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Some investigations on a class of nonlinear integrodifferential equations on the half-line

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We consider the second-order integrodifferential boundary value problem

$$\begin{cases} \nu(y)g(y) - \int_0^{+\infty} k(x)g(x)\,dx\,[D(y)g'(y)]' = p(y) \quad \text{for } y \geq 0, \\ g'(0) = 0, \quad g(+\infty) = 0, \end{cases}$$

arising from the kinetic theory of dusty plasmas, and we provide information on the existence and other qualitative properties of the solution that have been essential in the numerical investigation.

1. Introduction

In this paper we present an analytical study of a particular class of nonlinear integrodifferential equations given by

$$\begin{cases} \nu(y)g(y) - \int_0^{+\infty} k(x)g(x)\,dx\,[D(y)g'(y)]' = p(y) \quad \text{for } y \geq 0, \\ g'(0) = 0, \quad g(+\infty) = 0, \end{cases} \tag{1.1}$$

that is, a second-order boundary value problem on the half-line where the coefficients of the derivatives of the unknown function depend on the function itself by means of an integral over the semiaxis. This kind of problem arises from important applications such as kinetics of plasma, population dynamics and thermodynamical equilibrium [Laitinen and Tiihonen 1998; Ratynskaia et al. 2007; Ricci et al. 2001; de Angelis et al. 2006; Takeuchi et al. 2007; Cannon and Galiffa 2008; 2011; Junghanns et al. 2014]. However, the above mentioned dependence makes it difficult to approach both from an analytical and a numerical point of view.

The aim of the present investigation is a theoretical analysis of problem (1.1), which provides useful information about the solution itself and represents an essential preparation for the numerical approach to the problem [Basile et al. 2012].

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A complete analysis of some related problems has been performed in [Cannon and Galiffa 2008; 2011]. The problems taken into account in such papers are of the type

$$\alpha\left(\int_0^1 g(x)\,dx\right)g''(y) = p(y), \tag{1.2}$$

$$\alpha\left(\int_0^1 g(x)\,dx\right)g''(y) + (g(y))^{2\eta+1} = 0, \tag{1.3}$$

where $0 < y < 1$, $g(0)$ and $g(1)$ are given, η is a nonnegative integer, and α is a given function. Although these problems contain the same peculiarity as problem (1.1), there are some different characteristics (for example, the function α in (1.2) and (1.3), the nonlinearity $g^{2\eta+1}$ in (1.3), and the presence of g' and the infinite domain of integration in (1.1)). Because of these differences, the existence and uniqueness results obtained in [Cannon and Galiffa 2008; 2011] cannot be directly applied to our case, and a specific analysis is carried out in the following sections. First we set

$$q = \int_0^{+\infty} k(x)g(x)\,dx$$

and rewrite (1.1) as

$$\begin{cases} \nu(y)g(y) - q[D(y)g'(y)]' = p(y), & y \geq 0, \\ g'(0) = 0, \quad g(+\infty) = 0. \end{cases} \tag{1.4}$$

Of course the solution of (1.4) depends on q, and when we want to underscore this dependence we will use the notation $g(y,q)$ instead of $g(y)$. In Section 2 we will examine problem (1.4), taking $q > 0$ as a fixed parameter, and report results about existence, uniqueness, positiveness, regularity and boundedness of the solution. This section contains an elaboration of known results (see, for example, [Granas et al. 1978; 1986]) and will serve the investigations on the complete problem (1.1) carried out in Section 3, where we prove the existence of a nonnegative solution g that is uniformly bounded together with its derivatives. The properties of the solutions of (1.1) reported in that section are helpful in the comprehension of the problem itself and its numerical analysis.

2. Analysis of the solution of problem (1.4)

In this section we consider q fixed and positive so that our equation reduces to a classical Sturm–Liouville boundary value problem. We report some results on the existence and the uniqueness of the solution $g(y)$ of problem (1.4) along with the analysis of other useful properties such as the sign of g and the boundedness of $g(y)$, $g'(y)$ and $g''(y)$.

In order to make the aim of this paper clear, we want to specify that most of the results reported here are already known in the literature (and this will be evident throughout). However an effort has been made to fit them into the form of problem (1.4) in order to prepare the basis for the analysis of problem (1.1), which will be performed in the following section.

