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Spanning tree congestion of planar graphs

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This paper is devoted to estimates of the spanning tree congestion for some planar graphs. We present three main results: (1) We almost determined (up to ± 1) the maximal possible spanning tree congestion for planar graphs. (2) The value of congestion indicator introduced by Ostrovskii [*Discrete Math.* **310**, 1204–1209] can be very far from the value of the spanning tree congestion. (3) We find some more examples in which the congestion indicator can be used to find the exact value of the spanning tree congestion.

1. Introduction

Let G be a graph and let T be a spanning tree in G. We follow the terminology and notation of [Clark and Holton 1991]. For each edge e of T, let A_e and B_e be the vertex sets of the components of $T - e$ (see Figure 1). By $e_G(A_e, B_e)$ we denote the number of edges in G with one end vertex in A_e and the other end vertex in B_e. We define the *edge congestion* of G in T by

$$\mathrm{ec}(G:T) = \max_{e \in E(T)} e_G(A_e, B_e).$$

The number $e_G(A_e, B_e)$ is called the *congestion* in e. The name comes from the following analogy. Imagine that edges of G are roads, and edges of T are those roads which are cleaned of snow after snowstorms. If we assume that each edge in G bears the same amount of traffic, and that after a snowstorm each driver takes the corresponding (unique) detour in T, then $\mathrm{ec}(G:T)$ describes the traffic congestion at the most congested road of T. Clearly, it is interesting for applications to find a spanning tree which minimizes the congestion.

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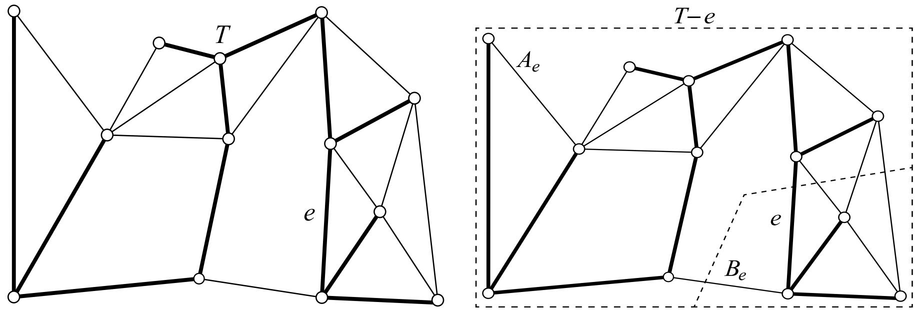

Figure 1. Left: spanning tree T of a graph G. Right: subgraph $T-e$ of G. In this case, $e_G(A_e, B_e) = 5$.

We define *the spanning tree congestion* of G by

$$s(G) = \min\{\mathrm{ec}(G:T) \colon T \text{ is a spanning tree of } G\}. \tag{1}$$

Each spanning tree T in G satisfying $\mathrm{ec}(G:T) = s(G)$ is called a *minimum congestion spanning tree*. The definitions of $\mathrm{ec}(G:T)$ and $s(G)$ were introduced and their study initiated in [Ostrovskii 2004]. Closely related parameters were introduced earlier in [Simonson 1987, p. 236; Khuller et al. 1993]. After the publication of [Ostrovskii 2004], the spanning tree congestion became the object of active study. As a result the spanning tree congestion was computed and estimated for many families of graphs — see [Law and Ostrovskii 2010; Otachi 2011] for surveys of such results and further references. Algorithmic issues of the problem were studied in [Bodlaender et al. 2012; Löwenstein 2010; Otachi et al. 2010]. In [Löwenstein 2010, Section 5.6] and [Otachi et al. 2010] it was independently discovered that the spanning tree congestion is computationally hard. The contents of the latter were incorporated in [Bodlaender et al. 2012], which contains a systematic analysis and the strongest known results on the algorithmic complexity of problems related to the spanning tree congestion. In a note to Lemma 8, we mention what seems to be the easiest known way to show that the spanning tree congestion problem is NP-hard even for planar graphs.

In this paper we restrict our attention to the study of the spanning tree congestion for planar graphs. In this case some additional tools are available, but computing the spanning tree congestion is still NP-hard and offers some challenging problems.

The main results of this paper:

(1) We almost determined (up to ± 1) the maximal possible spanning tree congestion for planar graphs; see Section 3.

(2) The computational hardness of the spanning tree congestion problem makes us interested in parameters which approximate the spanning tree congestion. We

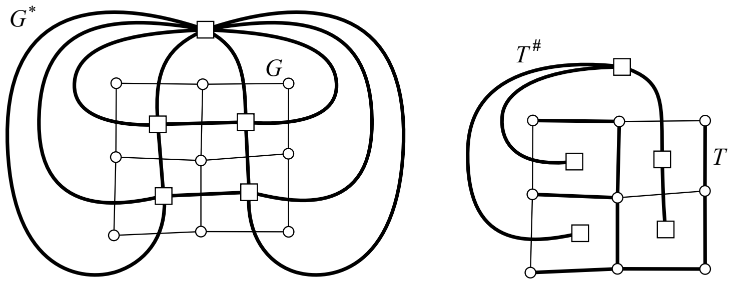

Figure 2. Left: The dual graph G^* of a graph G. Right: the dual tree $T^\sharp$ of a spanning tree T.

find that the value of the congestion indicator introduced in [Ostrovskii 2010] is very far from the value of the spanning tree congestion for some graphs; see Section 4.

(3) We find more examples in which the congestion indicator introduced in [Ostrovskii 2010] can be used to find the exact value of the spanning tree congestion; see Section 5.

2. Dual graphs, indices and center-tail systems

In this section we introduce tools which can be used to estimate the spanning tree congestion and which are available for planar graphs only. By a *plane graph* we mean a planar graph whose planar drawing is fixed.

Definition 1. The *dual graph* G^* of a plane graph G is defined to be the multigraph whose vertices correspond to the faces of G, including the exterior face O. Two faces are joined by an edge if and only if they have a common edge in their boundaries. (If two faces have several common edges in their boundaries, the corresponding edges are multiple edges.) Note that an edge $e^* \in E(G^*)$ corresponding to $e \in E(G)$ joins the faces of G (equal to the vertices of $V(G^*)$) whose boundaries contain e. If T is a spanning tree of G, then the *dual tree* $T^\sharp$ is defined as a spanning subgraph of G^* such that $e^* \in E(T^\sharp)$ if an only if $e \notin E(T)$ (see Figure 2). As is well known, $T^\sharp$ is a spanning tree in G^* (see [Lovász 2007, solution of Problem 5.23] for an explanation).

Definition 2 [Ostrovskii 2010]. An edge $e \in E(G)$ is said to be an *outer edge* of G if it lies on the boundary of the exterior face. The *index* $i(F, e)$, where F is a bounded face and e is an outer edge, is defined to be the length of the shortest path in G^* which joins the exterior face O with F and satisfies the condition that e^* is the first edge in the path.

Definition 3 [Ostrovskii 2010]. A *center-tail system* S in the dual graph G^* of a plane graph G consists of:

(1) A connected set C of vertices of G^*, which is called a *center*.

(2) A set of paths in G^* which join some vertices of the center C with the exterior face. Such a path is called a *tail*. The *tip* of a tail is the last vertex of the corresponding path before it reaches the exterior face.

(3) An assignment of *opposite tails* for outer edges of G. This means that for each outer edge e, a tail is assigned to be the opposite tail, which is denoted by $N(e)$ and its tip by $t(e)$.

