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Homogenization of a nonsymmetric embedding-dimension-three numerical semigroup

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Let n_1, n_2, n_3 be positive integers with $\gcd(n_1, n_2, n_3) = 1$. For $S = \langle n_1, n_2, n_3\rangle$ nonsymmetric, we give an alternative description, using elementary techniques, of a minimal presentation of its homogenization $\bar{S} = \langle(1, 0), (1, n_1), (1, n_2), (1, n_3)\rangle$. As a consequence, we show that this minimal presentation is unique. We recover Bresinsky's characterization of the Cohen–Macaulay property of $\bar{S}$ and present a procedure to compute all possible catenary degrees of the elements of $\bar{S}$.

Introduction

An *affine semigroup* is a finitely generated submonoid of $\mathbb{N}^k$ for some positive integer k, where $\mathbb{N}$ stands for the set of nonnegative integers. Every affine semigroup admits a unique minimal generating system (see Exercise 6 in [Rosales and García-Sánchez 1999, Chapter 3]). Let S be an affine semigroup and let $A = \{n_1, \dots, n_e\}$ be its unique minimal generating system. Then the monoid morphism $\varphi\colon \mathbb{N}^e \to S$ induced by $e_i \mapsto n_i$ (e_i stands for the i-th row of the $e \times e$ identity matrix) is an epimorphism. Therefore S is isomorphic as a monoid to $\mathbb{N}^e/\ker\varphi$, where $\ker\varphi = \{(a, b) \in \mathbb{N}^e \times \mathbb{N}^e \mid \varphi(a) = \varphi(b)\}$ is the kernel congruence of S. A generating set for $\ker\varphi$ is known as a presentation for S, and it is a *minimal presentation* if it is minimal with respect to set inclusion (or equivalently, if it is minimal with respect to cardinality in view of [Rosales and García-Sánchez 1999, Corollary 9.5], which is finite). The monoid S is said to be uniquely presented if it has a unique minimal presentation (see [García-Sánchez and Ojeda 2010]).

The monoid morphism φ is sometimes called the factorization morphism associated to S. This is because for $s \in S$, the set $\mathsf{Z}(s) = \varphi^{-1}(s)$ corresponds with

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the *set of factorizations* of s if we identify the free monoid on A with $\mathbb{N}^e$ (the elements in A are sometimes called the atoms or irreducible elements of S). The set of factorizations of s has finitely many elements (see, for instance, [Rosales and García-Sánchez 1999, Lemma 9.1]), and corresponds to the set of nonnegative integer solutions of a system of linear Diophantine equations $xB = s$ (where B denotes the matrix whose rows are $n_1, \dots, n_e$). An element $s \in S$ is said to have *unique expression* if the cardinality of $\mathsf{Z}(s)$ is one. If every element has unique expression, the monoid is *factorial*; in this case, $\ker\varphi$ is trivial and S is isomorphic to $\mathbb{N}^e$.

For a factorization $x = (x_1, \dots, x_e) \in \mathsf{Z}(s)$, its *support* is the set

$$\operatorname{supp}(x) = \{n_i \mid x_i \neq 0\},$$

that is, it is the set of atoms involved in the factorization x. For a given factorization $x = (x_1, \dots, x_e) \in \mathsf{Z}(s)$, its *length* is $|x| = x_1 + \cdots + x_e$. The *set of lengths* of s is $\mathsf{L}(s) = \{|x| \mid x \in \mathsf{Z}(s)\}$. When the set of lengths of all the elements have cardinality one, then the monoid is said to be *half-factorial*.

A minimal presentation of S can be computed as described in [Rosales and García-Sánchez 1999, Chapter 9]. We briefly explain this procedure. For $s \in S$, define the graph G_s whose vertices are

$$\mathrm{V}(\mathrm{G}_s) = \{a \in A \mid s - a \in S\}$$

(the atoms "dividing" s), and edges

$$\mathrm{E}(\mathrm{G}_s) = \{ab \mid a, b \in A \text{ and } s - (a + b) \in S\}.$$

On $\mathsf{Z}(s)$ define the relation $\mathcal{R}$ as follows: $x \,\mathcal{R}\, y$ if there exists $x_1, \dots, x_k \in \mathsf{Z}(s)$ such that

- $x_1 = x$, $x_k = y$, and
- for every $i \in \{1, \dots, k-1\}$, $x_i \cdot x_{i+1} \neq 0$ (or equivalently, $\operatorname{supp}(x_i) \cap \operatorname{supp}(x_{i+1})$ is not empty).

Proposition 9.7 in [Rosales and García-Sánchez 1999] states that there is a bijective map between the set of $\mathcal{R}$-classes of $\mathsf{Z}(s)$ and the set of nonconnected components of G_s: for every connected component C of G_s, there exists $x \in \mathsf{Z}(s)$ whose support is contained in the vertices of C; the map sends C to the $\mathcal{R}$-class containing x. Let $R_1, \dots, R_t$ be the different $\mathcal{R}$-classes of $\mathsf{Z}(s)$, and take $x_i \in R_i$ for every i. Define $\rho_s = \{(x_1, x_2), \dots, (x_{t-1}, x_t)\}$ (actually, one can choose any set of pairs corresponding to the edges of a spanning tree of the complete graph with vertices $\{x_1, \dots, x_t\}$; if $t = 1$, then $\rho_i = \varnothing$). Then

$$\rho = \bigcup_{s \in S} \rho_s$$

is a minimal presentation of S. This union in fact ranges only over the elements $s \in S$ such that G_s is not connected. These elements are called *Betti elements* of S, and the set of Betti elements of S will be denoted by $\mathrm{Betti}(S)$.

Let k be a field. The semigroup ring associated to S is $k[S] = \bigoplus_{s \in S} kt^s$, where t is an indeterminate. Addition is performed componentwise, while the product is defined by distributivity and the rule $t^s t^{s'} = t^{s+s'}$. The monoid morphism φ has a ring analog $\bar{\varphi}\colon k[x_1, \dots, x_e] \to k[S]$, which is the morphism induced by $x_i \mapsto t^{n_i}$, $i \in \{1, \dots, e\}$, where $x_1, \dots, x_e$ are unknowns. Its kernel I_S is generated by

$$\left\{x_1^{a_1} \cdots x_e^{a_e} - x_1^{b_1} \cdots x_e^{b_e} \mid \big((a_1, \dots, a_e), (b_1, \dots, b_e)\big) \in \ker \varphi\right\}.$$

Indeed, σ is a minimal presentation if and only if

$$\left\{x_1^{a_1} \cdots x_e^{a_e} - x_1^{b_1} \cdots x_e^{b_e} \mid \big((a_1, \dots, a_e), (b_1, \dots, b_e)\big) \in \sigma\right\}$$

is a minimal generating system of I_S (see [Herzog 1970]).

Let S be a *numerical semigroup*, that is, a submonoid of $\mathbb{N}$ with finite complement in $\mathbb{N}$ (or equivalently, $\gcd(S) = 1$). It is easy to show that S admits a unique *minimal generating set* with finitely many elements, and thus every numerical semigroup is an affine semigroup. The cardinality of the minimal generating set of S is known as the *embedding dimension* of S. The largest integer not belonging to S is the *Frobenius number* of S, denoted $\mathrm{F}(S)$. The numerical semigroup S is *symmetric* if for every integer z not in S, $\mathrm{F}(S) - z \in S$.

Let S be a numerical semigroup minimally generated by $\{n_1, n_2, n_3\}$, where $n_1 < n_2 < n_3$. Define

$$c_i = \min\left\{k \in \mathbb{N} \setminus \{0\} \mid kn_i \in \langle n_j, n_k \rangle\right\},$$

where $\{i, j, k\} = \{1, 2, 3\}$. Thus there exists $r_{ij} \in \mathbb{N}$ such that

$$c_i n_i = r_{ij} n_j + r_{ik} n_k.$$

Also, we have $\mathrm{Betti}(S) = \{c_1 n_1, c_2 n_2, c_3 n_3\}$ [Rosales and García-Sánchez 2009, Example 8.23]. If S is not symmetric, then these r_{ij} are unique (see [Herzog 1970]) and

$$\sigma = \left\{\big((c_1, 0, 0), (0, r_{12}, r_{13})\big), \big((0, c_2, 0), (r_{21}, 0, r_{23})\big), \big((0, 0, c_3), (r_{31}, r_{32}, 0)\big)\right\}$$

is essentially the unique minimal presentation of S (that is, if τ is any other minimal presentation and $(a, b) \in \tau$, then either $(a, b) \in \sigma$ or $(b, a) \in \sigma$). Moreover, we have

$$\mathsf{Z}(c_1 n_1) = \{(c_1, 0, 0), (0, r_{12}, r_{13})\},$$

$$\mathsf{Z}(c_2 n_2) = \{(0, c_2, 0), (r_{21}, 0, r_{23})\},$$

$$\mathsf{Z}(c_3 n_3) = \{(0, 0, c_3), (r_{31}, r_{32}, 0)\}.$$

We also have the following relations.