From now on we will consider boundary value problems of the kind (1.4) with q positive and fixed, and we make the following assumptions on the involved functions:

(1) $D \in C^1([0,+\infty))$, $\nu,\ p \in C([0,+\infty))$,

(2) $0 < D_{\text{inf}} \le D(y) \le D_{\text{sup}}$, $y \ge 0$,

(3) $0 < \sup_{y\ge 0} |D'(y)/D(y)| < +\infty$,

(4) $0 < \nu_{\text{inf}} \le \nu(y) \le \nu_{\text{sup}}$, $y \ge 0$,

(5) $0 \le p(y) \le P$, $y \ge 0$,

(6) $\int_0^{+\infty} p(y)\,dy < +\infty$,

(7) $\lim_{y\to+\infty} p(y) = 0$.

Theorem 2.1. *Assume* (1)–(7) *are satisfied. Then for any* $q > 0$, *the boundary value problem* (1.4) *has a unique nonnegative solution* $g \in C^2([0,+\infty))$.

Proof. Following a standard procedure (see, for example, the proof of Theorem 2.2 in [Granas et al. 1986]), starting from the solutions g_n of

$$\begin{cases} \nu(y)g_n(y) - q[D(y)g_n'(y)]' = p(y) & \text{for } 0 \le y \le n, n \in N, \\ g_n'(0) = 0, \quad g_n(n) = 0 \end{cases}$$

and applying inductively the Ascoli–Arzelà theorem we get the existence of a solution $g \in C^2([0,+\infty))$ of (1.4) satisfying only the first boundary condition.

Then using hypotheses (2) and (4), we set

$$m := (qD_{\text{inf}}\nu_{\text{inf}})^{1/2},$$

and consider the function [Agarwal and O'Regan 2001]

$$\omega(y) := \frac{1}{2m} e^{-\frac{m}{q}\int_0^y \frac{ds}{D(s)}} \left(\int_0^{+\infty} p(\tau) e^{-\frac{m}{q}\int_0^\tau \frac{ds}{D(s)}}\,d\tau + \int_0^y p(\tau) e^{\frac{m}{q}\int_0^\tau \frac{ds}{D(s)}}\,d\tau \right) + \frac{1}{2m} e^{\frac{m}{q}\int_0^y \frac{ds}{D(s)}} \int_y^{+\infty} p(\tau) e^{-\frac{m}{q}\int_0^\tau \frac{ds}{D(s)}}\,d\tau.$$

Hence, proceeding as in the proof of Theorem 1.8.3 in [Agarwal and O'Regan 2001], it can be proved that

$$0 \le g(y) \le \omega(y) \quad \text{and} \quad \lim_{y\to+\infty} \omega(y) = 0,$$

which ensures that the second boundary condition in (1.4) is satisfied too. Finally, the uniqueness of the solution of problem (1.4) can be proved by standard arguments showing that the homogeneous problem has only the trivial solution. □

In order to show the boundedness of $g(y)$ and its first and second derivatives, we note that hypotheses (2)–(5) allow us to define the constants

$$A := \left\| \frac{D'}{D} \right\|_\infty, \quad B := \left\| \frac{D'}{D} \right\|_\infty + \left\| \frac{\nu}{qD} \right\|_\infty \left\| \frac{p}{\nu} \right\|_\infty + \left\| \frac{p}{qD} \right\|_\infty, \tag{2.5}$$

where, as usual, $\|f\|_\infty = \sup_{y \geq 0} |f(y)|$. Moreover, we set

$$r_0 := \left\| \frac{p}{\nu} \right\|_\infty, \tag{2.6}$$

$$r_1 := \left[\frac{B}{A} (e^{2Ar_0} - 1) \right]^{1/2}, \tag{2.7}$$

$$r_2 := \left\| \frac{D'}{D} \right\|_\infty r_1 + \left\| \frac{\nu}{qD} \right\|_\infty r_0 + \left\| \frac{p}{qD} \right\|_\infty. \tag{2.8}$$