Definition 4 [Ostrovskii 2010]. The *congestion indicator* $\mathrm{CI}(S)$ of a center-tail system S is defined as the minimum of three numbers:

(1) $\min_{F,H,f,h}\big(i(F,f)+i(H,h)+1\big)$, where the minimum is taken over all pairs F, H of adjacent vertices in the center C and over all pairs f, h of outer edges with $f \neq h$. In the case where the center consists of just one vertex, we assume that the minimum is ∞.

(2) $\min_e i(t(e),e)+1$, where the minimum is taken over all outer edges of G.

(3) $\min_e \min_{F\in N(e)} \min_{\tilde{e}\neq e}\big(i(F,e)+i(H,\tilde{e})+1\big)$, where the first minimum is taken over all outer edges of G; the second minimum is over vertices F from the path $N(e)$ different from $t(e)$ and the exterior face, and H is the vertex in $N(e)$ which follows immediately after F if one moves along $N(e)$ from F to $t(e)$; and the third minimum is over all outer edges different from e.

Theorem 5 [Ostrovskii 2010]. *Let S be any center-tail system in a connected planar graph G. Then $s(G) \geq \mathrm{CI}(S)$.*

Definition 6 [Ostrovskii 2010]. The *absolute index* $i(F)$ of a face F is defined as $\min_e i(F,e)$, where the minimum is over all outer edges.

Theorem 7 [Ostrovskii 2010]. *For each connected planar graph G with at least two bounded adjacent faces, we have $s(G) \leq \max\big(i(F)+i(H)\big)+1$, where the maximum is over all pairs F, H of bounded faces which have a common edge in their boundaries.*

For the study of maximal spanning tree congestion, we make use of results on graph radius. Recall that given a connected graph G, the *radius* is

$$\mathrm{rad}(G) = \min_{x\in V(G)} \max_{y\in V(G)} d_G(x,y). \tag{2}$$

A vertex x for which the minimum in (2) is attained is called *central*. (Warning: this notion of centrality is not related to the center-tail systems introduced above.)

For planar graphs the spanning tree congestion is closely related to the widely used notion of *stretch*; see [Peleg 2000, p. 166].

If H is a connected spanning subgraph in G, then its *stretch* is defined by

$$\operatorname{Stretch}(H) = \max_{u,v \in V(G)} \frac{d_H(u,v)}{d_G(u,v)}. \tag{3}$$

The following observations can be found in [Ostrovskii 2010; Otachi et al. 2010; Peleg 2000].

Lemma 8. *Let G be a connected planar graph.*

(a) *If T is a spanning tree in G and $T^\sharp$ is its dual tree, then*

$$\operatorname{ec}(G:T) = \operatorname{Stretch}(T^\sharp) + 1.$$

(b) $s(G) = \inf_{T^\sharp} \operatorname{Stretch}(T^\sharp) + 1$, *where the infimum is over all spanning trees $T^\sharp$ in the dual graph G^*.*

(c) $\min_{T^\sharp} \operatorname{Stretch}(T^\sharp) \le 2\operatorname{rad}(G^*)$.

Proof. It is easy to see that the number of detours using an edge $e \in T$ is the length of the cycle obtained by adding the edge e^* to $T^\sharp$. On the other hand, the length of this cycle is exactly $d_{T^\sharp}(u,v) + 1$, where u, v are the ends of e^*. Therefore $\operatorname{ec}(G:T) = \operatorname{Stretch}(T^\sharp) + 1$, proving (a).

The statement (b) follows immediately from (a).

To prove (c) it suffices to observe that any breadth-first search (BFS) tree $T^\sharp$ in G^* rooted at one of its central vertices C satisfies $\operatorname{Stretch}(T^\sharp) \le 2\operatorname{rad}(G^*)$. (See [Rosen et al. 2000, Section 9.2.1] or [Nishizeki and Chiba 1988, p. 31] for information on BFS trees.) To see the inequality $\operatorname{Stretch}(T^\sharp) \le 2\operatorname{rad}(G^*)$ we need only the defining property of a BFS tree in G^* rooted at C: it is a spanning tree in G^* in which the distance between any vertex and C is the same as in G^*, and therefore is $\le \operatorname{rad}(G^*)$. □

Note. Fekete and Kremer [2001] proved that the determination of the least t for which a planar graph has a spanning tree T with $\operatorname{Stretch}(T) = t$ is NP-hard. Combining this with Lemma 8 we get that the problem of computation of $s(G)$ for planar graphs is also NP-hard.

3. On the maximal spanning tree congestion of planar graphs

The purpose of this section is to find sharp estimates of the quantity

$$\mu_p(n) = \max\{s(G) : G \text{ is a planar graph with } n \text{ vertices}\}.$$

Graphs G with n vertices satisfying $s(G) = \mu_p(n)$ can be called *the most congested planar graphs with n vertices.*

Note. A consequence of Euler's formula is that a simple planar graph with $n \geq 3$ vertices has at most $3n - 6$ edges. As $n - 1$ of them are in a spanning tree, they are detours for themselves. Therefore the spanning tree congestion cannot exceed $3n - 6 - (n - 1) + 1$. Thus $\mu_p(n) \leq 2n - 4$. Our purpose is to get more precise estimates for $\mu_p(n)$.

Theorem 9. *Let $n \geq 5$. If n is even, then $n \leq \mu_p(n) \leq n + 1$. If n is odd, then $n - 1 \leq \mu_p(n) \leq n$.*

The proof of this theorem naturally splits into two parts: estimates from above (Section 3.1) and estimates from below (Section 3.2).

Problem 10. Fill the gap of size 1 between the upper and lower estimates in Theorem 9.

3.1. ***Estimates from above.*** We need some terminology and notation of [Diestel 2000]. A plane graph is called a *plane triangulation* if all faces of it are triangles. Adding some edges (but not vertices) to an arbitrary planar graph G we get a plane triangulation G_t which we call a *triangulation* of G. It is easy to construct examples showing that G_t, in general, is not uniquely determined by G.

Lemma 11.
$$\operatorname{rad} G_t^* \geq \operatorname{rad} G^*.$$

Proof. To see this, it suffices to observe that G^* is a minor of G_t^*, obtained if sets of triangular faces of G_t that originated from the same face of G are considered as branch sets (see [Diestel 2000, p. 16] for minor-related definitions). It is clear that such sets are connected in G_t^* and the corresponding minor is isomorphic to G^*. Since in creating this minor we did not delete any edges or vertices, the radius of the resulting graph can only be less than the radius of G_t^*, and we get the desired inequality. □

The following two facts are well known; see, for example, [Diestel 2000, Section 4.4; Exercise 40 in Chapter 4].

Lemma 12. *A triangulation of a planar graph with at least* 4 *vertices is* 3*-connected.*

Lemma 13. *The dual graph of a* 3*-connected planar graph is a* 3*-connected planar graph.*

Finally we need the following tight estimate for a radius of a 3-connected graph obtained in [Iida 2007]. (See [Egawa and Inoue 1999; Harant 1993; Harant and Walther 1981; Iida and Kobayashi 2006; Inoue 1996] for preceding and related estimates.)

