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New results on an anti-Waring problem

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The number $N(k, r)$ is defined to be the first integer such that it and every subsequent integer can be written as the sum of the k-th powers of r or more distinct positive integers. For example, it is known that $N(2, 1) = 129$, and thus the last number that cannot be written as the sum of one or more distinct squares is 128. We give a proof of a theorem that states if certain conditions are met, a number can be verified to be $N(k, r)$. We then use that theorem to find $N(2, r)$ for $1 \leq r \leq 50$ and $N(3, r)$ for $1 \leq r \leq 30$.

1. Introduction

In 1770, Waring conjectured that for each positive integer k there exists a $g(k)$ such that every positive integer is a sum of $g(k)$ or fewer k-th powers of positive integers. After Hilbert proved this theorem true in 1909, the challenge that became known as Waring's problem was the question that asks, for each k, what is the smallest $g(k)$ such that the statement holds. For more information on Waring's problem, see [Weisstein].

Recently, two papers have tackled the following "anti-Waring" conjecture: *If k and r are positive integers, then every sufficiently large positive integer is the sum of r or more k-th powers of distinct positive integers.*

The fact that there must be r *or more* k-th powers motivated the choice of the designation *anti-Waring* in [Johnson and Laughlin 2011], where the conjecture was put forth. What sets this statement apart from Waring's problem is the word "distinct". The conjecture was later proved in [Looper and Saritzky 2012]. A natural anti-Waring problem arising from this proven conjecture is to find the smallest integer $N(k, r)$ such that it and every subsequent integer can be written as the sum of r or more k-th powers of distinct positive integers. Johnson and Laughlin proved that $N(2, 1) = N(2, 2) = N(2, 3) = 129$.

The following results are restricted to the case when $k = 2$ and $k = 3$. $N(2, r)$ is the smallest integer such that it and every subsequent integer can be written as the

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sum of r or more distinct squares. $N(2, r)$ has been found for $1 \le r \le 50$. $N(3, r)$ is the smallest integer such that it and every subsequent integer can be written as the sum of r or more distinct cubes. $N(3, r)$ has been found for $1 \le r \le 30$. For the purposes of this paper we use two definitions.

Definitions. An integer is (k, r)-*good* if it *can* be written as the sum of r or more k-th powers of distinct positive integers. An integer is (k, r)-*bad* if it *cannot* be written as the sum of r or more k-th powers of distinct positive integers.

To see an example of this idea, consider the case when $k = 2$ and $r = 4$. Since 129 can be written as $2^2 + 3^2 + 4^2 + 10^2$, 129 is $(2, 4)$-good. However, it is a brief exercise to verify that there is no way to write 128 as the sum of four or more distinct squares, and hence 128 is $(2, 4)$-bad. The fact that 129 is $(2, 4)$-good also directly implies that it is $(2, r)$-good for any integer $1 \le r \le 4$. Using these definitions, the problem of finding $N(2, r)$ can be reworded to be the problem of finding the first $(2, r)$-good integer such that every subsequent integer is also $(2, r)$-good. In the case when $r = 4$, the fact that 128 is $(2, 4)$-bad implies that $N(2, 4) \ge 129$.

As will be seen, an inductive argument used in the following theorems requires a consecutive list of (k, r)-good integers whose size grows as r does. Computer software was used to attain these large lists of (k, r)-good integers as well as to verify that certain key integers are in fact (k, r)-bad.

2. Results

Before stating the general result of this paper, it may be helpful to offer a less general theorem and proof that will serve as valuable context for Theorem 2.2.

Theorem 2.1. $N(2, 4) = 129.$

Proof. As shown previously, $N(2, 4) \ge 129$. It is also true that the consecutive integers $\{129, \ldots, 18^2\}$ are $(2, 4)$-good. Therefore, if $n \le 18^2$ and n is $(2, 4)$-bad, then $n \le 128$. The rest of the proof continues by induction on m with $m \ge 18$.

The induction statement: If $n \le m^2$ and n is $(2, 4)$-bad, then $n \le 128$. If $m = 18$, the statement is clearly true as we know the consecutive integers $\{129, \ldots, 18^2\}$ are $(2, 4)$-good.

Now suppose $n \le (m+1)^2$ and n is $(2, 4)$-bad. If $n \le m^2$, then by the induction hypothesis, $n \le 128$. Thus we can say

$$(m+1)^2 \ge n \ge m^2 + 1. \tag{1}$$

Consider the integer $n - (m-4)^2$. From (1) and the fact that $m \ge 18$, we know that

$$m^2 \ge n - (m-4)^2 \ge m^2 + 1 - (m-4)^2 \ge 129. \tag{2}$$

To see that $n-(m-4)^2$ is (2, 4)-bad, suppose that it is (2, 4)-good and hence

$$n-(m-4)^2 = a_1^2 + a_2^2 + \cdots + a_t^2 \quad \text{with } t \geq 4,\ a_i \neq a_j \text{ for all } i \text{ and } j,$$

or

$$n = a_1^2 + a_2^2 + \cdots + a_t^2 + (m-4)^2.$$

Since n is (2, 4)-bad, there is some $j \in \{1, 2, \ldots, t\}$ such that $a_j = (m-4)$. Therefore

$$n-(m-4)^2 \geq 1^2 + 2^2 + 3^2 + (m-4)^2,$$

and equivalently, $n - m^2 \geq m^2 - 16m + 46$.