Theorem 2.2. *Under the assumptions* (1)–(7), *for any fixed* $q > 0$, *the unique solution* $g(y)$ *of* (1.4) *satisfies the bounds*

$$g(y) \leq r_0 \quad \textit{for } y \geq 0, \tag{2.9}$$

$$|g'(y)| \leq r_1 \quad \textit{for } y \geq 0, \tag{2.10}$$

$$|g''(y)| \leq r_2 \quad \textit{for } y \geq 0. \tag{2.11}$$

Proof. The boundedness of g follows by observing that $g(+\infty) = 0$ and that g cannot have a local maximum point $\bar{y} \geq 0$ such that $g(\bar{y}) > r_0$ (see, for example, [Agarwal and O'Regan 2001, page 18]). The remaining part of the proof is based on an idea developed in [Granas et al. 1978, page 71]. Therefore, we give here only a brief sketch of it. From (1.4) and hypothesis (2), we have

$$g''(y) = -\frac{D'(y)}{D(y)} g'(y) + \frac{\nu(y)}{qD(y)} g(y) - \frac{p(y)}{qD(y)} \quad \text{for } y \geq 0. \tag{2.12}$$

Then we deduce the *Bernstein growth condition*

$$|g''(y)| \leq A g'^2(y) + B \quad \text{for } y \geq 0,$$

which, multiplying both sides by $2A|g'(y)|$, gives

$$\frac{2Ag'(y)g''(y)}{Ag'^2(y) + B} \leq 2A|g'(y)| \quad \text{for } y \geq 0, \tag{2.13}$$

with A and B defined in (2.5).

Observe that any $y > 0$ such that $g'(y) \neq 0$ belongs to an interval $[a, b]$ where g' does not change sign and $g'(a) = 0$. Hence, (2.10) follows by integrating (2.13) from a to y and by recalling that $0 \leq g(y) \leq r_0$ for all $y \geq 0$.

Finally, from (2.9), (2.10), (2.12) and the hypotheses (2)–(5), we easily obtain (2.11). □

Remark. The positiveness of the solution g arises from the positiveness of the right side p in (1.4). However, if no information on the sign of p is given, we can still say that a unique solution $g(y)$ of the problem (1.4) exists, and (2.9) becomes

$$|g(y)| < r_0 \quad \text{for } y \geq 0. \tag{2.14}$$

Corollary 2.3. *Assume* (1)–(7) *hold. Then* $\lim_{y\to+\infty} g'(y) = 0$.

Proof. From Theorem 2.2 we know that g'' is bounded for any fixed value of the parameter q; hence, g' is uniformly continuous for all $y \geq 0$. From here and Barbalat's lemma [Sun 2009], we have $\lim_{y\to+\infty} g'(y) = 0$. □

We now prove other useful properties of g. Denote by $\mathrm{BC}^{(r)}[0, +\infty)$ the space of functions $f(x)$ with $f^{(j)}(x)$, $j = 0, 1, \ldots, r$, bounded and continuous on $[0, +\infty)$. Then observe that if $g(y)$ is a solution of (1.4) it satisfies (2.12); hence, the proof of the following theorem is straightforward.

Theorem 2.4. *Let* $r \in N$. *In addition to* (1)–(7), *assume* $p, v \in \mathrm{BC}^r[0, +\infty)$ *and* $D \in \mathrm{BC}^{r+1}[0, +\infty)$. *Then, for any fixed* $q > 0$, *the solution* g *of* (1.4) *is in* $\mathrm{BC}^{r+2}[0, +\infty)$. *Moreover, for any* $\bar{q} > 0$, *the derivatives* $g^{(j)}(y)$, $j = 0, \ldots, r$, *are uniformly bounded with respect to* $q \in [\bar{q}, +\infty)$.

All these properties together with the uniform continuity of g as function of q, that we are going to prove in the following section represent the basic material to deal with the difficult task of proving the existence of the solution of the original problem (1.1).

3. Existence of the solution of problem (1.1)

In this section, the focus of our attention will be the model (1.1), whose analysis requires all the results already described for (1.4). Whereas the results in Section 2 for problem (1.4) with fixed $q > 0$ are mainly obtained by elaborations of existing studies, the investigations we start in this section represent new contributions.