Theorem 14 [Iida 2007]. *Let G be a* 3*-connected graph with radius r. Then*
$$|V(G)| \geq 4r - 4.$$

Lemma 15. *Let $n \geq 4$. Then*

$$\mu_p(n) \leq \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let G be a plane graph with n vertices satisfying $s(G) = \mu_p(n)$. By Lemma 12 the graph G_t is 3-connected. By Lemma 13 the graph G_t^* is also 3-connected. An easy computation with Euler's formula shows that G_t^* has $2n-4$ vertices. By Theorem 14 we get $\operatorname{rad}(G_t^*) \leq 2n/4 = n/2$. By Lemma 11 we get $\operatorname{rad}(G^*) \leq n/2$. Therefore $\operatorname{rad}(G^*) \leq n/2$ if n is even and $\operatorname{rad}(G^*) \leq (n-1)/2$ if n is odd. Combining these inequalities with Lemma 8 we get

$$s(G) \leq \begin{cases} n+1 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases} \qquad \square$$

3.2. ***Estimates from below.*** For $n \geq 5$, we let B_n be graphs of bipyramids whose bases are $(n-2)$-gons. These graphs can be constructed in the following way: we start with C_{n-2} (cycle of length $n-2$), then introduce two more vertices and join each of them with each of the vertices in the cycle.

Lemma 16. *Let $n \geq 5$. Then*

$$s(B_n) = \begin{cases} n & \text{if } n \text{ is even,} \\ n-1 & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

Proof. Observe that the dual of B_n is $B_n^* = C_{n-2} \times K_2$.

Denote by $\ell(n)$ the case-defined function given by the right-hand side of (4). By the proof of Lemma 8, in order to prove $s(B_n) \geq \ell(n)$ it suffices to show that for an arbitrary spanning tree $T^\sharp$ in B_n^* there is an edge e^* in B_n^* which is not in $T^\sharp$ and such that $T^\sharp \cup \{e^*\}$ contains a cycle of length $\geq \ell(n)$. The inequality $s(B_n) \leq \ell(n)$ will also follow from our argument, but it is clear that the main point of Lemma 16 is the lower estimate.

An edge in B_n^* is called *vertical* if its end vertices are (c, k_1) and (c, k_2), where c is a vertex of C_{n-2} and k_1, k_2 are vertices of K_2; otherwise, it is *horizontal*. Two horizontal edges form a *couple* if they correspond to the same edge in C_{n-2}.

If all vertical edges are in $T^\sharp$, then there is a couple e^*, f^* of horizontal edges which are both not in $T^\sharp$ (otherwise $T^\sharp$ would contain a cycle). Clearly, at least one of e^*, f^* creates together with edges of $T^\sharp$ a cycle of length at least $n \geq \ell(n)$.

Now suppose that there are vertical edges which are not in $T^\sharp$. Let e^* be one of the vertical edges in $E(B_n^*) \setminus E(T^\sharp)$. Then $T^\sharp \cup \{e^*\}$ contains a cycle. If this cycle contains an edge from each couple of the horizontal edges, we say that it *goes around*. It is clear that if the cycle contained in $T^\sharp \cup \{e^*\}$ goes around, then it has length $\geq n \geq \ell(n)$. If it does not go around, then it contains exactly one more vertical edge.

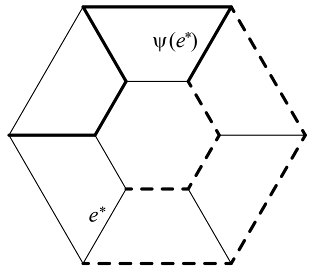

Figure 3. Different sides of e^* and $\psi(e^*)$ in B_8^*.

Therefore, if there are no cycles of the described type which go around, then there is a mapping ψ from the set of vertical edges which are not in $E(T^\sharp)$ to the set of vertical edges which are in $E(T^\sharp)$ satisfying this condition: all couples of horizontal edges on one of the "sides" between e^* and $\psi(e^*)$ belong to $T^\sharp$. To clarify the meaning of the word "sides" in the previous sentence we show different sides in Figure 3 using dashed and continuous lines, respectively, attribution of vertical edges to sides does not matter; the tree $T^\sharp$ is shown using thick lines, dashed or continuous. In this way, vertical edges split into groups having the common image under ψ. We include f^* in the group of edges e^* for which $\psi(e^*) = f^*$. It is clear that all vertical edges between e^* and $\psi(e^*)$ which are on the suitable side (see above) belong to the same group as e^*. Therefore, the groups partition the vertex set of the cycle C_{n-2} into connected pieces.

If there is just one connected piece, then there is just one vertical edge in $E(T^\sharp)$, and all but two horizontal edges are in $E(T^\sharp)$. It is clear that the missing horizontal edges should form a couple (otherwise there would be a vertical edge e^* for which the cycle in $T^\sharp \cup \{e^*\}$ goes around). It is in this case that we get a weaker estimate for odd n.

In fact, if the end vertices of the only vertical edge of $T^\sharp$ divide those pieces of $C_{n-2} \times \{k_1\}$ and $C_{n-2} \times \{k_2\}$ which are in $T^\sharp$ into parts of equal length (this is possible if n is odd), then the maximal length of the cycle in $T^\sharp \cup \{e^*\}$ over $e^* \in E(B_n^*) \setminus E(T^\sharp)$ is $n-1$. (Otherwise, the longest cycle in $T^\sharp \cup \{e^*\}$ has length at least $n+1$.)

On the other hand, if n is even, the cycle obtained by adding to $E(T^\sharp)$ the vertical edge which is most distant from the one contained in $E(T^\sharp)$ produces a cycle of length at least n.

Now we suppose that there are at least two connected pieces. We consider horizontal edges between the neighboring intervals. It is easy to check that if there are at least three intervals, there is a pair of neighboring intervals with no edges in

$T^\sharp$ between them. If there are two intervals, then on one side there are no edges in $T^\sharp$ between them.

Let e_1^* and e_2^* be the corresponding missing horizontal edges. Then $E(T^\sharp)\cup\{e_1^*\}$ or $E(T^\sharp)\cup\{e_2^*\}$ contains a cycle which contains vertical edges and therefore has length $\geq n \geq \ell(n)$. □

4. Limitations of center-tail systems

In this section we show that for some classes of planar graphs the estimates of the spanning tree congestion given by center-tail systems (see Theorem 5) are far from being sharp. More precisely we prove the following result.

Theorem 17. *There exists a sequence $\{G_n\}_{n=1}^\infty$ of planar graphs such that*

$$\lim_{n\to\infty} s(G_n)=\infty,$$

but for any center-tail system S_n in G_n, we have $\mathrm{CI}(S_n)\leq 6$.

Note. By a center-tail system for a planar graph we mean a center-tail system of any of its drawings. In particular, any of the faces of the graph can be regarded as its exterior face.

Proof. Before defining the graphs G_n, it is convenient to define a two-parametric family of graphs, which we denote $\{Q_{n,m}\}_{n,m=1}^\infty$. To construct the graph $Q_{n,m}$ we start with a family of $2n+m$ concentric circles. They cut out of the plane $2n+m-1$ concentric annuli. We cut both the outer and the inner annuli into 4 pieces each using radial cuts (see Figure 4). We next cut annuli, both from the inner and the outer side, into 4^2 equal pieces using radial cuts. We make these radial cuts in such a way that they extend the radial cuts done in the first step (see Figure 4). Continuing, for each $k\leq n$ we cut the k-th annuli, both from the inner and the outer

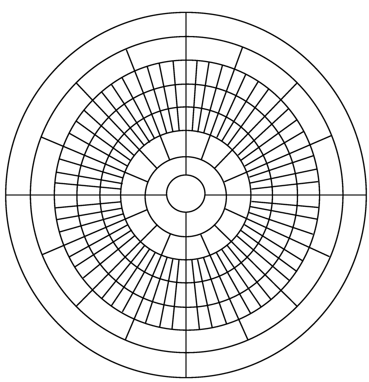

Figure 4. A planar graph $Q_{3,2}$.