- Since $c_1n_1 = r_{12}n_2 + r_{13}n_3$, we have $c_1n_1 > r_{12}n_1 + r_{13}n_1$. Hence
$$c_1 > r_{12} + r_{13},$$
and we set $\lambda = c_1 - r_{12} - r_{13}$.
- Since $c_3n_3 = r_{31}n_1 + r_{32}n_2$, we have $c_3n_3 < r_{31}n_3 + r_{32}n_3$. Hence
$$c_3 < r_{31} + r_{32},$$
and we set $\nu = r_{31} + r_{32} - c_3$.
- $c_i = r_{ji} + r_{ki}$ for every $\{i, j, k\} = \{1, 2, 3\}$ [Rosales and García-Sánchez 2009, Lemma 10.19].

Define $\bar{n}_i = (1, n_i)$, $i \in \{1, 2, 3\}$ and $\bar{n}_0 = (1, 0)$. Set $\bar{S} = \langle \bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3 \rangle$, which we call the homogenization of S since $I_{\bar{S}}$ corresponds with the homogenization of I_S (see [Cox et al. 2007, Chapter 8]; with the notation introduced there, $I_{\bar{S}} = I_S^h$). The ring $k[\bar{S}]$ is the coordinate ring of a monomial curve on $\mathbb{P}^3$.

We start with an example that illustrates Bresinsky's algorithm [1984] for computing a minimal presentation (and thus the Betti elements) of $\bar{S}$. We are going to make use of the Apéry set associated to an element in S. Let $m \in S \setminus \{0\}$. The *Apéry set* of m in S is defined as

$$\mathrm{Ap}(S, m) = \{s \in S \mid s - m \notin S\},$$

and has exactly m elements, one for each congruent class modulo m. (See [Rosales and García-Sánchez 2009, Chapter 1]; clearly, this definition applies to any monoid. We will use it later for $\bar{S}$, though in the general case this set might have infinitely many elements.)

Example 1. Let S_k be the numerical semigroup minimally generated by

$$\langle 10, 17 + 10k, 19 + 10k \rangle, \quad k \in \mathbb{N}.$$

In this setting, $n_1 = 10$, $n_2 = 17 + 10k$, and $n_3 = 19 + 10k$. This semigroup is not symmetric since its minimal generators are pairwise coprime (see [Rosales and García-Sánchez 2009, Chapter 9]).

First, we compute the values of $c_1, c_2, c_3, \lambda, \delta, \nu$ and r_{ij} for all k. Let us denote them with the superindex k. A minimal presentation for $S = S_0$ is

$$\{((4, 1, 0), (0, 0, 3)), ((3, 0, 2), (0, 4, 0)), ((7, 0, 0), (0, 3, 1))\},$$

and thus we know these values for $k = 0$. Also it is easy to check that

$$\mathrm{Ap}(S, 10) = \{0, n_2, 2n_2, 3n_2, n_3, 2n_3, n_2 + n_3, 2n_2 + n_3, n_2 + 2n_3, 2n_2 + 2n_4\}$$

(one can use the package `numericalsgps` [Delgado et al. 2013] to do these computations).

Now let $k \geq 1$.

• $c_1^k = 7 + k4$. Observe that $(7+4k)10 = 3(17+10k) + (19+10k)$, which gives us $c_1^k \leq 7+4k$. If $x10 = a(17+10k) + b(19+10k)$, with $0 \neq x, a, b \in \mathbb{N}$, then we have $x10 = a17 + b19 + (a+b)k10$. We can deduce that if $x \leq (a+b)k$, then $a17+b19+(ak+bk-x)10 = 0$, and this implies that $a = 0$, $b = 0$ and $x = 0$, and this is impossible. If $x > (a+b)k$, then $(x-(a+b)k)10 = a17+b19$. This shows that $x-(a+b)k \geq c_1^0 = 7$. Hence $x \geq 7+(a+b)k$, so it remains to show that $a+b \geq 4$. So assume to the contrary that $a+b \leq 3$. Clearly $a17+b19 = (x-(a+b)k)10$ and $x-(a+b)k \geq 0$ imply that $a17+b19 \notin \mathrm{Ap}(S, 10)$. According to the shape of $\mathrm{Ap}(S, 10)$, this forces $a = 0$ and $b = 3$. However $3 \times 19 \neq (x-3k)10$ for any k. This proves that $x \geq 7+4k$, and consequently $c_1^k = 7+k4$. Since S^k is uniquely presented, we also have $r_{12}^k = 3$ and $r_{13}^k = 1$, whence $\lambda = 3+4k$.

• $c_2^k = 4$. Note that $4(17+10k) = (3+2k)10 + 2(19+10k)$. Assume that $y(17+10k) = a10 + b(19+10k)$ for some $0 \neq y, a, b \in \mathbb{N}$. Then $y17 = (a+bk-yk)10 + b19$. If $a+bk-yk \geq 0$, this implies that $y \geq c_2^0 = 4$. For $a+bk-yk < 0$, we get $b19 = y17 + (yk-a-bk)10$. Thus $b \geq c_3^0 = 3$. It follows that $y > a/k + b > b \geq 3$, and thus $y \geq 4$. Hence $c_2^k = 4$. Also we obtain that $r_{21}^k = 3+2k$, $r_{23}^k = 2$ and $\delta = 1+2k$.

• $c_3^k = 3$. We already know that $c_3^k = r_{13}^k + r_{23}^k = 1+2 = 3$.

Hence, we have

$$(7+4k)n_1 = 3n_2 + n_3, \quad 4n_2 = (3+2k)n_1 + 2n_3, \quad 3n_3 = (4+2k)n_1 + n_2,$$

and a minimal presentation for S^k is

$$\{((7+4k, 0, 0), (0, 3, 1)), ((0, 4, 0), (3+2k, 0, 2)), ((0, 0, 3), (4+2k, 1, 0))\}.$$

If we apply Bresinsky's algorithm to these equalities, from $3n_3 = (4+2k)n_1 + n_2$ and $4n_2 = (3+2k)n_1 + 2n_3$ $(4+2k \geq 3+3k)$ we obtain $5n_3 = n_1 + 5n_2$. We now proceed with $4n_2 = (3+2k)n_1 + 2n_3$ and $5n_3 = n_1 + 5n_2$, getting

$$(5+4)n_2 = (3+2k-1)n_1 + (5+2)n_3.$$

Then we continue with $(5+4)n_2 = (3+2k-1)n_1 + (5+2)n_3$ and $5n_3 = n_1 + 5n_2$, obtaining $(2\times 5+4)n_2 = (3+2k-2)n_1 + (2\times 5+2)n_3$. By repeating these steps we obtain the general term $(5i+4)n_2 = (3+2k-i)n_1 + (5i+2)n_3$, and we must stop whenever $5i+4 \geq 3+2k-i+5i+2$, or equivalently $i \geq 2k+1$. Hence we need $2k+1$ steps to end after the initial step $5n_3 = n_1 + 5n_2$, which together with the three initial relations yield $2k+5$ relators in a minimal presentation of $\bar{S}_k$.

Observe that each of these relations come from a different element in $\bar{S}_k$, and thus we also deduce that $\#\operatorname{Betti}(\bar{S}_k) = 2k+5$ for all $k \in \mathbb{N}$.

In particular this also shows that even if the cardinality of a minimal presentation of a nonsymmetric embedding-dimension-three numerical semigroup S is always three, the cardinality of a minimal presentation of $\bar{S}$ can be arbitrarily large.

Alternatively, we can use Theorem 4 in [Cox et al. 2007, Chapter 8] to compute a presentation of $\bar{S}$ from a minimal presentation of S.

Example 2. Let $S = \langle 10, 17, 19\rangle$. A minimal presentation for S is

$$\big\{\big((4,1,0),(0,0,3)\big), \big((3,0,2),(0,4,0)\big), \big((7,0,0),(0,3,1)\big)\big\}.$$

Hence, a minimal generating system of I_S is

$$\{x_1^4x_2 - x_3^3,\, x_1^3x_3^2 - x_2^4,\, x_1^7 - x_2^3x_3\}.$$

We compute a Gröbner basis of I_S with respect to the graded lexicographic ordering and obtain

$$\{x_1^4x_2 - x_3^3,\, x_1^3x_3^2 - x_2^4,\, x_1^7 - x_2^3x_3,\, x_1x_2^5 - x_3^5,\, x_1^2x_3^7 - x_2^9,\, x_2^{14} - x_1x_3^{12}\}.$$

Hence

$$\{x_1^4x_2 - x_0^2x_3^3,\, x_1^3x_3^2 - x_0x_2^4,\, x_1^7 - x_0^3x_2^3x_3,\, x_1x_2^5 - x_0x_3^5,\, x_1^2x_3^7 - x_2^9,\, x_2^{14} - x_0x_1x_3^{12}\}$$

is a generating system for $I_{\bar{S}}$. By Herzog's correspondence,

$$\big\{\big((0,4,1,0),(2,0,0,3)\big), \big((0,3,0,2),(1,0,4,0)\big), \big((0,7,0,0),(3,0,3,1)\big), \big((0,1,5,0),(1,0,0,5)\big), \big((0,2,0,7),(0,0,9,0)\big), \big((0,0,14,0),(1,1,0,12)\big)\big\}$$

is a presentation of $\bar{S}$, though not a minimal presentation, since we saw in Example 1 that the cardinality of a minimal presentation is 5.