In Section 2, we showed that for any q fixed and positive there exists a unique solution $g(y, q)$ of (1.4). Thus, the function

$$F(q) := q - \int_0^{+\infty} k(x) g(x, q)\, dx \quad \text{for } q > 0 \tag{3.15}$$

is well defined, where the kernel k is assumed to satisfy

(8) $k \in C^2([0, +\infty))$,

(9) $\int_0^{+\infty} k(x)\, dx < +\infty$,

(10) $k(x) \geq 0$ for $x \in [0, +\infty)$.

To prove the existence of a solution of (1.1), we show that there exists a solution of the equation $F(q) = 0$. Specifically, if F is continuous and there exist two positive values a and b such that $F(a)F(b) < 0$, then by the intermediate value theorem, equation $F(q) = 0$ has at least one solution q^* and the corresponding function $g(y, q^*)$ is the solution of (1.1).

Theorem 3.1. *Assume that hypotheses* (1)–(10) *hold. Then* $F(q)$ *is uniformly continuous on* $[\bar{q}, +\infty)$, *for all* $\bar{q} > 0$.

Proof. Let us prove that $g(y, q)$ is uniformly continuous with respect to $q \geq \bar{q}$ and $y \geq 0$; that is, for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$|g(y, q_1) - g(y, q_2)| < \epsilon \quad \forall q_1, q_2 \text{ such that } |q_1 - q_2| < \delta_\epsilon \text{ and } \forall y \geq 0. \tag{3.16}$$

Let $q_1,\ q_2 \geq \bar{q}$ be arbitrarily fixed. Then the functions $g(y, q_1)$ and $g(y, q_2)$ satisfy, respectively,

$$\begin{cases} \nu(y)g(y, q_1) = q_1[D(y)g'(y, q_1)]' + p(y) & \text{for } y \geq 0, \\ g'(0, q_1) = 0, \qquad g(+\infty, q_1) = 0, \end{cases} \tag{3.17}$$

$$\begin{cases} \nu(y)g(y, q_2) = q_2[D(y)g'(y, q_2)]' + p(y) & \text{for } y \geq 0, \\ g'(0, q_2) = 0, \qquad g(+\infty, q_2) = 0, \end{cases} \tag{3.18}$$

Subtracting both sides of (3.17) and (3.18), we see that

$$e(y) = g(y, q_1) - g(y, q_2)$$

is a solution of

$$\begin{cases} \nu(y)e(y) = q_1[D(y)e'(y)]' + (q_1 - q_2)[D(y)g'(y, q_2)]' & \text{for } y \geq 0, \\ e'(0) = 0, \qquad e(+\infty) = 0, \end{cases}$$

Hence, thanks to (2.14) the inequality

$$|e(y)| \leq |q_1 - q_2| \sup_{y \geq 0} \frac{\big|[D(y)g'(y, q_2)]'\big|}{\nu(y)} \tag{3.19}$$

holds, where, by using hypotheses (2) and (4) and Theorem 2.4, it comes out that

$$\sup_{y \geq 0} \frac{\big|[D(y)g'(y, q_2)]'\big|}{\nu(y)} \leq M_0, \tag{3.20}$$

with M_0 independent of the parameters q_1, $q_2 \geq \bar{q}$. Thus, we conclude that for all $\epsilon > 0$, there exists $\delta_\epsilon = \epsilon/M_0 > 0$ such that (3.16) holds. The desired result on F is achieved by noting that from (2.9) and hypotheses (9) and (10), the improper integral $\int_0^{+\infty} k(x)g(x,q)\,dx$ is uniformly convergent with respect to q and we are allowed to take the limit under the integral. □

In the following theorem, we find an interval $[a,b]$ where the function F changes its sign. In order to provide the explicit values for a and b, the additional hypotheses (11)–(14) are required.

Theorem 3.2. *Let $F(q)$ be the function defined in* (3.15), *assume that hypotheses* (1)–(10) *hold*, *and that*

(11) $\nu \in C^2([0,+\infty))$,

(12) $|\nu'(y)| < c$ *for all* $y \geq 0$,

(13) $|k'(y)| < k_1$ *for all* $y \geq 0$,

(14) $\int_0^{+\infty} \left| \left[\left(\frac{k(y)}{\nu(y)} \right)' D(y) \right]' \right| dy = C_1 < \infty, \quad \int_0^{+\infty} \frac{k(y)}{\nu(y)} p(y)\,dy = C_2 < \infty.$

Then there exist $a, b \in (0,+\infty)$ such that $F(a)F(b) \leq 0$.