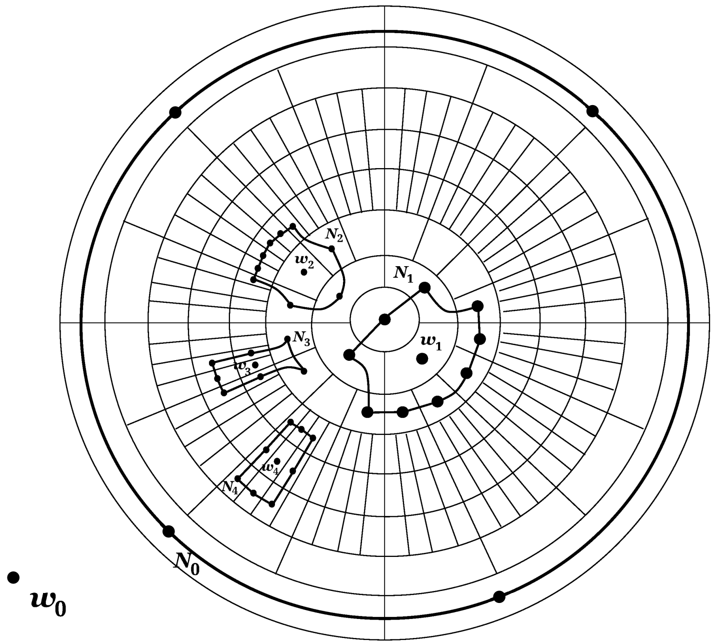

Figure 5. A planar graph $Q_{3,2}$ with pieces of the dual graph needed to estimate the congestion indicator.

side, into 4^k equal pieces using radial cuts. We cut the remaining $m-1$ annuli in the same way as the annuli in the last set, that is, using 4^n radial cuts. In Figure 4, we show the resulting graph in the case where $n=3$ and $m=2$.

Then G_n is defined as $Q_{n,4^n-2}$. Now we estimate the congestion indicator. Recall that CI is a minimum of three terms, one of which is

$$\min_e i(t(e), e) + 1.$$

Clearly this term, in the case where the face playing the role of the exterior face is denoted by w, does not exceed

$$\max_{u,v} d(u, v) + 2, \tag{5}$$

where the maximum is over pairs u, v of vertices in G_n^*, both of which are adjacent to w, and d is the graph distance in $G_n^* - w$. To estimate from above the value of (5), we observe that vertices adjacent to w in G_n^* belong to a cycle in $G_n^* - w$ whose length is between 4 and 9. See Figure 5, in which we denote several possible choices of w by w_0, w_1, w_2, w_3, and w_4, and denote the cycles described in the previous sentence by N_0, N_1, N_2, N_3, and N_4, respectively. It is clear that the distance between any two vertices of such a cycle of length ≤ 9 does not exceed 4, so the maximum in (5) does not exceed 6. We get the desired estimate: the congestion indicator of any center-tail system in any of the graphs $Q_{n,m}$, and therefore in any of the graphs G_n, does not exceed 6.

Now we turn to spanning tree congestion estimates. Here we use the approach suggested in [Ostrovskii 2004] using centroids and isoperimetric estimates.

Definition 18 [Jordan 1869]. Let u be a vertex of a tree T. Let the *weight of* T *at* u be the maximal number of vertices in components of $T-u$. A vertex v of T is called a *centroid* vertex if the weight of T at v is minimal.

Let T be an optimal tree in G_n so that $\mathrm{ec}(G_n : T) = s(G_n)$. Let u be a centroid of T. Since the maximum degree of G_n is 4, there are at most 4 edges incident with u. Let

$$o_{G_n} = \left\lceil \frac{|V(G_n)|-1}{4} \right\rceil.$$

Since u is a centroid, it is not hard to see that there is a component of $T-u$ whose vertex set A satisfies

$$o_{G_n} \le |A| \le \frac{|V(G_n)|}{2}.$$

As the edge connecting u with A is used in $e_{G_n}(A, V(G_n)-A)$ detours, any lower bound of this number, where A runs over sets of size within the above range, is a lower bound of $s(G_n)$.

We use the following special case of the isoperimetric result of Bollobás and Leader [1991, Theorem 3]. Let $R(k)$ be the graph with vertex set

$$[k]^2 = \{0, 1, 2, 3, \dots, k-1\}^2$$

in which $x = (x_1, x_2)$ is adjacent to $y = (y_1, y_2)$ if and only if $|x_i - y_i| = 1$ for some i and $x_j = y_j$ for $j \ne i$.

Theorem 19. *Let B be a subset of $[k]^2$ with $|B| \le k^2/2$. Then*

$$e_{R(k)}(B, \bar{B}) \ge \min\{2\sqrt{|B|}, k\}. \tag{6}$$

Let us introduce the function

$$f_k(t) = \min\{k, 2\sqrt{t}\} \quad \text{for } t \in \left[0, \frac{k^2}{2}\right].$$

Observe that the graph G_n has a subgraph S_n isomorphic to $R(4^n)$. Indeed, we may take S_n to contain all vertices of the 4^n central circles and all the corresponding edges except one "radial" set of 4^n edges. The subgraph S_n has $4^n \times 4^n = 4^{2n}$ vertices. In addition, G_n has $2(4+4^2+\cdots+4^{n-1}) = \frac{8}{3}(4^{n-1}-1)$ vertices on the $2(n-1)$ circles which are not in S_n. It is clear that the intersection of the set A

with the vertex set of S_n has at most $2 \cdot 4^{2n-1} + \frac{4}{3}(4^{n-1} - 1)$ vertices. We need also the inequality

$$|A \cap V(S_n)| \geq 4^{2n-1} - 2(4^{n-1} - 1).$$

To get this inequality we recall that

$$|A| \geq \left\lceil \frac{|V(G_n)|-1}{4} \right\rceil \geq 4^{2n-1} + \tfrac{2}{3}(4^{n-1} - 1) - \tfrac{1}{4},$$

and observe that

$$\begin{aligned} |A \cap V(S_n)| &\geq |A| - (|V(G_n)| - |V(S_n)|) \\ &\geq 4^{2n-1} + \tfrac{2}{3}(4^{n-1} - 1) - \tfrac{1}{4} - \tfrac{8}{3}(4^{n-1} - 1) \\ &= 4^{2n-1} - 2(4^{n-1} - 1) - \tfrac{1}{4}. \end{aligned}$$

We may drop $\frac{1}{4}$ since $|A \cap V(S_n)|$ is an integer.

Applying Theorem 19 to the smaller of $A \cap V(S_n)$ and $V(S_n) \setminus A$, we get that the number of edges joining $A \cap V(S_n)$ with $V(S_n) \setminus A$ can be estimated from below by

$$\min_t \{ f_{4^n}(t) \},$$

where t ranges from $\min\bigl\{4^{2n-1} - 2(4^{n-1} - 1), 4^{2n} - \bigl(2 \cdot 4^{2n-1} + \frac{4}{3}(4^{n-1} - 1)\bigr)\bigr\}$ to $2 \cdot 4^{2n-1}$. It is clear that these minima approach ∞ as $n \to \infty$. Therefore $\lim_{n\to\infty} s(G_n) = \infty$. This completes the proof of the theorem. □

Note. It is known that for some planar graphs, center-tail systems and the corresponding congestion indicators give sharp lower bounds of the spanning tree congestion. However, as the above example shows, in some cases the lower bound given by center-tail systems is very far from the actual value of the spanning tree congestion.