If we use the graded inverse lexicographic ordering instead, we obtain

$$\{x_1^4x_2 - x_3^3,\, x_1^3x_3^2 - x_2^4,\, x_1^7 - x_2^3x_3,\, x_1x_2^5 - x_3^5,\, x_1^2x_3^7 - x_2^9\},$$

which yields a minimal presentation for $\bar{S}$:

$$\big\{\big((0,4,1,0),(2,0,0,3)\big), \big((0,3,0,2),(1,0,4,0)\big), \big((0,7,0,0),(3,0,3,1)\big), \big((0,1,5,0),(1,0,0,5)\big), \big((0,2,0,7),(0,0,9,0)\big)\big\}.$$

The Gröbner basis computations in this example have been performed with Maxima (http://maxima.sourceforge.net).

In the first section we describe the Betti elements of $\bar{S}$ and its unique minimal presentation. The second section recovers a test due to Bresinsky for the Cohen–Macaulay property of $\bar{S}$. Section 3 shows how the catenary degree of $\bar{S}$ (and thus the homogeneous catenary degree of S) can be computed.

1. Determining the set of Betti elements

In this section we depict Betti($\bar{S}$), the set of elements $\bar{n} \in \bar{S}$ such that $\mathrm{G}_{\bar{n}}$ is not connected, or equivalently, $\mathsf{Z}(\bar{n})$ has more than one $\mathcal{R}$-class. Theorems 2.7 and 2.9 in [Li et al. 2012] determine Betti($\bar{S}$) just by imposing that $\gcd\{n_1, n_2, n_3\} = 1$ (notice that $\bar{S}$ is isomorphic to $\langle (n_3, 0), (n_3 - n_1, n_1), (n_3 - n - 2, n_2), (0, n_3)\rangle$ [Rosales et al. 1998, Example 1.4]). Here we present an alternative description for the case $S = \langle n_1, n_2, n_3\rangle$ is a nonsymmetric embedding-three numerical semigroup, and we obtain that in this setting $\bar{S}$ is uniquely presented.

Lemma 3. $\mathsf{Z}(c_1\bar{n}_1) = \{(0, c_1, 0, 0), (\lambda, 0, r_{12}, r_{13})\}$. *In particular, the graph* $\mathrm{G}_{c_1\bar{n}_1}$ *is not connected.*

Proof. We already know that $\{(0, c_1, 0, 0), (\lambda, 0, r_{12}, r_{13})\} \subseteq \mathsf{Z}(c_1\bar{n}_1)$. So assume that $(a_0, a_1, a_2, a_3) \in \mathsf{Z}(c_1\bar{n}_1)$. Then

$$a_0\bar{n}_0 + a_1\bar{n}_1 + a_2\bar{n}_2 + a_3\bar{n}_3 = c_1\bar{n}_1 = \lambda\bar{n}_0 + r_{12}\bar{n}_2 + r_{13}\bar{n}_3,$$

and in particular $c_1 n_1 = a_1 n_1 + a_2 n_2 + a_3 n_3$, which means that

$$(a_1, a_2, a_3) \in \mathsf{Z}(c_1 n_1) = \{(c_1, 0, 0), (0, r_{12}, r_{13})\}.$$

It follows that if $(a_1, a_2, a_3) = (c_1, 0, 0)$, then $(a_0, a_1, a_2, a_3) = (0, c_1, 0, 0)$, and if $(a_1, a_2, a_3) = (0, r_{12}, r_{13})$, we get $(a_0, a_1, a_2, a_3) = (\lambda, 0, r_{12}, r_{13})$. □

Lemma 4. *Let* $\bar{n} = a_0\bar{n}_0 + a_1\bar{n}_1 \neq c_1\bar{n}_1$, $a_0, a_1 \in \mathbb{N}$. *Then the graph* $\mathrm{G}_{\bar{n}}$ *is connected.*

Proof. Notice that if $a_1 = c_1$, then

$$a_0\bar{n}_0 + a_1\bar{n}_1 = a_0\bar{n}_0 + c_1\bar{n}_1 = (\lambda + a_0)\bar{n}_0 + r_{21}\bar{n}_2 + r_{13}\bar{n}_3.$$

As $\bar{n} \neq c_1\bar{n}_1$, $a_0 > 0$, and we get that $\mathrm{V}(\mathrm{G}_{\bar{n}}) = \{\bar{n}_0, \bar{n}_1, \bar{n}_2, \bar{n}_3\}$, and $\bar{n}_0\bar{n}_2$, $\bar{n}_0\bar{n}_3$, $\bar{n}_0\bar{n}_1 \in \mathrm{E}(\mathrm{G}_{\bar{n}})$, and thus $\mathrm{G}_{\bar{n}}$ is connected.

If $a_1 < c_1$, then $\bar{n}$ has unique expression, since if

$$a_0\bar{n}_0 + a_1\bar{n}_1 = b_0\bar{n}_0 + b_1\bar{n}_1 + b_2\bar{n}_2 + b_3\bar{n}_3$$

for some $b_0, b_1, b_2, b_3 \in \mathbb{N}$, then $a_1 n_1 = b_1 n_1 + b_2 n_2 + b_3 n_3$. By the minimality of c_1, we deduce that $b_1 \geq a_1$. But then $0 = (b_1 - a_1)n_1 + b_2 n_2 + b_3 n_3$, which leads to $a_1 = b_1$, $b_2 = b_3 = 0$. Since $\bar{n}$ has unique expression, the graph $\mathrm{G}_{\bar{n}}$ is connected.

Finally, if $a_1 > c_1$, then $a_0\bar{n}_0 + a_1\bar{n}_1 = (a_0+\lambda)\bar{n}_0 + (a_1-c_1)\bar{n}_1 + r_{21}\bar{n}_2 + r_{13}\bar{n}_3$. In this setting, the graph $\mathrm{G}_{\bar{n}}$ is K_4, the complete graph on four vertices, whence connected. □

Lemma 5. $\mathsf{Z}(\nu\bar{n}_0 + c_3\bar{n}_3) = \{(r_{31}, r_{32}, 0, 0), (\nu, 0, 0, c_3)\}$. *In particular, the graph* $\mathrm{G}_{\nu\bar{n}_0+c_3\bar{n}_3}$ *is not connected.*

Proof. The proof goes as in Lemma 3. □

Lemma 6. *For every positive integer k, we have $k\bar{n}_3 \notin \langle \bar{n}_0, \bar{n}_1, \bar{n}_2\rangle$.*

Proof. This is because $\bar{n}_3$ is not in the cone spanned by $\{\bar{n}_0, \bar{n}_1, \bar{n}_2\}$ (which is the cone spanned by $\{\bar{n}_0, \bar{n}_2\}$). □

Let

$$c_2' = \min\{k \in \mathbb{N} \setminus \{0\} \mid k\bar{n}_2 \in \langle \bar{n}_0, \bar{n}_1, \bar{n}_3\rangle\}.$$

Assume that

$$c_2'\bar{n}_2 = \gamma\bar{n}_0 + r_{21}'\bar{n}_1 + r_{23}'\bar{n}_3,$$

with $\gamma, r_{21}', r_{23}' \in \mathbb{N}$.

Lemma 7. $\mathsf{Z}(c_2'\bar{n}_2) = \{(0, 0, c_2', 0), (\gamma, r_{21}', 0, r_{23}')\}$. *In particular,* $\mathrm{G}_{c_2'\bar{n}_2}$ *is not connected. Moreover,*

(1) $r_{23}' \neq 0$,

(2) *if* $r_{21}' = 0$, *then*

$$c_2' = \frac{n_3}{\gcd\{n_2, n_3\}} \quad \textit{and} \quad r_{23}' = \frac{n_2}{\gcd\{n_2, n_3\}}.$$

Proof. Assume that $c_2'\bar{n}_2 = a_0\bar{n}_0 + a_1\bar{n}_1 + a_2\bar{n}_2 + a_3\bar{n}_3$ for some $a_0, a_1, a_2, a_3 \in \mathbb{N}$. The minimality of c_2' forces $a_2 = 0$. If $(a_0, a_1, a_3) \neq (\gamma, r_{21}', r_{23}')$, then assume without loss of generality that $a_0 \le \gamma$. Then $(\gamma - a_0)\bar{n}_0 + r_{21}'\bar{n}_1 + r_{23}'\bar{n}_3 = a_1\bar{n}_1 + a_3\bar{n}_3$. Notice that $(a_1, a_3) \not\le (r_{21}', r_{23}')$, since otherwise we would obtain

$$(\gamma - a_0)\bar{n}_0 + (r_{21}' - a_1)\bar{n}_1 + (r_{23}' - a_3)\bar{n}_3 = 0,$$

and consequently $(a_0, a_1, a_3) = (\gamma, r_{21}', r_{23}')$, a contradiction. Hence either $a_1 \ge r_{21}'$ and $a_3 < r_{23}'$, or $a_1 < r_{21}'$ and $a_3 \ge r_{23}'$. By Lemma 6, we have $a_1 \not\le r_{21}'$. This leads to $a_3 \le r_{23}'$ and $(a_1 - r_{21}')\bar{n}_1 = (\gamma - a_0)\bar{n}_0 + (r_{23}' - a_3)\bar{n}_3$. Hence $a_1 \ge c_1$, and consequently $c_2'\bar{n}_2 = (a_0+\lambda)\bar{n}_0 + (a_1 - c_1)\bar{n}_1 + r_{12}\bar{n}_2 + (a_3 + r_{13})\bar{n}_3$. But $r_{13} \neq 0$, and we have that $r_{12} \neq 0$, and this forces $c_2' > r_{12}$. Hence

$$(c_2' - r_{12})\bar{n}_2 = (a_0+\lambda)\bar{n}_0 + (a_1 - c_1)\bar{n}_1 + r_{12}\bar{n}_2 + r_{13}\bar{n}_3,$$

contradicting once more the minimality of c_2'. This shows that

$$\mathsf{Z}(c_2'\bar{n}_2) = \{(0, 0, c_2', 0), (\gamma, r_{21}', 0, r_{23}')\}.$$

Observe that $r'_{23} \neq 0$, since otherwise on the one hand $c'_2 = \gamma + r'_{21} \geq r'_{21}$, while on the other $c'_2 n_2 = r'_{21} n_1 < r'_{21} n_2$, which leads to $c'_2 < r'_{21}$, a contradiction.