Combining this with (1), we get

$$(m+1)^2 \geq n \geq 2m^2 - 16m + 46$$

or

$$0 \geq m^2 - 18m + 45,$$

which is untrue when $m \geq 18$. Therefore $n-(m-4)^2$ must be (2, 4)-bad, and by (2) and the inductive hypothesis, $n-(m-4)^2 \leq 128$. However, this is a contradiction since by (2) it is also true that $n-(m-4)^2 \geq 129$, and thus there are no n that are (2, 4)-bad and satisfy (1). □

In Theorem 2.1, 129 was the expected result for $N(2, 4)$ after using computer software to generate a long list of consecutive (2, 4)-good integers that began with 129. The aim of Theorem 2.2 is to offer a theorem such that under given conditions, expected results for $N(k, r)$ can be proven for any positive integers k and r. To simplify the notation $S_k(z)$ will be used to represent $\sum_{i=1}^{z} i^k$.

Theorem 2.2. *If the consecutive integers $\{\hat{N}(k, r), \ldots, b^k\}$ are all (k, r)-good, $\hat{N}(k, r) - 1$ is (k, r)-bad, and if there exists an integer x such that*

(i) $0 < S_k(r-1) + 2(m-x)^k - (m+1)^k$ *for all* $m \geq b$,

(ii) $(m+1)^k - (m-x)^k \leq m^k$ *for all* $m \geq b$,

(iii) $m^k + 1 - (m-x)^k \geq \hat{N}(k, r)$ *for all* $m \geq b$, *and*

(iv) $0 < x < b - r$,

then $\hat{N}(k, r) = N(k, r)$.

Proof. We use induction on $m \in \mathbb{N}$ with $m \geq b$. The induction statement: If $n \leq m^k$ and n is (k, r)-bad, then $n \leq \hat{N}(k, r) - 1$.

If $m = b$, the statement is clearly true as we know the consecutive integers $\{\hat{N}(k, r), \ldots, b^k\}$ are all (k, r)-good.

Now suppose $n \leq (m+1)^k$ and n is (k, r)-bad. If $n \leq m^k$, then by the induction hypothesis, $n \leq \hat{N}(k, r) - 1$. Thus we can say

$$(m+1)^k \geq n \geq m^k + 1. \tag{3}$$

We will show that n cannot satisfy (3), and hence all cases have been addressed.

Consider the integer $n - (m - x)^k$. Using (3) and condition (iii), we know that

$$n - (m - x)^k \geq m^k + 1 - (m - x)^k \geq \hat{N}(k, r)$$

or

$$n - (m - x)^k \geq \hat{N}(k, r). \tag{4}$$

To see that $n - (m - x)^k$ is (k, r)-bad, suppose it is (k, r)-good. Then

$$n - (m - x)^k = a_1^k + a_2^k + \cdots + a_t^k \quad \text{with } t \geq r,\ a_i \neq a_j \text{ for all } i \neq j,$$

or

$$n = a_1^k + a_2^k + \cdots + a_t^k + (m - x)^k.$$

Since n is (k, r)-bad, $a_j = m - x$ for some $j \in \{1, 2, \ldots, t\}$. This, along with condition (iv), implies that $n - (m - x)^k \geq S_k(r - 1) + (m - x)^k$. Combining this with (3), we get

$$(m+1)^k \geq n \geq S_k(r - 1) + 2(m - x)^k,$$

or

$$0 \geq S_k(r - 1) + 2(m - x)^k - (m + 1)^k.$$

This contradiction of condition (i) means $n - (m - x)^k$ must be (k, r)-bad.

Now from (3) and condition (ii),

$$n - (m - x)^k \leq (m + 1)^k - (m - x)^k \leq m^k.$$

Thus by the induction hypothesis, $n - (m - x)^k \leq \hat{N}(k, r) - 1$. This contradicts (4) and means that there are no n that are (k, r)-bad and satisfy (3). □

As a result of Theorem 2.2, in order to find $N(k, r)$ one must simply find a suitable list of (k, r)-good consecutive integers $\{\hat{N}(k, r), \ldots, b^k\}$ such that $\hat{N}(k, r) - 1$ is (k, r)-bad and an integer x that satisfies the four conditions of the theorem. It is this strategy that gives way to the tables of values in Theorems 2.3 and 2.4. Again, computer software was a valuable tool in determining whether a given number was (k, r)-good or (k, r)-bad for $k \in \{2, 3\}$. For each r in the following two theorems, corresponding values for x and b are listed in Tables 1 and 2 rather than in the proof of the theorem.