Problem 20. Is it possible to define a flexible version of the congestion indicator (FCI) such that for some function $f : \mathbb{N} \to \mathbb{N}$ and any planar graph G we have $s(G) \leq f(n)$ if the maximal possible value of FCI (on the corresponding analogue of the center-tail system in G) has value $\leq n$?

5. Computing spanning tree congestion by center-tail systems

Center-tail systems were introduced in [Ostrovskii 2010] as a tool to compute or estimate the spanning tree congestion of some plane graphs. In [Ostrovskii 2010] the computation was performed for the triangular grids. Another grid for which center-tail systems give the exact value of the spanning tree congestion was found in [Bodlaender et al. 2011, Theorem 3.7]. In this section we use the center-tail systems and Theorems 5 and 7 to find the spanning tree congestion of other sets of planar graphs.

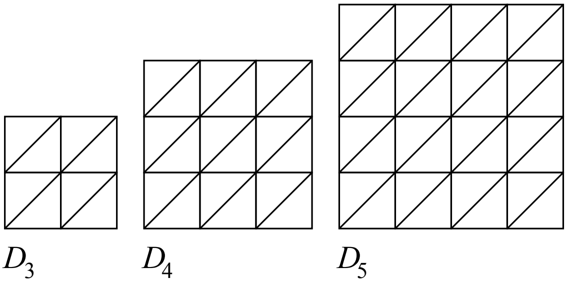

Figure 6. A sequence of square-triangular graphs.

5.1. ***Square-triangular grids.*** Consider the sequence of square-triangular graphs in Figure 6. In this figure, there is a vertex at each intersection of the line segments. The spanning tree congestion of these graphs is computed in the next theorem.

Theorem 21. *Let* $n \in \mathbb{N}$. *Then*

$$s(D_k) = \begin{cases} 4n & \text{if } k = 2n+1, \\ 4n+3 & \text{if } k = 2n+2. \end{cases}$$

Proof. Case 1. $k = 2n+1$, where $n \in \mathbb{N}$. We start by considering the graph D_3 and its center-tail system S_3 shown in Figure 7. The center for the system S_3 consists of one vertex and is marked with the letter C. The tail whose tip points to the upper-left corner is assigned as the opposite tail for the outer edges on the right and at the bottom of D_3. The tail with right-most tip is assigned as the opposite tail for those outer edges on the left. The tail with bottom-most tip is assigned to those outer edges on top. It is easy to see that the congestion indicator $\mathrm{CI}(S_3)$ (see Definition 4) of the center-tail system S_3 is the minimum of three numbers: (i) ∞,

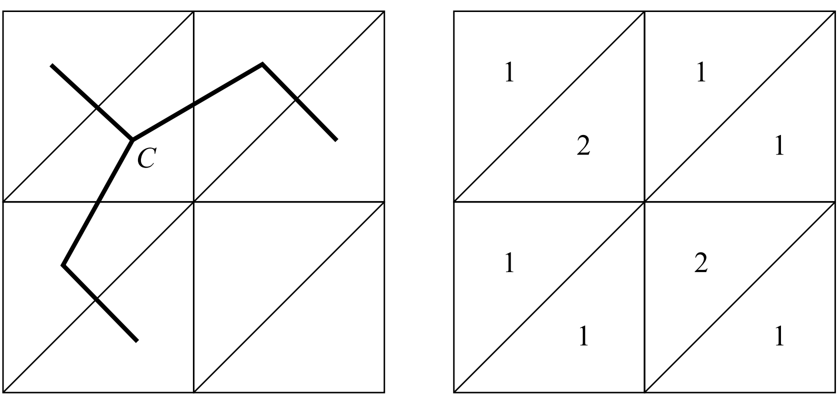

Figure 7. Left: D_3 with a center-tail system S_3. Right: absolute indices for D_3.

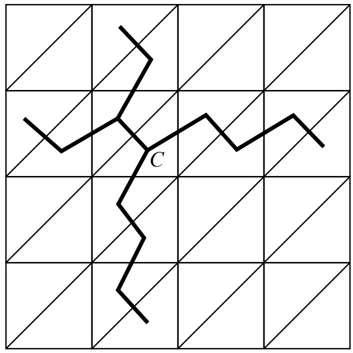

Figure 8. D_5 with a center-tail system S_5.

(ii) 5 and (iii) 4. Hence $\mathrm{CI}(S_3) = 4$ and, by Theorem 5, $s(D_3) \geq 4$. According to Theorem 7, we have $s(D_3) \leq 4$ (see the values of the absolute indices in Figure 7), therefore $s(D_3) = 4$.

Adding a row on each side of D_3 gives us the graph D_5. We consider what can be regarded as a natural extension of S_3 to S_5; the only feature of this extension which is not completely predictable is that the tail whose tip points to the upper-left corner in S_3 is now splitting into two tails in S_5 (see Figure 8). The tail whose tip points to the left is assigned to the outer edges on the right, and the tail whose tip points upward is assigned to those outer edges at the bottom. Since the indices of the central triangles increase by two as we add a row on each side of the graph, the spanning tree congestion increases by four, so $s(D_5) = s(D_3) + 4 = 4 + 4 = 8$. It is clear by induction that $s(D_{2n+1}) = 4n$ for each $n \in \mathbb{N}$.

Case 2: $k = 2n + 2$. First we consider the graph D_4 and its center-tail system S_4, described as follows. The center of S_4 consists of two vertices which are labeled C (see Figure 9). There are four tails, which are drawn in Figure 9 with thick lines. The assignments of opposite tails for outer edges are done in the natural way. For

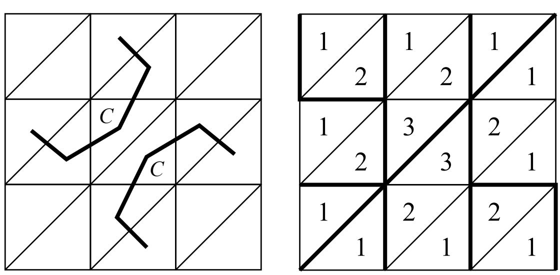

Figure 9. Left: center-tail system S_4. Right: absolute indices for D_4 and a minimum congestion spanning tree for D_4.

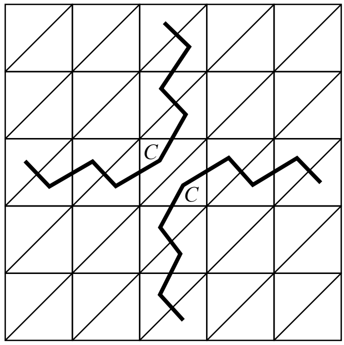

Figure 10. D_6 with a center-tail system S_6.

example, the tail whose tip points to the left is assigned to the outer edges on the right. The tail whose tip points upward is assigned to the outer edges at the bottom of the graph.

It is easy to see that the congestion indicator $\mathrm{CI}(S_4)$ of the center-tail system S_4 is the minimum of the following three numbers: (i) $3+3+1=7$, (ii) $6+1=7$ and (iii) 7. Hence $\mathrm{CI}(S_4)=\min\{7,7,7\}=7$. By Theorem 5, $s(D_4)\geq 7$. The values of absolute indices $i(F)$ for D_4 are shown in Figure 9. According to Theorem 7, $s(D_4)\leq \max\bigl(i(F)+i(H)\bigr)+1=3+3+1=7$, where F and H are bounded faces with an edge in common, and the maximum is taken over F and H. Hence, $s(D_4)=7$. Following the argument of the proof of Theorem 7 in [Ostrovskii 2010], we sketch one of the spanning trees for which the congestion is 7; see Figure 9.