If $r'_{21} = 0$, then $c'_2 n_2 = r'_{23} n_3$. Whenever $a_2 n_2 = a_3 n_3$ for some $a_2, a_3 \in \mathbb{N}$, we get $a_2 n_2 = a_3 n_3 > a_3 n_2$, whence $a_2 > a_3$. So $c'_2 n_2$ is the least multiple of n_2 that is a multiple of n_3, and we obtain $c'_2 = n_3/\gcd\{n_2, n_3\}$. □

Lemma 8. *Let $a_0, a_2 \in \mathbb{N}$, with $a_2 > c'_2$. Then $\mathrm{G}_{a_0\bar{n}_0 + a_2\bar{n}_2}$ is connected.*

Proof. Set $\bar{n} = a_0\bar{n}_0 + a_2\bar{n}_2$.

Observe that $a_0\bar{n}_0 + a_2\bar{n}_2 = (a_0 + \gamma)\bar{n}_0 + r'_{21}\bar{n}_1 + (a_2 - c'_2)\bar{n}_2 + r'_{23}\bar{n}_3$, and thus $\bar{n}_0$, $\bar{n}_2$ and $\bar{n}_3$ are in the same connected component (and so is $\bar{n}_1$ if $r'_{21} \neq 0$).

We distinguish two cases.

- If $\bar{n}_1 \notin \mathrm{V}(\mathrm{G}_{\bar{n}})$, then r'_{21} must be zero and $\mathrm{G}_{\bar{n}}$ is connected with set of vertices $\{\bar{n}_0, \bar{n}_2, \bar{n}_3\}$.
- If $\bar{n}_1 \in \mathrm{V}(\mathrm{G}_{\bar{n}})$, then there must exist $b_0, b_1, b_2, b_3 \in \mathbb{N}$, $b_1 \neq 0$, such that $\bar{n} = b_0\bar{n}_0 + b_1\bar{n}_1 + b_2\bar{n}_2 + b_3\bar{n}_3$. If $b_0 + b_2 + b_3 \neq 0$, then $\bar{n}_1$ is in the same component as $\bar{n}_0$, $\bar{n}_2$ and $\bar{n}_3$, and thus $\mathrm{G}_{\bar{n}}$ is connected. If $b_0 = b_2 = b_3 = 0$, then $b_1\bar{n}_1 = a_0\bar{n}_0 + a_2\bar{n}_2$, which is clearly different from $c_1\bar{n}_1$, and thus Lemma 4 asserts that $\mathrm{G}_{\bar{n}}$ is connected. □

Lemma 9. *The only $k \in \mathbb{N}$ for which $\mathrm{G}_{k\bar{n}_2}$ is not connected is $k = c'_2$.*

Proof. If $k < c'_2$, then by the minimality of c'_2, $k\bar{n}_2$ has unique expression, whence $\mathrm{G}_{k\bar{n}_2}$ is connected. If $k > c'_2$, then Lemma 8 with $a_0 = 0$ and $a_2 = k$ asserts that $\mathrm{G}_{k\bar{n}_2}$ is connected. Finally, for $k = c'_2$, Lemma 7 ensures that $\mathrm{G}_{k\bar{n}_2}$ is not connected. □

For the rest of the discussion we need to distinguish between $c_2 \geq r_{21} + r_{23}$ and $c_2 < r_{21} + r_{23}$.

1.1. *The case $c_2 \geq r_{21} + r_{23}$.*

Under the standing hypothesis, we have

$$c_1\bar{n}_1 = \lambda\bar{n}_0 + r_{12}\bar{n}_2 + r_{13}\bar{n}_3,$$
$$c_2\bar{n}_2 = \delta\bar{n}_0 + r_{21}\bar{n}_1 + r_{23}\bar{n}_3,$$
$$\nu\bar{n}_0 + c_3\bar{n}_3 = r_{31}\bar{n}_1 + r_{32}\bar{n}_2,$$

and all the coefficients appearing in these equations are nonzero, except eventually δ.

Lemma 10. $\mathsf{Z}(c_2\bar{n}_2) = \{(\delta, r_{21}, 0, r_{23}), (0, 0, c_2, 0)\}$. *In particular, the graph $\mathrm{G}_{c_2\bar{n}_2}$ is not connected.*

Proof. In this setting, $c'_2 = c_2$, and the proof follows from Lemma 7. □

Lemma 11. *Let $a_0, a_2 \in \mathbb{N}$, and let $\bar{n} = a_0\bar{n}_0 + a_2\bar{n}_2$. Assume that $\bar{n} \neq c_2\bar{n}_2$. Then the graph $\mathrm{G}_{\bar{n}}$ is connected.*

Proof. The proof goes as in Lemma 4, except for the case $a_2 > c_2 = c'_2$, for which we use Lemma 8. □

Lemma 12. *Let $a_0, a_3 \in \mathbb{N}$. Assume that $a_0\bar{n}_0 + a_3\bar{n}_3 \neq \nu\bar{n}_0 + c_3\bar{n}_3$. Then $\mathrm{G}_{a_0\bar{n}_0+a_3\bar{n}_3}$ is connected.*

Proof. Let $\bar{n} = a_0\bar{n}_0 + a_3\bar{n}_3$, and assume to the contrary that $\mathrm{G}_{\bar{n}}$ is not connected. Hence $\bar{n}$ admits at least another expression with support disjoint to the support of $a_0\bar{n}_0 + a_3\bar{n}_3$. This in particular means that $a_0 \neq 0$ by Lemma 6. Hence there exists $a_1, a_2 \in \mathbb{N}$ such that $a_0\bar{n}_0 + a_3\bar{n}_3 = a_1\bar{n}_1 + a_2\bar{n}_2$.

Since $a_0\bar{n}_0 + a_3\bar{n}_3 = a_1\bar{n}_1 + a_2\bar{n}_2$, we get $a_3n_3 = a_1n_1 + a_2n_2$. By the minimality of c_3, we have $a_3 \geq c_3$. If $a_3 = c_3$, since $\mathsf{Z}(c_3n_3) = \{(0, 0, c_3), (r_{31}, r_{32}, 0)\}$, we deduce $a_1 = r_{31}$ and $a_2 = r_{32}$. If follows that $a_0 = \nu$, contradicting $\bar{n} \neq \nu\bar{n}_0 + c_3\bar{n}_3$. Hence $a_3 > c_3$.

If $a_1 \geq c_1$, then $a_0\bar{n}_0 + a_3\bar{n}_3 = a_1\bar{n}_1 + a_2\bar{n}_2 = (a_1 - c_1)\bar{n}_1 + (a_2 + r_{12})\bar{n}_2 + r_{13}\bar{n}_3$. For $a_1 > c_1$ we get that $\mathrm{G}_{\bar{n}}$ is connected. If $a_1 = c_1$, then a_2 cannot be zero, since otherwise $c_1n_1 = a_3n_3$, and c_1n_1 does not admit a factorization of the form $(0, 0, a_3)$. Again, in this setting we obtain that $\mathrm{G}_{\bar{n}}$ is connected, a contradiction.

