(V) \cdot u_H(J_j, J_k)}$$

and $u_H(J_j, V) \cdot u_H(J_k, V) \le \frac{4}{9}$, we obtain that the sequence $\{J_n\}$ converges at infinity.

Finally, we let $a \in \partial\boldsymbol{X}$ to be the equivalence class of sequences converging at infinity and equivalent to $\{J_n\}$. Then it follows from the first part that

$$\lim_{n\to\infty} u_H(J_n, \{x\}) = 0 \quad \text{for each } J_n \in a,$$

completing the proof of the lemma. □

Lemma 3.1 implies that the map $f : \partial\boldsymbol{X} \to \mathscr{C}$, given by $f(a) = x_a$, is a well defined, bijective map. Now we define a metric d on $\partial\boldsymbol{X}$ by setting $d(a, b) = |x_a - x_b|$.

Lemma 3.2. The metric d is a visual metric. More precisely,

$$\tfrac{1}{3} e^{-(a|b)_V} \le d(a, b) \le 3e^{-(a|b)_V} \quad \text{for all } a, b \in \partial\boldsymbol{X}.$$

Proof. Recall that $V = \left(\frac{1}{3}, \frac{2}{3}\right)$ and

$$(a|b)_V = \inf\{\liminf_{n\to\infty} (I_n|J_n)_V : I_n \in a,\ J_n \in b\}.$$

Given $a, b \in \partial\boldsymbol{X}$, let $I_n \in a$ and $J_n \in b$ be arbitrary sequences. Then

$$(I_n|J_n)_V = \log \frac{u_H(I_n, V) \cdot u_H(I_n, V)}{l(V) \cdot u_H(I_n, J_n)}.$$

Lemma 3.1 implies that

$$\lim_{n\to\infty} u_H(I_n, \{x_a\}) = 0 \quad \text{and} \quad \lim_{n\to\infty} u_H(J_n, \{x_b\}) = 0.$$

In particular, since

$$\big|u_H(I_n, J_n) - |x_a - x_b|\big| \le u_H(I_n, \{x_a\}) + u_H(J_n, \{b_a\}),$$

we have

$$\lim_{n\to\infty} u_H(I_n, J_n) = |x_a - x_b| = d(a, b).$$

Also, since

$$\big|u_H(V, I_n) - u_H(V, \{x_a\})\big| \le u_H(I_n, \{x_a\}),$$

we have

$$\lim_{n\to\infty} u_H(V, I_n) = u_H(V, \{x_a\}) \quad \text{and} \quad \lim_{n\to\infty} u_H(V, J_n) = u_H(V, \{x_b\}).$$

Therefore, as the sequences $\{I_n\} \in a$ and $\{J_n\} \in b$ were arbitrary, we obtain

$$(a|b)_V = \log \frac{u_H(V, \{x_a\}) \cdot u_H(V, \{x_b\})}{l(V) \cdot d(a, b)}.$$

Finally, since $l(V) = \frac{1}{3}$ and since $\frac{1}{3} \le u_H(V, \{x\}) \le \frac{2}{3}$ for all $x \in [0, 1]$, we have

$$\tfrac{1}{3} d(a, b) \le \tfrac{3}{4} d(a, b) = \frac{\frac{1}{3}}{\frac{2}{3} \cdot \frac{2}{3}} d(a, b) \le e^{-(a|b)_V} \le \frac{\frac{1}{3}}{\frac{1}{3} \cdot \frac{1}{3}} d(a, b) = 3d(a, b).$$

Equivalently,

$$\tfrac{1}{3} e^{-(a|b)_V} \le d(a, b) \le 3e^{-(a|b)_V},$$

completing the proof. □

As an immediate consequence of Lemma 3.2 we obtain our main result.

Theorem 3.1. The spaces $(\partial \boldsymbol{X}, d)$ and $(\mathscr{C}, |-|)$ are isometric.

4. Further remarks

Although this particular geometric approach was successful, there is no guarantee that *any* such construction will produce the desired results. Consider, for example, the following seemingly natural distance function $\hat{h}$, defined for any $I, J \in \boldsymbol{X}$ by

$$\hat{h}(I, J) = 2 \log \frac{l(I \cup J)}{\sqrt{l(I) \cdot l(J)}}.$$

Since the distinct intervals in $\boldsymbol{X}$ are disjoint, $l(I \cup J) = l(I) + l(J)$ whenever $I \neq J$, from which it follows that $\hat{h}(I, J) \le \hat{h}(I, K) + \hat{h}(K, J)$, for all $I, J, K \in \boldsymbol{X}$. Hence the space $(\boldsymbol{X}, \hat{h})$ is a metric space. In fact, it is Gromov hyperbolic. Indeed, by setting $\mu(I, J) = l(I \cup J)$, we find that $\mu(I, I) > 0$, $\mu(I, J) = \mu(J, I)$

and $\mu(I, J) \le \mu(I, K) + \mu(K, J)$, for all $I, J, K \in \boldsymbol{X}$. By [Ibragimov 2011a, Lemma 3.7], we have

$$\mu(I, J) \cdot \mu(K, L) \le 4\big[(\mu(I, K) \cdot \mu(J, L)) \vee (\mu(I, L) \cdot \mu(J, K))\big],$$

for all $I, J, K, L \in \boldsymbol{X}$. Hence the space $(\boldsymbol{X}, \hat{h})$ is δ-hyperbolic with $\delta \le \log 4$ (see, for example, the proof of [Ibragimov 2011b, Theorem 2.1(2)]).