By adding one row on each side of the graph, we obtain the square-triangular grid D_6 (see Figure 10). Addition of a row on each side increases the indices of central triangles by two. Straightforward computation shows that all of the estimates increase by 4, hence $s(D_6)=s(D_4)+4=11$. We use induction to show that $s(D_{2n+2})=4n+3$ for each $n\in\mathbb{N}$. □

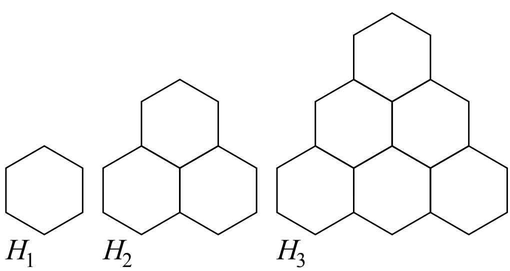

Figure 11. A sequence of hexagonal grids.

5.2. *Hexagonal grids.* A *hexagonal grid* H_k is constructed following the pattern shown in Figure 11. Our next purpose is to compute $s(H_k)$.

In fact, the following theorem was stated in [Castejon et al. 2007], but its proof was insufficient. The authors of [Castejon et al. 2007] wrote that the proof is the same as their proof for rectangular grids; errors of their proof for rectangular grids were described in [Ostrovskii 2010, p. 1209]. We provide a proof of this theorem using center-tail systems.

Theorem 22. *Let $n \geq 0$ be an integer. Then*

$$s(H_k) = \begin{cases} 2n+2 & \text{if } k = 3n+1, \\ 2n+3 & \text{if } k = 3n+2, \\ 2n+3 & \text{if } k = 3n+3. \end{cases}$$

Proof. *Case 1*: $k = 3n+1$. Since H_1 is isomorphic to C_6, it is easy to see that $s(H_1) = 2$.

By adding one row on each side of H_1, we obtain the graph H_4 (see Figure 12). Here the center of the center-tail system S_4 consists of one vertex, labeled C. The tails are drawn with thick lines. The assignments of opposite tails to the outer edges are done in the natural way. The tail whose tip points to the left, downward and upward is assigned to outer edges on the right, the left and at the bottom, respectively (see Figure 12). According to the center-tail system S_4, we have the three numbers defined in Definition 4: (i) ∞, since there is only one face in the center, (ii) $3+1=4$, witnessed by an outer edge e in the middle of any of the three sides since $i(t(e), e)$ is 3, and (iii) $2+1+1=4$. We pick an outer edge e in the middle of one of the three sides, let $F = C$, and H be the face that contains $t(e)$. Then $i(F, e) = 2$ and $i(H, \tilde{e}) = 1$, where $\tilde{e}$ is an outer edge on the boundary of the face that contains H. So $\mathrm{CI}(S_4) = \min\{\infty, 4, 4\} = 4$. By Theorem 5, $s(H_4) \geq 4$.

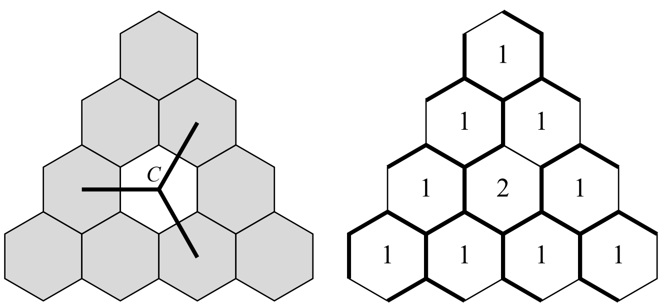

Figure 12. Left: H_4 with center-tail system S_4. The shaded region represents the additional rows added on each side of H_1. Right: absolute indices for H_4 and minimum spanning congestion tree for H_4.

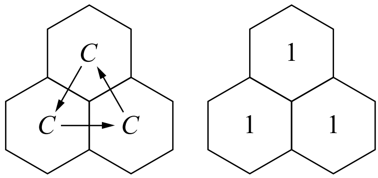

Figure 13. Left: H_2 with center-tail system S_2. Right: absolute indices for H_2.

The absolute indices of faces are shown in Figure 12 (right). The sum of indices of adjacent faces never exceeds 3. Therefore, by Theorem 7, we have $s(H_4) \le 4$. So $s(H_4) = 4$.

Now we prove that

$$s(H_{3n+1}) = 2n + 2 \tag{7}$$

for each $n \in \mathbb{N}$. We use induction. We have shown that (7) holds for $n = 0, 1$. It remains to show that $s(H_{3n+1}) = 2n+2$ implies $s(H_{3(n+1)+1}) = 2(n+1)+2$. To see this, we observe that if an additional row is added on each side of H_{3n+1}, then each side has three more hexagons and the graph becomes $H_{3n+1+3} = H_{3(n+1)+1}$. An increment of one row on each side increases the indices of central vertices by one, so each of the three numbers defined in Definition 4, as well as the number $\max\bigl(i(F)+i(H)\bigr)+1$ (see Theorem 7), increase by two. Hence, $s(H_{3(n+1)+1}) = 2(n+1)+2$.

Case 2: $k = 3n+2$. Now consider the graph H_2 with the center-tail system S_2 (see Figure 13). The center of S_2 consists of three vertices labeled C. The tails are represented by the arrows; their tips correspond to the arrow heads. The tail whose tip points to the right, downward and upward is assigned to the outer edges on the left, the right and the bottom, respectively. According to Definition 4, $\mathrm{CI}(S_2)$ is the minimum of the following three numbers: (i) $1+1+1 = 3$, since the distance from the exterior face O to any face that contains a vertex of the center is 1. (ii) $2+1 = 3$, since every tail has length 1, and the distance from O to any face that contains a vertex of the center is also 1. (iii) $1+1+1 = 3$, based on the same reasoning as in (ii). So $\mathrm{CI}(S_2) = \min\{3, 3, 3\} = 3$. By Theorem 5, $s(H_2) \ge 3$. Since there are only three faces in H_2 and each face is adjacent to one another, by Theorem 7, we have $s(H_2) \le 1+1+1 = 3$. Therefore, $s(H_2) = 3$.

We can obtain the graph H_5 from H_2 by simply adding a row on each side of H_2 (see Figure 14). Notice that the configuration of the center-tail system S_5 for H_5 is different than S_2. The center of S_5 also consists of three vertices (labeled C), and they are located in the middle of the graph (see Figure 14). The tails are drawn with thick lines. The assignment of opposite tails to the outer edge is done in the natural way. The tail whose tip points to the left, downward and upward is assigned to

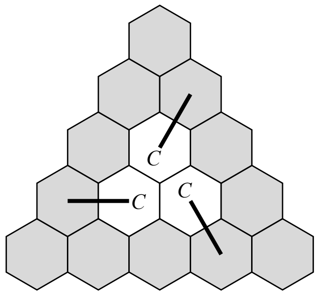

Figure 14. H_5 with center-tail system S_5. The shaded region represents the additional rows added on each side of H_2.

the outer edges on the right, the left and the bottom, respectively. By Definition 4, $\mathrm{CI}(S_5)$ is the minimum of (i) $2+2+1=5$, (ii) $4+1=5$ and (iii) 5. By Theorem 5, $s(H_5) \geq \mathrm{CI}(S_5) = 5$, and by Theorem 7, $s(H_5) \leq 2+2+1=5$. So $s(H_5)=5$. As in the previous case, we can use the natural extensions of the center-tail system S_5 to prove that $s(H_{3n+2}) = 2n+3$ for each $n \in \{0\} \cup \mathbb{N}$.