In the same way we obtain a contradiction if $a_2 \geq c_2$. Hence $a_1 < c_1$ and $a_2 < c_2$. As $a_3n_3 = a_1n_1 + a_2n_2$ and σ is the unique minimal presentation of S, it can be deduced that $(r_{31}, r_{32}) < (a_1, a_2)$ (with the usual partial order; the equality does not hold since otherwise we would obtain $c_3 = a_3$). Hence

$$a_0\bar{n}_0 + a_3\bar{n}_3 = a_1\bar{n}_1 + a_2\bar{n}_2 = \nu\bar{n}_0 + (a_1 - r_{31})\bar{n}_1 + (a_2 - r_{32})\bar{n}_2 + c_3\bar{n}_3.$$

This forces $\mathrm{G}_{\bar{n}}$ to be connected (even if $a_0 = 0$; recall that $\{n_0\}$ is not a connected component), a contradiction. □

Theorem 13. *Let S be a nonsymmetric embedding-dimension-three numerical semigroup, with $c_2 \geq r_{21} + r_{23}$. Let $\bar{n} \in \bar{S}$. The graph $\mathrm{G}_{\bar{n}}$ is not connected if and only if*

$$\bar{n} \in \{c_1\bar{n}_1, c_2\bar{n}_2, \nu\bar{n}_0 + c_3\bar{n}_3\}.$$

Proof. The proof follows from Lemmas 3 to 12. □

Notice also that this result follows as a consequence of Bresinsky's algorithm, since in this setting, as $c_2 \geq r_{21} + r_{23}$, the procedure stops in the first step, and then we only have to homogenize the relations.

Example 14. Let $S = \langle 10, 13, 19\rangle$. The unique minimal presentation for S is

$$\big\{\big((2, 0, 1), (0, 3, 0)\big), \big((7, 0, 0), (0, 1, 3)\big), \big((5, 2, 0), (0, 0, 4)\big)\big\}.$$

In this example, $c_2 = 3 = r_{21} + r_{23}$. The Betti elements of S are 39, 70 and 76, while the Betti elements of $\bar{S}$ are $(3, 39)$, $(7, 76)$ and $(7, 70)$.

Remark 15. Notice that if $c_2 \geq r_{21} + r_{23}$, then, by using Buchberger's criterion (see, for instance, [Cox et al. 2007, Chapter 3]), it is not hard to show that

$$G = \left\{x_1^{c_1} - x_2^{r_{12}}x_3^{r_{13}}, x_2^{c_2} - x_1^{r_{21}}x_3^{r_{23}}, x_1^{r_{31}}x_2^{r_{32}} - x_3^{c_3}\right\}$$

is a reduced Gröbner basis with respect to any total degree ordering. Hence, in view of Theorem 4 in [Cox et al. 2007, Chapter 8], the homogenization of G

$$\{x_1^{c_1} - x_0^{\lambda} x_2^{r_{12}} x_3^{r_{13}},\ x_2^{c_2} - x_0^{\delta} x_1^{r_{21}} x_3^{r_{23}},\ x_1^{r_{31}} x_2^{r_{32}} - x_0^{\nu} x_3^{c_3}\}$$

would contain a minimal generating set for $I_{\bar{S}}$. None of the elements in this set are redundant, since they correspond to binomials associated to factorizations of different Betti elements of $\bar{S}$ (Lemmas 3, 10 and 5). This gives an alternative proof to Theorem 13 without using Lemmas 4, 6, 9, 8, 11 and 12.

Since all the elements in $\mathrm{Betti}(S)$ have two factorizations, we get the following as a consequence of [García-Sánchez and Ojeda 2010, Corollary 5].

Corollary 16. *Let S be a nonsymmetric embedding-dimension-three numerical semigroup, with $c_2 \geq r_{21} + r_{23}$. Then*

$$\Big\{\big((0, c_1, 0, 0), (\lambda, 0, r_{12}, r_{13})\big), \big((0, 0, c_2, 0), (\delta, r_{21}, 0, r_{31})\big), \big((0, 0, 0, c_3), (\nu, r_{31}, r_{32}, 0)\big)\Big\}$$

is the unique minimal presentation of $\bar{S}$.

1.2. ***The case $c_2 < r_{21} + r_{23}$.*** Recall that in this setting we have

$$\begin{aligned} c_1\bar{n}_1 &= \lambda\bar{n}_0 + r_{12}\bar{n}_2 + r_{13}\bar{n}_3, \\ \delta\bar{n}_0 + c_2\bar{n}_2 &= r_{21}\bar{n}_1 + r_{23}\bar{n}_3, \\ \nu\bar{n}_0 + c_3\bar{n}_3 &= r_{31}\bar{n}_1 + r_{32}\bar{n}_2. \end{aligned}$$

Lemma 17. $\mathsf{Z}(\delta n_0 + c_2\bar{n}_2) = \{(0, r_{21}, 0, r_{23}), (\delta, 0, c_2, 0)\}$. *In particular, the graph $\mathrm{G}_{\delta\bar{n}_0 + c_2\bar{n}_2}$ is not connected.*

Proof. Similar to the proof of Lemma 3. □

Remark 18. Observe that

$$d_2\bar{n}_2 = d_1\bar{n}_1 + d_3\bar{n}_3,$$

with $d_i = (n_j - n_k)/\gcd\{n_3 - n_2, n_2 - n_1\}$, $\{i, k < j\} = \{1, 2, 3\}$. Notice that the set of rational solutions of $\bar{n}_1 x_1 - \bar{n}_2 x_2 + \bar{n}_3 x_3 = 0$ is spanned by (d_1, d_2, d_3). And since $\gcd(d_1, d_2, d_3) = 1$, every integer solution (x_1, x_2, x_2) is a multiple of (d_1, d_2, d_3).

Observe also that

$$\frac{n_3}{\gcd\{n_2, n_3\}} n_2 = \frac{n_2}{\gcd\{n_2, n_3\}} n_3,$$

and thus

$$\frac{n_3}{\gcd\{n_2, n_3\}} \bar{n}_2 = \eta\bar{n}_0 + \frac{n_2}{\gcd\{n_2, n_3\}} \bar{n}_3$$

for some positive integer η. Hence

$$c_2' \leq \min\left\{d_2, \frac{n_3}{\gcd\{n_2, n_3\}}\right\}.$$

Lemma 19. *Let $a_0, a_1, a_2, a_3 \in \mathbb{N}$. Assume that*

$$\bar{n} = a_0\bar{n}_0 + a_2\bar{n}_2 = a_1\bar{n}_1 + a_3\bar{n}_3 \notin \{c_2'\bar{n}_2, \delta\bar{n}_0 + c_2\bar{n}_2\}$$

yields a nonconnected graph. Then (a_1, a_2, a_3) belongs to

$$C_2 = \left\{ (x_1, x_2, x_3) \in \mathbb{N}^3 \;\middle|\; \begin{array}{c} n_1x_1 - n_2x_2 + n_3x_3 = 0, \\ x_2 < x_1 + x_3 < x_2 + \delta, \\ 0 < x_1 < r_{21},\; c_3 \leq x_3, \\ c_2 < x_2 < c_2' \end{array} \right\}.$$

Moreover,

(1) $(a_1, a_3) \in M_2 := \text{Minimals}_{\leq}\{(x_1, x_3) \mid (x_1, x_2, x_3) \in C_2$ *for some* $x_2 \in \mathbb{N}\}$,

(2) $\mathsf{Z}(\bar{n}) = \{(a_0, 0, a_2, 0), (0, a_1, 0, a_3)\}$.

Proof. If $a_0 = 0$, we know by Lemma 9 that the only nonconnected graph $\mathrm{G}_{a_2\bar{n}_2}$ is $\mathrm{G}_{c_2'\bar{n}_2}$. Hence $a_0 \neq 0$.

From

$$a_0\bar{n}_0 + a_2\bar{n}_2 = a_1\bar{n}_1 + a_3\bar{n}_3,$$

we deduce

$$a_0 + a_2 = a_1 + a_3 \quad \text{and} \quad a_2n_2 = a_1n_1 + a_3n_3.$$

The minimality of c_2 yields $a_2 \geq c_2$. If $c_2 = a_2$, then we get $\delta = a_0$, which is not possible by hypothesis. Hence (a_1, a_2, a_3) is a solution of

$$n_1x_1 - n_2x_2 + n_3x_3 = 0, \quad c_2 < x_2 < x_1 + x_3.$$

If $a_1 \geq c_1$, then $a_0\bar{n}_0 + a_2\bar{n}_2 = a_1\bar{n}_1 + a_3\bar{n}_3 = (a_1 - c_1)\bar{n}_1 + r_{12}\bar{n}_2 + (a_3 + r_{13})\bar{n}_3$. If $a_1 > c_1$, we easily derive that $\mathrm{G}_{\bar{n}}$ is connected. If $a_1 = c_1$, then a_3 cannot be zero, since otherwise $c_1n_1 = a_2n_2$, contradicting that $\mathsf{Z}(c_1n_1) = \{(c_1, 0, 0), (r_{12}, 0, r_{13})\}$. Again, the connectedness of $\mathrm{G}_{\bar{n}}$ follows easily. Hence $a_1 < c_1$.

If $a_1 = 0$, then $a_0 + a_2 = a_3$, and this implies that $a_2 \leq a_3$. However, we have $a_2n_2 = a_3n_3 > a_3n_2$, which yields $a_2 > a_3$, a contradiction.