Theorem 2.3. *Table 1 is a list of* $N(2, r)$ *for integers* $1 \leq r \leq 50$.

r	$N(2,r)$	x	b	r	$N(2,r)$	x	b	r	$N(2,r)$	x	b	r	$N(2,r)$	x	b
1	129	4	18	14	1398	19	47	27	7953	54	101	40	23679	100	169
2	129	4	18	15	1723	21	52	28	8677	57	105	41	25348	104	174
3	129	4	18	16	1991	24	54	29	9538	61	109	42	27208	108	180
4	129	4	18	17	2312	26	58	30	10394	63	114	43	29093	112	186
5	198	6	22	18	2673	28	62	31	11559	67	120	44	31229	116	193
6	238	6	23	19	3048	31	65	32	12603	71	125	45	33298	120	199
7	331	8	26	20	3493	34	69	33	13744	74	130	46	35290	123	205
8	383	9	27	21	4094	36	75	34	14864	78	135	47	37654	127	212
9	528	10	32	22	4614	39	79	35	16253	81	141	48	40043	132	218
10	648	12	33	23	5139	42	83	36	17529	85	146	49	42488	135	225
11	889	14	39	24	5719	44	87	37	18958	89	151	50	45024	140	231
12	989	15	41	25	6380	48	91	38	20482	92	158				
13	1178	17	44	26	7124	51	96	39	22043	96	163				

Table 1. For each r listed, $N(2,r)-1$ is $(2,r)$-bad, and the list of consecutive integers $\{N(2,r),\ldots,b^2\}$ is $(2,r)$-good. The three necessary conditions of Theorem 2.2 are satisfied by x.

Proof. For $1 \le r \le 4$, $N(2,r) = 129$ by [Johnson and Laughlin 2011] and Theorem 2.1. For each r, $N(2,r)-1$ has been shown to be $(2,r)$-bad. There exist b and x such that the consecutive integers $\{N(2,r),\ldots,b^2\}$ are $(2,r)$-good, and x satisfies the four conditions of Theorem 2.2. □

Theorem 2.4. *Table 2 is a list of $N(3,r)$ for integers $1 \le r \le 30$.*

Proof. For each r, $N(3,r)-1$ has been shown to be $(3,r)$-bad. There exist b and x such that the consecutive integers $\{N(3,r),\ldots,b^3\}$ are $(3,r)$-good, and x satisfies the four conditions listed in Theorem 2.2. □

r	$N(3,r)$	x	b	r	$N(3,r)$	x	b	r	$N(3,r)$	x	b	r	$N(3,r)$	x	b
1	12759	5	32	9	16224	6	33	17	56076	11	47	25	179520	18	67
2	12759	5	32	10	18149	6	35	18	66534	12	50	26	201921	19	69
3	12759	5	32	11	22398	7	37	19	75912	12	52	27	227400	20	72
4	12759	5	32	12	24855	7	38	20	87567	13	54	28	256254	22	73
5	12759	5	32	13	28887	8	39	21	101093	14	56	29	289869	23	76
6	15279	6	33	14	36951	9	42	22	122064	15	60	30	325590	24	79
7	15279	6	33	15	39660	9	43	23	138696	16	62				
8	15279	6	33	16	49083	10	46	24	156498	17	64				

Table 2. For each r listed, $N(3,r)-1$ is $(3,r)$-bad, and the list of consecutive integers $\{N(3,r),\ldots,b^3\}$ is $(3,r)$-good. The three necessary conditions of Theorem 2.2 are satisfied by x.

3. Future work

The list of values of $N(k, r)$ can be extended indefinitely for any value of k. Currently we are only limited by our computing speed. A natural direction for further research would be to attempt to find an explicit formula for $N(k, r)$ for a specific k. In [Johnson and Laughlin 2011], it was noticed that $N(1, r) = r(r+1)/2$. However, we have not found a formula for $N(2, r)$ or $N(3, r)$.

Another area that seems natural is to attempt to find $N(k, r)$ for values of k greater than 3. We have attempted to use our current software to find $N(4, 1)$ and $N(5, 1)$, but our methods appear to be too inefficient. At this point, all that can be said confidently is that $N(4, 1)$ is greater than 4.3 million, $N(5, 1)$ is greater than 26.25 million, and perhaps they are both much larger.

It is also clear that $N(k, i) \leq N(k, j)$ when $i \leq j$, and it seems natural to conjecture that $N(x, r) \leq N(y, r)$ when $x \leq y$. Since $N(1, r) = (r(r+1))/2$, $N(1, r) \leq S_k(r) \leq N(k, r)$ for any integer $k \geq 1$. However, it is possible for an integer that it is (k, r)-bad to be (l, r)-good with $k < l$. For example, 9 is (2, 2)-bad but (3, 2)-good. Thus, a proof of this conjecture eludes us currently.

Note. After finishing this paper, it was brought to our attention that [Deering and Jamieson] had recently been submitted for publication. This paper has some of the same results as ours. In particular, our method of discovering $N(k, r)$, with proof, is very much like that of Deering and Jamieson. However, we feel that our method is sufficiently different and easier to use to merit publication.

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