Case 3: $n = 3n+3$. The hexagonal grid H_3 and the center-tail system S_3 are shown in Figure 15. The center of S_3 consists of three vertices, labeled C. The tails for the system are drawn with thick lines. The assignments of opposite tails to the outer edges are natural. The tail whose tip points to the right, upward and downward is assigned to the outer edges on the left, the bottom and the right, respectively (see Figure 15). The congestion indicator $\mathrm{CI}(D_3)$ for H_3 is determined as the minimum of the following three numbers defined in Definition 4: (i) $1+1+1=3$, (ii) $3+1=4$ and (iii) $2+1+1=4$. Hence, by Theorem 5, $s(H_3) \geq \mathrm{CI}(S_3) = 3$. According to Theorem 7, $s(H_3) \leq 1+1+1=3$ (see Figure 15). So $s(H_3)=3$.

The graph H_6 can be obtained by adding a row on each side of the graph H_3. The configuration of the center-tail system S_6 is shown in Figure 16, where the

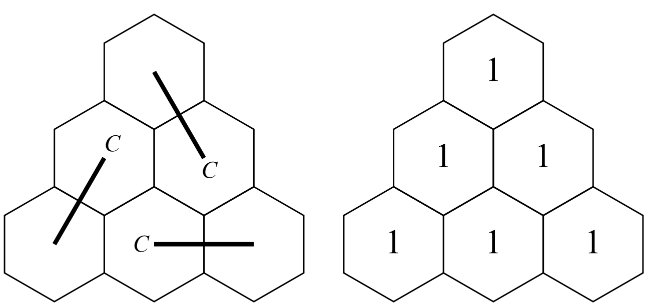

Figure 15. Left: H_3 with center-tail system S_3. Right: absolute indices for H_3.

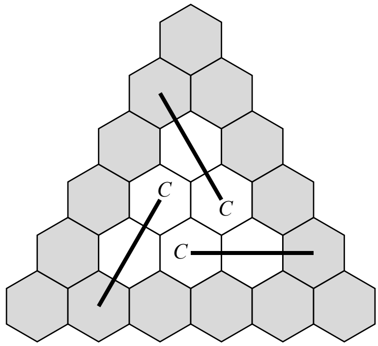

Figure 16. H_6 with center-tail system S_6. The shaded region represents the additional rows added on each side of H_3.

assignment of opposite tail to the outer edges is done in the obvious and natural way, that is, each tail is assigned to the outer edges in the opposite direction. By Definition 4, $\mathrm{CI}(S_6)$ is the minimum of (i) $2+2+1=5$, (ii) $5+1=6$ and (iii) $3+2+1=6$. By Theorem 5, $s(H_6) \geq \mathrm{CI}(S_6) = 5$, and by Theorem 7, $s(H_6) \leq 2+2+1=5$. So $s(H_6)=5$. Using induction (as in the previous cases) we get $s(H_{3n+3}) = 2n+3$ for each $n \in \{0\} \cup \mathbb{N}$. This concludes our proof of Theorem 22. □

5.3. ***Rectangular grids.*** Let $R_{m,n}$ denote the rectangular grid consisting of m horizontal lines and n vertical lines. The purpose of this section is to show that center-tail systems can be used to prove the following result of Hruska.

Theorem 23 [Hruska 2008]. *Suppose $m < n$, where m and n are natural numbers. Then*

$$s(R_{m,n}) = \begin{cases} m & \text{if } m \text{ is odd,} \\ m+1 & \text{if } m \text{ is even.} \end{cases}$$

Proof. Case 1: *m is odd.*

Subcase 1: *n is also odd.* As an instructive example, we consider $R_{5,7}$ with the center-tail system $S_{5,7}$ as shown in Figure 17. The center of $S_{5,7}$ consists of four vertices, labeled C. Each tail is assigned to the diagonally opposite outer edges, for example, the tail which is on the left half of the graph and whose tip points upward is assigned to the outer edges at the bottom of the right half of the graph.

The three numbers corresponding (according to Definition 4) to the center-tail system $S_{m,n}$, $m < n$, m and n are odd, are (i) $(m-1)/2+(m-1)/2+1=m$; (ii) $m+1$; and (iii) $m+1$. Hence, $\mathrm{CI}(S_{m,n}) = m$ in the described case. Then by Theorem 5, $s(R_{m,n}) \geq m$. On the other hand, by Theorem 7, the values of the absolute indices (see Figure 17 for the absolute indices in the case $R_{5,7}$) imply that $s(R_{m,n}) \leq m$. Thus $s(R_{m,n}) = m$ if both m and n are odd and $m < n$.

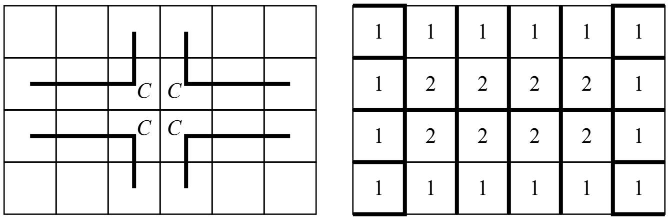

Figure 17. Left: $R_{5,7}$ with center-tail system $S_{5,7}$. Right: absolute indices for $R_{5,7}$ and a minimum spanning congestion tree for $R_{5,7}$.

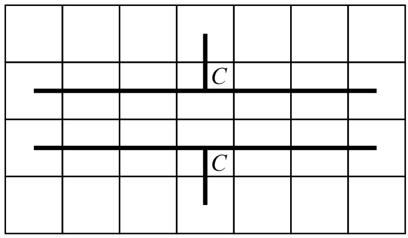

Figure 18. $R_{5,8}$ with center-tail system $S_{5,8}$.

Subcase 2: *m is odd and n is even*. An instructive example of this type is shown in Figure 18.

We assign the tail pointing upward and downward to the outer edges at the bottom and the top, respectively. Assignments of the tails to the vertical outer edges are the same as before. It is easy to see that the congestion indicator of this center-tail system is equal to 5 in the case of $R_{5,8}$ and m in general. Also, it is easy to see that computing the absolute indices (as in Figure 17), we get that $s(R_{m,n}) = m$ in Subcase 2.

Case 2: *m is even*. As instructive examples, we consider the cases $R_{6,9}$ and $R_{6,10}$ (see Figure 19). The center-tail systems $S_{6,9}$ and $S_{6,10}$ are also shown in Figure 19. The centers of $S_{6,9}$ and $S_{6,10}$ consist of two vertices, labeled C. The tails S, T, U, V are assigned to the outer edges in the regions $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, $\mathcal{V}$, respectively. Finally, the tail whose tip points to the left and the right is assigned to those outer edges on the right and the left, respectively. In this case, the three numbers defined in Definition 4 are (i) $3+3+1=7$, (ii) $6+1=7$ and (iii) $6+1=7$. So $\mathrm{CI}(S_{6,9}) = 7$ and hence, by Theorem 5, $s(R_{6,9}) \geq 7$ and $s(R_{6,10}) \geq 7$. By Theorem 7, we have $s(R_{6,9}) \leq 7$ and $s(R_{6,10}) \leq 7$. Thus $s(R_{6,9}) = s(R_{6,10}) = 7$. Observe that the length of the longest side of the rectangular grid does not play an important role in this computation, since we assume $n > m$. It is clear that similar center-tail systems can be used to show that $s(R_{m,n}) = m+1$ for any even m and any n satisfying $n > m$. □

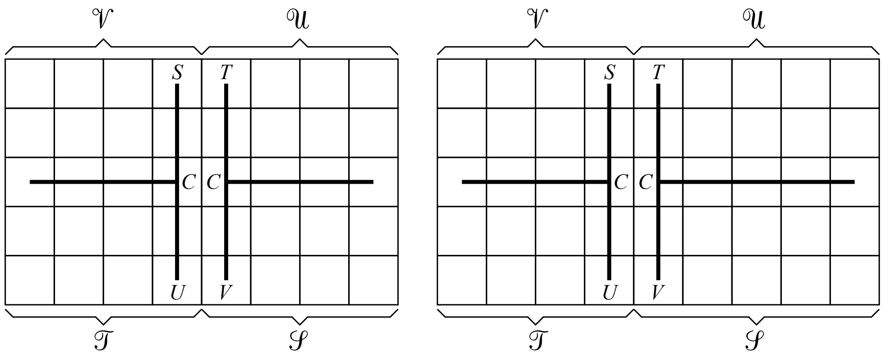

Figure 19. Left: $R_{6,9}$ with center-tail system $S_{6,9}$. Right: $R_{6,10}$ with center-tail system $S_{6,10}$.