Assume that $a_3 < c_3$. As $a_2n_2 = a_1n_1 + a_3n_3$, and σ is a minimal presentation for S, we can deduce that $r_{21} \leq a_1$ and $r_{23} \leq a_3$. Note that both equalities cannot hold, since $a_2 \neq c_2$. Hence

$$a_0\bar{n}_0 + a_2\bar{n}_2 = a_1\bar{n}_1 + a_3\bar{n}_3 = (a_1 - r_{21})\bar{n}_1 + (a_3 - r_{23})\bar{n}_3 + \delta a_0 + c_2\bar{n}_2,$$

which leads once more to the connectedness of $\mathrm{G}_{\bar{n}}$. This proves that $a_3 \geq c_3$. As $c_3 = r_{13} + r_{23} > r_{23}$, if $a_1 \geq r_{21}$, then we have

$$a_0\bar{n}_0 + a_2\bar{n}_2 = a_1\bar{n}_1 + a_3\bar{n}_3 = (a_1 - r_{21})\bar{n}_1 + (a_3 - r_{23})\bar{n}_3 + \delta\bar{n}_0 + c_2\bar{n}_2,$$

obtaining once more a connected graph. This shows that $a_1 < r_{21}$.

Hence for the rest of the proof we may assume that $a_0a_1a_2a_3 \neq 0$.

We now focus on (2), which will be used later. If

$$(a'_0, a'_1, a'_2, a'_3) \in \mathsf{Z}(\bar{n}) \setminus \{(a_0, 0, a_2, 0), (0, a_1, 0, a_3)\},$$

then as $\mathrm{G}_{\bar{n}}$ is not connected and $a_0a_1a_2a_3 \neq 0$, either $a'_0 = a'_2 = 0$ or $a'_1 = a'_3 = 0$.

- If $a'_0 = a'_2 = 0$, then $a_0\bar{n}_0 + a_2\bar{n}_2 = a_1\bar{n}_1 + a_3\bar{n}_3 = a'_1\bar{n}'_1 + a'_3\bar{n}'_3$. This in particular means that $(a_1 - a'_1)\bar{n}_1 + (a_3 - a'_3)\bar{n}_3 = 0$. Since $\bar{n}_1$ and $\bar{n}_3$ are linearly independent, $a_1 - a'_1 = 0$ and $a_3 - a'_3 = 0$, that is, $a_1 = a'_1$ and $a_3 = a'_3$, a contradiction.
- The case $a'_1 = a'_3 = 0$ follows analogously, since $\bar{n}_0$ and $\bar{n}_2$ are also linearly independent.

Now, if $a_0 \geq \delta$, as $a_2 > c_2$, we get

$$a_0\bar{n}_0 + a_2\bar{n}_2 = (a_0 - \delta)\bar{n}_0 + (a_2 - c_2)\bar{n}_2 + r_{21}\bar{n}_1 + r_{23}\bar{n}_3 = a_1\bar{n}_1 + a_3\bar{n}_3,$$

obtaining again three different factorizations of $\bar{n}$, a contradiction. Hence $a_0 < \delta$. This also implies that $a_1 + a_3 = a_0 + a_2 < \delta + a_2$.

If $a_2 \geq c'_2$, then

$$a_0\bar{n}_0 + a_2\bar{n}_2 = a_1\bar{n}_1 + a_3\bar{n}_3 = (\gamma + a_0)\bar{n}_0 + r'_{21}\bar{n}_1 + (a_2 - c'_2)\bar{n}_2 + r'_{23}\bar{n}_3,$$

which yields three factorizations of $\bar{n}$, in contradiction with (2).

To prove (1), assume there exists $(b_1, b_2, b_3) \in C_2$ such that $(b_1, b_3) \lneq (a_1, a_2)$. Then $a_0\bar{n}_0 + a_2\bar{n}_2 = a_1\bar{n}_1 + a_3\bar{n}_3 = (a_1 - b_1)\bar{n}_1 + (a_3 - b_3)\bar{n}_3 + a_0\bar{n}_0 + a_2\bar{n}_2$. Thus we get three different expressions of $\bar{n}$, a contradiction. □

Lemma 20. *Let $(a_1, a_3) \in M_2$, and let $\bar{n} = a_1\bar{n}_1 + a_3\bar{n}_3$. Then $\mathrm{G}_{\bar{n}}$ is not connected.*

Proof. As $(a_1, a_3) \in M_2$, there exists positive integers a_0 and a_2 such that $\bar{n} = a_0\bar{n}_0 + a_2\bar{n}_2$, $a_0 < \delta$ and $c_2 < a_2 < c'_2$. Assume to the contrary that $\mathrm{G}_{\bar{n}}$ is connected. Then there exists $(b_0, b_1, b_2, b_3) \in \mathsf{Z}(\bar{n}) \setminus \{(a_0, 0, a_2, 0), (0, a_1, 0, a_3)\}$.

From $a_0\bar{n}_0 + a_2\bar{n}_2 = b_0\bar{n}_0 + b_1\bar{n}_1 + b_2\bar{n}_2 + b_3\bar{n}_3$ we deduce the following.

- As $a_2 < c'_2$, we have $b_0 < a_0$, and consequently $b_0 < \delta$.
- Since $a_0 \neq 0$, we have $b_2 < a_2$. We obtain $b_2 < c'_2$.

Now, from $a_1\bar{n}_1 + a_3\bar{n}_3 = b_0\bar{n}_0 + b_1\bar{n}_1 + b_2\bar{n}_2 + b_3\bar{n}_3$ and Lemma 6, we deduce that $a_1 > b_1$. If $a_3 \geq b_3$, then $(a_1 - b_1)\bar{n}_1 + (a_3 - b_3)\bar{n}_3 = b_0\bar{n}_0 + b_2\bar{n}_2$. Notice that

$0 < a_1 - b_1 \le a_1 < r_{21}$, and that $b_2 \ge c_2$ because $b_2 n_2 = (a_1 - b_1)n_1 + (a_3 - b_3)n_3$, and if $b_2 = c_2$ this forces $a_1 - b_1 = r_{21}$, which is impossible. Hence $c_2 < b_2 < c_2'$. Arguing as in the proof of Lemma 19 we get that $c_3 \le a_2 - b_3$. This means that $(a_1 - b_1, b_2, a_3 - b_3) \in C_2$, but this contradicts $(a_1, b_1) \in M_2$.

Thus $a_3 > b_3$ and $(a_1 - b_1)\bar{n}_1 = b_0\bar{n}_0 + b_2\bar{n}_2 + (b_3 - a_3)\bar{n}_3$. But this contradicts the minimality of c_1, because

$$a_1 - b_1 \le a_1 < r_{21} < c_1 \quad \text{and} \quad (a_1 - b_1)n_1 = b_2 n_2 + (b_3 - a_3)n_3. \qquad \square$$

Lemma 21. *Let $a_0, a_1, a_2, a_3 \in \mathbb{N}$. Assume that*

$$\bar{n} = a_0\bar{n}_0 + a_3\bar{n}_3 = a_1\bar{n}_1 + a_2\bar{n}_2 \notin \{c_2'\bar{n}_2, \nu\bar{n}_0 + c_3\bar{n}_3\}$$

yields a nonconnected graph. Then (a_1, a_2, a_3) belongs to

$$C_3 = \left\{ (x_1, x_2, x_3) \in \mathbb{N}^3 \;\middle|\; \begin{array}{c} n_1x_1 + n_2x_2 - n_3x_3 = 0, \\ x_3 < x_1 + x_2 < x_3 + \nu, \\ 0 < x_1 < r_{31},\ c_3 < x_3, \\ c_2 \le x_2 < c_2' \end{array} \right\}.$$

Moreover,

(1) $(a_1, a_2) \in M_3 := \text{Minimals}_{\le}\{(x_1, x_2) \mid (x_1, x_2, x_3) \in C_3$ *for some* $x_3 \in \mathbb{N}\}$,

(2) $\mathsf{Z}(\bar{n}) = \{(a_0, 0, 0, a_3), (0, a_1, a_2, 0)\}$.

Proof. From Lemma 6, we know that $a_0 \neq 0$. Assume that $a_1 = 0$. Then $a_2\bar{n}_2$ is a nonconnected graph, which according to Lemma 9 means that $a_2 = c_2'$, which is excluded in the hypothesis. Hence a_1 is also not zero. The rest of the proof goes as in Lemma 19. $\square$

Lemma 22. *Let $(a_1, a_2) \in M_3$, and let $\bar{n} = a_1\bar{n}_2 + a_2\bar{n}_2$. Then $\mathrm{G}_{\bar{n}}$ is not connected.*

Proof. According to Lemma 21, there exists positive integers a_0 and a_3 such that $\bar{n} = a_0\bar{n}_0 + a_3\bar{n}_3$, $a_0 < \nu$ and $c_3 < a_3$. We argue as in Lemma 20. Assume that there exists an expression $b_0\bar{n}_0 + b_1\bar{n}_1 \bar{+} b_2\bar{n}_2 + b_3\bar{n}_3$ other than $a_0\bar{n}_0 + a_3\bar{n}_3$ and $a_1\bar{n}_1 + a_2\bar{n}_2$. Then $a_1\bar{n}_1 + a_2\bar{n}_2 = b_0\bar{n}_0 + b_1\bar{n}_1 + b_2\bar{n}_2 + b_3\bar{n}_3$. From $a_1 < c_1$, we deduce that $a_2 > b_2$, and from $a_2 < c_2'$ that $a_1 > b_1$. Thus

$$0 \neq (a_1 - b_1)\bar{n}_1 + (a_2 - b_2)\bar{n}_2 = b_0\bar{n}_0 + b_3\bar{n}_3.$$