Next, we investigate the boundary at infinity of $(\boldsymbol{X}, \hat{h})$. Observe that

$$m(I, J) \le \hat{h}(I, J) \le m(I, J) + \log 4 \quad \text{for all } I, J \in \boldsymbol{X},$$

where

$$m(I, J) = \log \frac{\max\{l(I),\ l(J)\}}{\min\{l(I),\ l(J)\}}.$$

Fix $V = \left(\frac{1}{3}, \frac{2}{3}\right) \in \boldsymbol{X}$ to be the base point. Then we have the following estimates for the Gromov products in $(\boldsymbol{X}, \hat{h})$ with respect to V:

$$(I|J)_V = \tfrac{1}{2}[\hat{h}(I, V) + \hat{h}(J, V) - \hat{h}(I, J)] \le \tfrac{1}{2}[m(I, V) + m(J, V) - m(I, J)] + \log 4,$$

for all $I, J \in \boldsymbol{X}$ and, similarly

$$(I|J)_V = \tfrac{1}{2}[\hat{h}(I, V) + \hat{h}(J, V) - \hat{h}(I, J)] \ge \tfrac{1}{2}[m(I, V) + m(J, V) - m(I, J)] - \log 2.$$

Since

$$\tfrac{1}{2}[m(I, V) + m(J, V) - m(I, J)] = \log \frac{1}{l(I) \vee l(J)} - \log 3,$$

we find

$$\log \frac{1}{l(I) \vee l(J)} - \log 6 \le (I|J)_V \le \log \frac{1}{l(I) \vee l(J)} + \log \frac{4}{3}.$$

Hence a sequence $\{I_n\}$ in $(\boldsymbol{X}, \hat{h})$ converges at infinity if and only if

$$\max\{l(I_n),\ l(I_k)\} \to 0 \qquad \text{as } n, k \to \infty.$$

But all such sequences are equivalent and, consequently, we obtain that the boundary at infinity of $(\boldsymbol{X}, \hat{h})$ consists of a single point.

We would like to also point out that this geometric construction differs from a topological approach. Topologically, the Cantor set can be viewed as the end space of the infinite binary tree, known as the Cantor tree (Figure 2), when the latter is endowed with a path metric. The end space of such a tree is the collection of all possible infinite branches emanating from its root, and is an ultrametric space when equipped with a visual metric (see [Hughes 2004] for details). As the end space is an ultrametric space, it can not be isometric to the Cantor set, although it is homeomorphic to it.

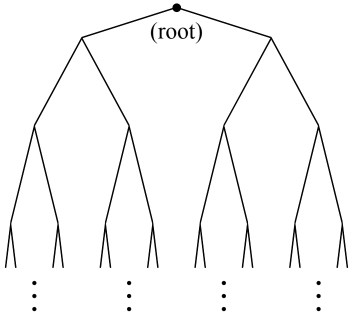


Figure 2. The Cantor tree.

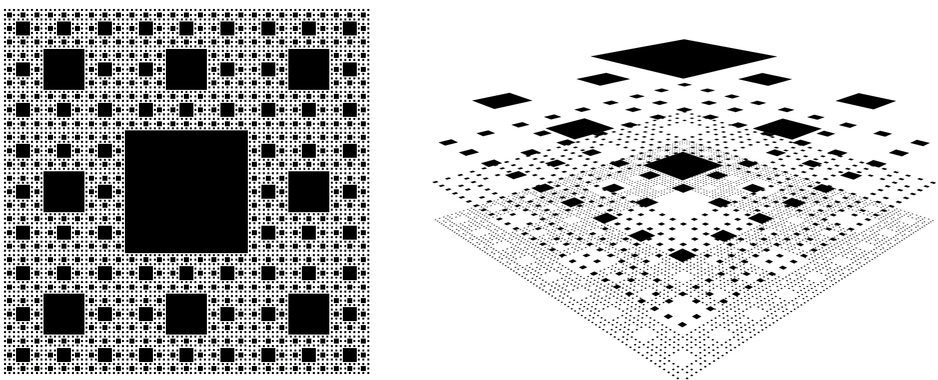


Figure 3. The standard Sierpiński carpet and some of its removed squares.

Finally, although we will not pursue it in this paper, many other fractals, such as Sierpiński carpets, can also be isometrically identified with the boundary at infinity of a similarly hyperbolized collection of removed squares (Figure 3).

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