References

[Bodlaender et al. 2011] H. L. Bodlaender, K. Kozawa, T. Matsushima, and Y. Otachi, "Spanning tree congestion of k-outerplanar graphs", *Discrete Math.* **311**:12 (2011), 1040–1045. MR 2012d:05105 Zbl 1223.05017

[Bodlaender et al. 2012] H. L. Bodlaender, F. V. Fomin, P. A. Golovach, Y. Otachi, and E. J. van Leeuwen, "Parameterized complexity of the spanning tree congestion problem", *Algorithmica* **64**:1 (2012), 85–111. MR 2923963 Zbl 1253.68163

[Bollobás and Leader 1991] B. Bollobás and I. Leader, "Edge-isoperimetric inequalities in the grid", *Combinatorica* **11**:4 (1991), 299–314. MR 92m:51042 Zbl 0755.05045

[Castejon et al. 2007] A. Castejon, E. Corbacho, and R. Vidal, "Árboles óptimos en rejillas", pp. 79–86 in *Avances en matemática discreta en Andalucía* (Cádiz, 2007), edited by J. C. Camacho Moreno et al., Serv. Publ. Univ. Cádiz, Cádiz, 2007. MR 2401733 Zbl 1193.05057

[Clark and Holton 1991] J. Clark and D. A. Holton, *A first look at graph theory*, World Scientific, Teaneck, NJ, 1991. MR 92i:05077 Zbl 0765.05001

[Diestel 2000] R. Diestel, *Graph theory*, 2nd ed., Graduate Texts in Mathematics **173**, Springer, New York, 2000. MR 1743598 Zbl 0945.05002

[Egawa and Inoue 1999] Y. Egawa and K. Inoue, "Radius of $(2k-1)$-connected graphs", *Ars Combin.* **51** (1999), 89–95. MR 99j:05061 Zbl 0977.05040

[Fekete and Kremer 2001] S. P. Fekete and J. Kremer, "Tree spanners in planar graphs", *Discrete Appl. Math.* **108**:1-2 (2001), 85–103. MR 2003d:05048 Zbl 0969.68111

[Harant 1993] J. Harant, "An upper bound for the radius of a 3-connected graph", *Discrete Math.* **122**:1-3 (1993), 335–341. MR 94i:05046 Zbl 0787.05055

[Harant and Walther 1981] J. Harant and H. Walther, "On the radius of graphs", *J. Combin. Theory Ser. B* **30**:1 (1981), 113–117. MR 82e:05081 Zbl 0483.05039

[Hruska 2008] S. W. Hruska, "On tree congestion of graphs", *Discrete Math.* **308**:10 (2008), 1801–1809. MR 2009b:05069 Zbl 1151.05014

[Iida 2007] T. Iida, "Order and radius of 3-connected graphs", *AKCE Int. J. Graphs Comb.* **4**:1 (2007), 21–49. MR 2008c:05047 Zbl 1138.05018

[Iida and Kobayashi 2006] T. Iida and K. Kobayashi, "Order and radius of $(2k-1)$-connected graphs", *SUT Journal of Math.* **42**:2 (2006), 335–356. MR 2010a:05073 Zbl 1132.05017

[Inoue 1996] K. Inoue, "Radius of 3-connected graphs", *SUT Journal of Math.* **32**:1 (1996), 83–90. MR 97d:05089 Zbl 0890.05025

[Jordan 1869] C. Jordan, "Sur les assemblages de lignes", *J. Reine Angew. Math.* **70** (1869), 185–190. JFM 02.0344.01

[Khuller et al. 1993] S. Khuller, B. Raghavachari, and N. Young, "Designing multi-commodity flow trees", pp. 433–441 in *Algorithms and data structures* (Montréal, 1993), edited by F. Dehne et al., Lecture Notes in Comput. Sci. **709**, Springer, Berlin, 1993. MR 1257430 Zbl 0803.68087

[Law and Ostrovskii 2010] H.-F. Law and M. I. Ostrovskii, "Spanning tree congestion: duality and isoperimetry, with an application to multipartite graphs", *Graph Theory Notes N. Y.* **58** (2010), 18–26. MR 2012j:05238

[Lovász 2007] L. Lovász, *Combinatorial problems and exercises*, 2nd ed., AMS Chelsea Publishing, Providence, RI, 2007. MR 2321240 Zbl 1120.05001

[Löwenstein 2010] C. Löwenstein, *In the complement of a dominating set*, PhD thesis, Technische Universität, Ilmenau, 2010, Available at http://tinyurl.com/LowensteinPhD.

[Nishizeki and Chiba 1988] T. Nishizeki and N. Chiba, *Planar graphs: theory and algorithms*, North-Holland Math. Studies **140**, North-Holland, Amsterdam, 1988. MR 89f:05068 Zbl 0647.05001

[Ostrovskii 2004] M. I. Ostrovskii, "Minimal congestion trees", *Discrete Math.* **285**:1-3 (2004), 219–226. MR 2005f:05111 Zbl 1051.05032

[Ostrovskii 2010] M. I. Ostrovskii, "Minimum congestion spanning trees in planar graphs", *Discrete Math.* **310**:6-7 (2010), 1204–1209. MR 2010m:05082 Zbl 1230.05112

[Otachi 2011] Y. Otachi, "Designing low-congestion networks with structural graph theory", *Interdiscip. Inform. Sci.* **17**:3 (2011), 197–216. MR 2012k:05371 Zbl 1247.68199

[Otachi et al. 2010] Y. Otachi, H. L. Bodlaender, and E. J. van Leeuwen, "Complexity results for the spanning tree congestion problem", pp. 3–14 in *Graph-theoretic concepts in computer science* (Zarós, 2010), edited by D. M. Thilikos, Lecture Notes in Comput. Sci. **6410**, Springer, Berlin, 2010. MR 2012c:05222 Zbl 05816303

[Peleg 2000] D. Peleg, *Distributed computing: a locality-sensitive approach*, SIAM Monographs on Discrete Mathematics and Applications **5**, Society for Industrial and Applied Mathematics, Philadelphia, 2000. MR 2001g:68010 Zbl 0959.68042

[Rosen et al. 2000] K. H. Rosen, J. G. Michaels, J. L. Gross, J. W. Grossman, and D. R. Shier (editors), *Handbook of discrete and combinatorial mathematics*, CRC Press, Boca Raton, FL, 2000. MR 2000g:05001 Zbl 1044.00002

[Simonson 1987] S. Simonson, "A variation on the min cut linear arrangement problem", *Math. Systems Theory* **20**:4 (1987), 235–252. MR 89g:05065 Zbl 0643.68094

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