Hence $b_3 n_3 = (a_1 - b_1)n_1 + (a_2 - b_2)n_2$, which implies that $b_3 \ge c_3$, and if $c_3 = b_3$ we would get $a_1 - b_1 = r_{31}$, contradicting that $a_1 < r_{31}$. Therefore $b_3 > c_3$. Also $a_1 - b_1 < r_{31}$, and from this it is not difficult to deduce that $a_2 - b_2$ must be greater than or equal to c_2, since otherwise there will be no way by using the relations in σ to get from $(a_1 - b_1, a_2 - b_2, 0)$ to $(0, 0, b_3)$. Gathering all this information, we obtain that $(a_1 - b_1, a_2 - b_2, b_3) \in C_3$ and $(a_1 - b_1, a_2 - b_2) < (a_1, a_2)$, contradicting $(a_1, a_2) \in M_3$. $\square$

Example 23. Let $S = \langle 11, 18, 21\rangle$. A minimal presentation for S is

$$\{((3,0,1),(0,3,0)),\ ((6,1,0),(0,0,4)),\ ((9,0,0),(0,2,3))\}.$$

The Betti elements of S are $\{54, 84, 99\}$, while those of $\bar{S}$ are

$$\{(4,54),(7,84),(9,99),(7,126),(7,105)\}.$$

In this example C_2 is empty, and $C_3 = \{(3,4,5),(3,8,7),(3,25,23)\}$. The minimality condition imposed to the first two coordinates reduces this set to $\{(3,4,5)\}$.

A minimal presentation for $\bar{S}$ is

$$\{((0,3,0,1),(1,0,3,0)),\ ((0,6,1,0),(3,0,0,4)),\ ((0,9,0,0),(4,0,2,3)),\\ ((1,0,0,6),(0,0,7,0)),\ ((0,3,4,0),(2,0,0,5))\}.$$

Notice that this semigroup is no longer generic (in all relations all atoms occur), but it is uniquely presented. The set of integers belonging to C_2 and C_3 can be computed by using [Wolfram Alpha 2013] by simply typing in the search field "find integer solutions to" and then the set of inequalities separated by "and."

Theorem 24. *Let S be a nonsymmetric embedding-dimension-three numerical semigroup, with $c_2 < r_{21} + r_{23}$. Then*

$$\mathrm{Betti}(\bar{S}) = \{c_1\bar{n}_1,\ \delta\bar{n}_0 + c_2\bar{n}_2,\ c_2'\bar{n}_2,\ \nu\bar{n}_0 + c_3\bar{n}_3\}\\ \cup \{a_1\bar{n}_1 + a_3\bar{n}_3 \mid (a_1,a_3)\in M_2\} \cup \{a_1\bar{n}_1 + a_2\bar{n}_2 \mid (a_1,a_2)\in M_3\}.$$

Moreover, $\bar{S}$ is uniquely presented.

Proof. If $\bar{n} \in \mathrm{Betti}(\bar{S})$, then at least $\mathsf{Z}(\bar{n})$ has two $\mathcal{R}$-classes. Thus in one of them there are at most two atoms of $\bar{S}$, and neither $\bar{n}_0$ nor $\bar{n}_3$ (Lemma 6) are alone. So we have that the set of atoms involved in one of the $\mathcal{R}$-classes is any of these sets: $\{n_0, n_1\}$, $\{n_0, n_2\}$, $\{n_0, n_3\}$, $\{n_1\}$ and $\{n_2\}$. Lemmas 3 to 9, 17, 19, 20, 21 and 22 cover all possibilities. Moreover, in all cases $\#\mathsf{Z}(\bar{n}) = 2$, and thus according to [García-Sánchez and Ojeda 2010, Corollary 5], $\bar{S}$ is uniquely presented. □

Example 25. Recall that a minimal presentation for $S = \langle 10, 17, 19\rangle$ is

$$\{((4,1,0),(0,0,3)),\ ((3,0,2),(0,4,0)),\ ((7,0,0),(0,3,1))\}$$

(Example 2). Moreover, $C_2 = \varnothing$ and $C_3 = \{(1,5,5)\}$. Thus the set of Betti elements of $\bar{S}$ is

$$\{7\bar{n}_1 = (7,70),\ \bar{n}_0 + 4\bar{n}_2 = (5,68),\ 2\bar{n}_0 + 3\bar{n}_3 = (5,57),\\ 9\bar{n}_2 = (9,153),\ \bar{n}_0 + 5\bar{n}_3 = (6,95)\}.$$

Example 26. Let $S = \langle 10, 27, 29\rangle$. In view of Example 1 with $k = 1$, a minimal presentation for S is

$$\{((6,1,0),(0,0,3)),\ ((5,0,2),(0,4,0)),\ ((11,0,0),(0,3,1))\}.$$

Here, $C_2 = \{(3,14,12),(4,9,7)\}$ and $C_3 = \{(1,5,5)\}$. Thus

$$\begin{aligned}\mathrm{Betti}(\bar S) = \{11\bar n_1 = (11,110),\ 3\bar n_0 + 4\bar n_2 = (7,108),\\ 4\bar n_0 + 3\bar n_3 = (7,87),\ 19\bar n_2 = (19,513),\\ \bar n_0 + 14\bar n_2 = (15,378),\ 2\bar n_0 + 9\bar n_2 = (11,243)\}.\end{aligned}$$

Remark 27. The uniqueness of the minimal presentation can be derived in a different way. As a consequence of Bresinsky's algorithm the cardinality of $\mathrm{Betti}(\bar S)$ equals the cardinality of a minimal presentation for $\bar S$ (this is also stated in [Li et al. 2012, Lemma 2.2] without using Bresinsky's procedure; there are no two relations in a minimal presentation corresponding to the same element in $\bar S$). Thus for every $b \in \mathrm{Betti}(\bar S)$, $\mathsf{Z}(b)$ has two $\mathcal{R}$-classes. This does not show that the minimal presentation is unique, because some of these $\mathcal{R}$-classes could have more than one element (see, for instance, [Li et al. 2012, Example 2.5]). However it can be shown that in our setting $\pm(b - b') \notin \bar S$ for every $b, b' \in \mathrm{Betti}(\bar S)$, that is to say, all Betti elements of $\bar S$ are Betti-minimal. Hence in view of [García-Sánchez and Ojeda 2010, Proposition 3] every $\mathcal{R}$-class of $\mathsf{Z}(b)$ for every $b \in \mathrm{Betti}(S)$ is a singleton (see also [Charalambous et al. 2007, Theorem 3.4]).

2. The Cohen–Macaulay property

We say that an affine semigroup is Cohen–Macaulay if the semigroup ring $k[S]$ is Cohen–Macaulay. The corollary on page 127 of [Bresinsky 1984] gives a characterization of the Cohen–Macaulay property. Also Remark 2.17 in [Li et al. 2012] offers another characterization of the Cohen–Macaulay property. We will use the test proposed in [Rosales et al. 1998] for affine subsemigroups of $\mathbb{N}^2$ to give an alternative proof of Bresinsky's characterization in our scope (S is not symmetric).

Observe that the (rational) cone spanned by $\{\bar n_0, \bar n_3\}$ equals the cone spanned by $\bar S$. Thus a_1 in [Rosales et al. 1998, Section 1] is n_3. Also μ in [Rosales et al. 1998, Lemma 1.1.3] corresponds with $\mu(s) = \min \mathsf{L}(s)$ for every $s \in S$.

Let G be a reduced Gröbner basis of I_S with respect to any total degree ordering and $(a_1, a_2, a_3) \in \mathsf{Z}(s)$ (observe that G consists also of binomial ideals). For a polynomial $f \in k[x_1, x_2, x_3]$, denote by $\mathrm{NF}_G(f)$ the remainder of the division of f by G. It follows that for $s \in S$ and $(a_1, a_2, a_3) \in \mathsf{Z}(s)$, $\mathrm{NF}_G(x_1^{a_1} x_2^{a_2} x_3^{a_3})$ is a monomial, and if

$$\mathrm{NF}_G(x_1^{a_1} x_2^{a_2} x_3^{a_3}) = x_1^{b_1} x_2^{b_2} x_3^{b_3},$$

then $\mu(s) = b_1 + b_2 + b_3$, the total degree of $\mathrm{NF}_G(x_1^{a_1} x_2^{a_2} x_3^{a_3})$.