Proof. By (1.4) we have

$$\begin{aligned} F(q) &= q - \int_0^{+\infty} k(x)g(x,q)\,dx \\ &= q\left(1 - \int_0^{+\infty} \frac{k(x)}{\nu(x)} \big(D(x)g'(x,q)\big)'\,dx\right) - \int_0^{+\infty} \frac{k(x)}{\nu(x)} p(x)\,dx. \end{aligned}$$

Integrating twice by parts, by (11), (13) and Corollary 2.3 we get

$$\begin{aligned} &\int_0^{+\infty} \frac{k(x)}{\nu(x)} \big(D(x)g'(x,q)\big)'\,dx \\ &\qquad = \left[\frac{k}{\nu}\right]'(0)g(0)D(0) + \int_0^{+\infty} \left[\left(\frac{k(x)}{\nu(x)} \right)' D(x) \right]' g(x,q)\,dx. \end{aligned}$$

Thus

$$F(q) \leq q\big[1 + r_0(C_0 + C_1)\big] - C_2, \tag{3.21}$$

where r_0 and C_1, C_2 are defined, respectively, in (2.6) and (14), and

$$C_0 = \left| \left[\frac{k}{\nu}\right]'(0) \right| D(0).$$

By (3.21), $F(q) \leq 0$ for any $q \leq a$ with

$$a := \frac{C_2}{1 + r_0(C_0 + C_1)}. \tag{3.22}$$

Finally observe that from (2.9) and hypothesis (10) we have

$$F(q) = q - \int_0^{+\infty} k(x) g(x,q)\, dx \geq q - r_0 \int_0^{+\infty} k(x)\, dx, \tag{3.23}$$

which gives $F(q) \geq 0$ for any $q \geq b$ with

$$b := r_0 \int_0^{+\infty} k(x)\, dx, \tag{3.24}$$

completing the proof. □

From Theorem 3.2, by using the intermediate value theorem, we get our main result.

Theorem 3.3. *Assume that* (1)–(14) *hold. Then there exists at least one solution g of problem* (1.1) *such that*

$$a \leq \int_0^{+\infty} k(x) g(x)\, dx \leq b,$$

where a and b are defined in (3.22) *and* (3.24).

Observe that this theorem requires only the continuity of F, and it gives the existence but does not assure the uniqueness of the solution of (1.1). By exploiting the uniform continuity of F the following uniqueness result can be proved.

Theorem 3.4. *Assume that* (1)–(10) *and*

(15) $M_0 \|k\|_1 \leq 1$

hold, where M_0 is given in (3.20). *Then problem* (1.1) *has a unique solution.*

Proof. The statement follows easily from (3.19), (15) and the Banach fixed point theorem. □

Since a solution of (1.1) is a solution of (1.4), it satisfies all the properties reported in Section 2. In particular, under the hypotheses of Theorem 3.2 and from (2.6), we define

$$r1_a := \left\{ \left[1 + \left(\left\| \frac{\nu}{aD} \right\|_\infty r_0 + \left\| \frac{p}{aD} \right\|_\infty \right) \left\| \frac{D'}{D} \right\|_\infty^{-1} \right] \left(e^{2\|D'/D\|_\infty r_0} - 1 \right) \right\}^{1/2},$$

$$r2_a := \left\| \frac{D'}{D} \right\|_\infty r1_a + \left\| \frac{\nu}{aD} \right\|_\infty r_0 + \left\| \frac{p}{aD} \right\|_\infty.$$

Thus we have for $y \geq 0$

$$0 \leq g(y) \leq r_0, \quad |g'(y)| \leq r1_a, \quad |g''(y)| \leq r2_a.$$

Compared to (2.6)–(2.8) these bounds are independent of q and they turn to be useful in the numerical analysis of the problem that we carried out in [Basile et al. 2012].

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