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An interesting proof of the nonexistence  
of a continuous bijection between  
 $\mathbb{R}^n$  and  $\mathbb{R}^2$  for  $n \neq 2$

Hamid Reza Daneshpajouh, Hamed Daneshpajouh and Fereshte Malek



# An interesting proof of the nonexistence of a continuous bijection between $\mathbb{R}^n$ and $\mathbb{R}^2$ for $n \neq 2$

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(Communicated by Joel Foisy)

We show that there is no continuous bijection from  $\mathbb{R}^n$  onto  $\mathbb{R}^2$  for  $n \neq 2$  by an elementary method. This proof is based on showing that for any cardinal number  $\beta \leq 2^{\aleph_0}$ , there is a partition of  $\mathbb{R}^n$  ( $n \geq 3$ ) into  $\beta$  arcwise connected dense subsets.

## 1. Introduction

In 1877 Cantor discovered a bijection of  $\mathbb{R}$  onto  $\mathbb{R}^n$  for any  $n \in \mathbb{N}$ . Cantor's map was discontinuous, but the discovery of the Peano curve in 1890 showed that there existed continuous (although not injective) maps of  $\mathbb{R}$  onto  $\mathbb{R}^n$ . Between then and 1910, several mathematicians showed that there does not exist a bicontinuous bijection (homeomorphism) from  $\mathbb{R}^m$  onto  $\mathbb{R}^n$  for the cases  $m = 2$  and  $m = 3$  and  $n > m$ . Finally in 1911, Brouwer showed that there does not exist a homeomorphism between  $\mathbb{R}^m$  and  $\mathbb{R}^n$  for  $n \neq m$  (for a modern treatment, see [Munkres 1984, p. 109]). The present paper proves the nonexistence of a continuous bijection from  $\mathbb{R}^n$  onto  $\mathbb{R}^2$  for  $n \neq 2$  by an elementary method.

Rudin [1963] showed that for any countable cardinal  $\alpha > 2$ , we cannot partition the plane into  $\alpha$  arcwise connected dense subsets. In this paper we show that for any cardinal number  $\beta \leq 2^{\aleph_0}$ , there is a partition of  $\mathbb{R}^n$  ( $n \geq 3$ ) into  $\beta$  arcwise connected dense subsets; then by using this we show that there is no continuous bijection from  $\mathbb{R}^n$  onto  $\mathbb{R}^2$  for  $n \neq 2$ .

**Lemma 1.** *There is a partition of  $\mathbb{R}^+$  into  $2^{\aleph_0}$  dense subsets.*

*Proof.* Consider the additive group  $(\mathbb{R}, +)$ . The quotient group  $\mathbb{R}/\mathbb{Q}$  has  $2^{\aleph_0}$  elements which are dense subsets of  $\mathbb{R}$ . Intersect them with  $\mathbb{R}^+$ .  $\square$

**Theorem 1.** *There is a partition of  $\mathbb{R}^3$  into  $2^{\aleph_0}$  arcwise connected dense subsets.*

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*Keywords:* arcwise connected, dense subset, homeomorphism.

*Proof.* Let  $\{S_i \mid i \in I\}$  be a partition of  $\mathbb{R}^+$  into  $2^{\aleph_0}$  dense subsets. The set  $I$  is just an index set, so we may suppose that  $I = (0, 1)$ . Define  $L_i = \{(t, it, 0) \mid t > 0\}$  and  $M = \bigcup_{i \in I} L_i$  and let  $A_i$  be the union of all spheres with center at the origin and radius from  $S_i$ , that is,  $A_i = \{x \in \mathbb{R}^3 \mid \|x\| \in S_i\}$ . Let  $B_i = (A_i \setminus M) \cup L_i$ . If  $S$  is a sphere centered at the origin, then  $S \setminus M$  is a sphere with a small arc removed. Therefore  $A_i \setminus M$  is the union of some arcwise connected punctured spheres. Open half-line  $L_i$  pastes these punctured spheres together, so  $B_i$  is arcwise connected. It is obvious that  $\{B_i \mid i \in I\}$  is a partition of  $\mathbb{R}^3$  with size  $2^{\aleph_0}$ . Since  $S_i$  is dense in  $\mathbb{R}^+$ ,  $A_i$  and consequently  $B_i$  are dense in  $\mathbb{R}^3$ .  $\square$

**Corollary 1.** *There is a partition of  $\mathbb{R}^n$  into  $2^{\aleph_0}$  arcwise connected dense subsets for  $n \geq 3$ .*

*Proof.* It is enough to set  $B_i^{(n)} = B_i \times \mathbb{R}^{n-3}$ , in which  $B_i$  is as above. The collection  $\{B_i^{(n)} \mid i \in I\}$  is a partition of  $\mathbb{R}^n$  satisfying the claim.  $\square$

Note that the union of any number of the sets  $B_i^{(n)}$  is an arcwise connected dense subset of  $\mathbb{R}^n$ , hence:

**Corollary 2.** *For any cardinal number  $\beta \leq 2^{\aleph_0}$ , there is a partition of  $\mathbb{R}^n$  ( $n \geq 3$ ) into  $\beta$  arcwise connected dense subsets.*

**Theorem 2.** *For any countable cardinal  $\alpha > 2$ , we cannot partition the plane into  $\alpha$  arcwise connected dense subsets.*

*Proof.* This statement is proved in [Rudin 1963].  $\square$

**Lemma 2.** *Let  $X, Y$  be metric spaces and  $T : X \rightarrow Y$  be a continuous map.*

- (a) *If  $A$  is dense in  $X$  and  $T$  is surjective, then  $T(A)$  is dense in  $Y$ .*
- (b) *If  $B \subset X$  is arcwise connected, then  $T(B)$  is also arcwise connected.*

**Theorem 3.** *There is no continuous bijection from  $\mathbb{R}$  onto  $\mathbb{R}^m$  for  $m \neq 1$ .*

*Proof.* Suppose the contrary: Let  $g : \mathbb{R} \rightarrow \mathbb{R}^m$  be a continuous bijective map. We put  $B_n = [-n, n]$ , and so we have  $\mathbb{R}^m = g(\bigcup_{n=1}^{\infty} B_n) = \bigcup_{n=1}^{\infty} g(B_n)$ . Since  $\mathbb{R}^m$  is not in the first category, at least one of the  $g(B_n)$ , for example  $g(B_k)$ , has nonempty interior in  $\mathbb{R}^m$ . Suppose  $B(x, r) \subset g(B_k)$ . Since  $B_k$  is compact,  $f : B_k \rightarrow g(B_k)$  is a homeomorphism. It follows that  $B(x, r)$  is homeomorphic with an interval in  $\mathbb{R}$ . This is a contradiction, because if we remove 3 points from  $B(x, r)$  it remains connected, but this is not the case for the intervals in  $\mathbb{R}$ .  $\square$

**Theorem 4.** *There is no continuous bijection from  $\mathbb{R}^n$  onto  $\mathbb{R}^2$  for  $n \neq 2$ .*

*Proof.* Suppose the contrary:

- (a) If  $n > 2$ , then according to Corollary 2 and Lemma 2 we can partition  $\mathbb{R}^2$  into 3 arcwise connected dense subsets, and this contradicts Theorem 2.
- (b) If  $n = 1$ , then this contradicts Theorem 3.  $\square$

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
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