Proposition 28. *Let S be a nonsymmetric embedding-dimension-three numerical semigroup. Then $\bar{S}$ is Cohen–Macaulay if and only if $c_2 \geq r_{21} + r_{23}$.*

Proof. Notice that if $c_2 \geq r_{21} + r_{23}$, then by Remark 15,

$$G = \left\{x_1^{c_1} - x_2^{r_{12}} x_3^{r_{13}},\, x_2^{x_2} - x_1^{r_{21}} x_3^{r_{23}},\, x_1^{r_{31}} x_2^{r_{32}} - x_3^{c_3}\right\}$$

is a reduced Gröbner basis with respect to any total degree ordering. Let $B = \mathrm{Ap}(\bar{S}, \bar{n}_0) \cap \mathrm{Ap}(\bar{S}, \bar{n}_3)$. We are going to show that $B = \{(\mu(s), s) \mid s \in \mathrm{Ap}(S, n_3)\}$ and thus by [Rosales et al. 1998, Theorem 1.2], $\bar{S}$ is Cohen–Macaulay (in particular the cardinality of B is n_3 and the Cohen–Macaulayness of $\bar{S}$ also follows from [Li et al. 2012, Theorem 1.2]). It is easy to see that if $(n, s) \in \mathrm{Ap}(\bar{S}, \bar{n}_0)$, then $n = \mu(s)$, and thus the inclusion $\{(\mu(s), s) \mid s \in \mathrm{Ap}(S, n_3)\} \subseteq B$ is clear. Now assume that there exists $(\mu(s), s) \in B$ with $s \notin \mathrm{Ap}(S, n_3)$. Then $s = n_3 + t$ for some $t \in S$ and $(\mu(s) - 1, t) \notin \bar{S}$. It is easy to see that this can only occur if and only if $\mu(t) > \mu(s) - 1$. Let $(b_1, b_2, b_3) \in \mathsf{Z}(t)$ be such that $\mathrm{NF}_G(x_1^{b_1} x_2^{b_2} x_3^{b_3}) = x_1^{b_1} x_2^{b_2} x_3^{b_3}$. Hence

$$\mu(t) = b_1 + b_2 + b_3 \quad \text{and} \quad (b_1, b_2, b_3 + 1) \in \mathsf{Z}(s).$$

As $\mu(t) = b_1 + b_2 + b_3 > \mu(s) - 1$, this means that $\mu(s) < b_1 + b_2 + b_3 + 1$, and consequently

$$\mathrm{NF}_G(x_1^{b_1} x_2^{b_2} x_3^{b_3+1}) \neq x_1^{b_1} x_2^{b_2} x_3^{b_3+1}.$$

This implies that either $x_1^{c_1}$ or $x_2^{c_2}$ or $x_1^{r_{31}} x_2^{r_{32}}$ divide $x_1^{b_1} x_2^{b_2} x_3^{b_3+1}$. As x_3 does not occur in $\{x_1^{c_1}, x_2^{c_2}, x_1^{r_{31}} x_2^{r_{32}}\}$, this means that either $x_1^{c_1}$ or $x_2^{c_2}$ or $x_1^{r_{31}} x_2^{r_{32}}$ divide $x_1^{b_1} x_2^{b_2} x_3^{b_3}$, yielding $\mathrm{NF}_G(x_1^{b_1} x_2^{b_2} x_3^{b_3}) \neq x_1^{b_1} x_2^{b_2} x_3^{b_3}$, a contradiction.

If $c_2 < r_{21} + r_{23}$, then $\mu(c_2 n_2) = c_2$ (recall that $\mathsf{Z}(c_2 n_2) = \{(0, c_2, 0), (r_{21}, 0, r_{23})\}$). Notice that $r_{21} n_1$ has unique expression, and consequently $r_{21} n_1 \in \mathrm{Ap}(S, n_3)$. Hence

$$c_2 = \mu(c_2 n_2) = \mu(r_{21} n_1 + r_{23} n_3) \quad \text{and} \quad \mu(r_{21} n_1) + r_{23} \mu(n_3) = r_{21} + r_{23}.$$

Since $c_2 \neq r_{21} + r_{23}$, Proposition 1.6 in [Rosales et al. 1998] states that $\bar{S}$ cannot be Cohen–Macaulay. □

Corollary 29. *Let S be a nonsymmetric embedding-dimension-three numerical semigroup. Then $\bar{S}$ is Cohen–Macaulay if and only if the cardinality of the minimal presentation of S coincides with the cardinality of the minimal presentation of $\bar{S}$.*

3. The catenary degree of $\bar{S}$

Let $S \subset \mathbb{N}^k$ be an affine semigroup. Let $s \in S$, and let

$$a = (a_1, \ldots, a_k),\, b = (b_1, \ldots, b_k) \in \mathsf{Z}(s).$$

The *distance* between a and b is $\mathsf{d}(a, b) = \max\{|a - (a \wedge b)|, |b - (a \wedge b)|\}$, where $a \wedge b = (\min(a_1, b_1), \ldots, \min(a_k, b_k))$, the common part to the factorizations a and b. For $N \in \mathbb{N}$, an N*-chain* of factorizations joining a and b is a sequence $a_1, \ldots, a_t \in \mathsf{Z}(s)$ such that $\mathsf{d}(a_i, a_{i+1}) \leq N$ for all $i \in \{1, \ldots, t-1\}$. The *catenary degree* of s, $\mathsf{c}(s)$, is the minimum N such for any $a, b \in \mathsf{Z}(s)$, there exists an N-chain of factorizations joining a and b. The catenary degree of S is defined as

$$\mathsf{c}(S) = \sup_{s \in S} \mathsf{c}(s).$$

As a consequence of [Chapman et al. 2006, Section 3], this supremum is a maximum and indeed

$$\mathsf{c}(S) = \max_{s \in \mathrm{Betti}(S)} \mathsf{c}(s).$$

If S is a numerical semigroup, as $\bar{S}$ is half-factorial, [García-Sánchez et al. 2013, Theorem 2.3] states that for every $s \in \bar{S}$, there exists $b \in \mathrm{Betti}(\bar{S})$ such that $\mathsf{c}(s) = \mathsf{c}(b)$. Hence in our setting we get the following corollary.

Corollary 30. *Let S be a nonsymmetric embedding-dimension-three numerical semigroup and let $s \in \bar{S}$.*

- *If $c_2 \geq r_{21} + r_{23}$, then $\mathsf{c}(s) \in \{c_1, c_2, \nu + c_3\}$.*
- *If $c_2 < r_{21} + r_{23}$, then*

$$\mathsf{c}(s) \in \{c_1, c_2 + \delta, c_2', \nu + c_3\} \cup \{(x + y) \mid (x, y) \in M_2 \cup M_3\}.$$

The catenary degree of $\bar{S}$ corresponds with the homogeneous catenary degree of S ([García-Sánchez et al. 2013, Proposition 3.5]; the concept of homogeneous catenary degree is introduced in that paper). Hence this result gives a description also of the homogeneous catenary degree of S. Also, the homogeneous catenary degree is a lower bound for the monotone catenary degree [García-Sánchez et al. 2013, Proposition 3.9].

Example 31. We apply the above corollary to the semigroups in Example 1. Recall that $S^k = \langle 10, 17 + 10k, 19 + 10k \rangle$ and that the minimal presentation for S is

$$\big\{\big((7 + 4k, 0, 0), (0, 3, 1)\big), \big((0, 4, 0), (3 + 2k, 0, 2)\big), \big((0, 0, 3), (4 + 2k, 1, 0)\big)\big\}.$$

Hence the catenary degree of S is $\mathsf{c}(S) = 7 + 4k$ (the catenary degree of an element with two factorizations with disjoint support is just the maximum of the lengths of these factorizations). The minimal presentation of $\bar{S}$ is

$$\big\{\big((0, 7 + 4k, 0, 0), (3 + 4k, 0, 3, 1)\big), \big((1 + 2k, 0, 4, 0), (0, 3 + 2k, 0, 2)\big), \big((0, 1, 5, 0), (1, 0, 0, 5)\big)\big\} \cup \big\{\big((2k + 1 - i, 0, 5i + 4, 0), (0, 3 + 2k - i, 0, 5i + 2)\big) \mid i \in \{0, \ldots, 2k + 1\}\big\}.$$

Hence $\mathsf{c}(\bar{S}) = 9 + 10k$.

4. The nonsymmetric case

If S is not symmetric, then we know (see, for instance, [Rosales and García-Sánchez 2009, Example 8.23]) that some of the following cases can occur (these also include the possibility that $\{n_1, n_2, n_3\}$ is not a minimal generating system, that is, some of the c_i are equal to one):

(1) $c_1n_1 = c_2n_2 = c_3n_3$,

(2) $c_1n_1 = r_{12}n_2 + r_{13}n_3 \neq c_2n_2 = c_3n_3$ $(r_{12}r_{13} \neq 0)$,

(3) $c_1n_1 = c_2n_2 \neq c_3n_3 = r_{31}n_1 + r_{32}n_2$ $(r_{31}r_{32} \neq 0)$,

(4) $c_1n_1 = c_3n_3 \neq c_2n_2 = r_{21}n_1 + r_{23}n_3$ $(r_{21}r_{23} \neq 0)$ and $c_2 \geq r_{21} + r_{23}$,

(5) $c_1n_1 = c_3n_3 \neq c_2n_2 = r_{21}n_1 + r_{23}n_3$ $(r_{21}r_{23} \neq 0)$ and $c_2 < r_{21} + r_{23}$.

For the cases (1), (2) and (4), Bresinsky's algorithm stops in the first step, and thus both $\bar{S}$ and S have a minimal presentation with two elements.

For (3) and (5), the discussion follows as in the similar case in the nonsymmetric setting.

Observe that the uniqueness of a minimal presentation for $\bar{S}$ is not ensured since S might have more than two minimal presentations.

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