involve
a journal of mathematics

Investigating root multiplicities in the indefinite Kac–Moody algebra E_{10}

Vicky Klima, Timothy Shatley, Kyle Thomas and Andrew Wilson

msp

2014 vol. 7, no. 4

INVOLVE 7:4 (2014)
dx.doi.org/10.2140/involve.2014.7.529

Investigating root multiplicities in the indefinite Kac–Moody algebra E_{10}

Vicky Klima, Timothy Shatley, Kyle Thomas and Andrew Wilson

(Communicated by Jim Haglund)

Following a procedure outlined by Kang, we view the generalized eigenspaces, known as root spaces, of the infinite dimensional Kac–Moody algebra E_{10} as generalized eigenspaces for representations of the finite dimensional special linear algebra A_9. Then, using the combinatorial representation theory of the special linear Lie algebras, we determine the dimensions of certain root spaces in E_{10}.

1. Introduction

For the past forty years, Kac–Moody algebras have been a rich area of study due to their numerous applications to other areas of mathematics and physics. Kac–Moody algebras are of one of three types (i) finite, (ii) affine, or (iii) indefinite. While the root multiplicities of finite and affine Kac–Moody algebras are well known [Kac 1990], we still do not have a general knowledge of root multiplicities in indefinite type algebras.

Building on the work of Feingold and Frenkel [1983], Kac, Moody, and Wakimoto [Kac et al. 1988] studied root multiplicities of the indefinite algebra E_{10} by considering this algebra as an extension of the affine algebra E_9. Later, Kang [1993a] developed a general construction for a class of indefinite algebras in which he built the larger algebra from a related affine or finite algebra and certain modules over that algebra. Kang's construction has been used to study the multiplicities of the indefinite algebras $HA_n^{(1)}$ [Benkart et al. 1995; Kang 1993b; 1994b; 1994a; Kang and Melville 1994; Hontz and Misra 2002a], $HC_n^{(1)}$ [Klima and Misra 2008], $HG_2^{(1)}$[Hontz and Misra 2002b] and $HD_4^{(3)}$ [Hontz and Misra 2002b], as well as $EHA_1^{(1)}$ and $EHA_2^{(2)}$ [Sthanumoorthy and Uma Maheswari 2012]. In this paper, as in [Kac et al. 1988], we consider the multiplicities of the indefinite algebra E_{10}.

MSC2010: 17B67.
Keywords: Kac–Moody, representation theory, combinatorial representation theory, root multiplicity.
This research was partially supported by a CURM mini-grant funded by the NSF grant DMS-063664. Vicky Klima would like to thank Kailash Misra for introducing her to this problem and Jennifer Hontz for her helpful introduction to the programming necessary to complete the project.

However, in applying Kang's construction we choose to build E_{10} not from the infinite-dimensional E_9 but rather from the finite-dimensional simple algebra A_9. Using a multiplicity formula also due to Kang [1994b] along with the combinatorial representation theory for A_9 we determine multiplicities for roots up to degree -5. We recover some of the results in [Kac et al. 1988] and develop a recursive procedure to extend these results.

2. Background

A *Lie algebra* over $\mathbb{C}$ is a vector space $\mathfrak{g}$ over $\mathbb{C}$, with an antisymmetric, bilinear operation $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, called the bracket, such that the following property — the Jacobi identity — holds: $[a,[b,c]]=[b,[a,c]]+[[a,b],c]$ for all $a,b,c\in\mathfrak{g}$.

Example 1. The Lie algebra of 2×2 trace zero complex matrices with the commutator bracket, $[A,B]=AB-BA$, is know as $sl(2,\mathbb{C})$. This Lie algebra has basis

$$\left\{e=\begin{pmatrix}0&1\\0&0\end{pmatrix}, f=\begin{pmatrix}0&0\\1&0\end{pmatrix}, h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}$$

and relations $[e,f]=h,\ [h,e]=2e,\ [h,f]=-2f$.

Any subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is a vector space over $\mathbb{C}$ and thus a linear functional α on $\mathfrak{s}$ is simply a linear function that assigns to each element of $\mathfrak{s}$ a corresponding complex number. The set of all linear functionals on $\mathfrak{s}$ is itself a vector space over $\mathbb{C}$, denoted $\mathfrak{s}^*$. A Lie algebra $\mathfrak{g}$ is $\mathfrak{s}$-*diagonalizable* if $\mathfrak{g}$ can be written as a direct sum of subspaces

$$\mathfrak{g}_\alpha=\left\{g\in\mathfrak{g}\mid [s,g]=\alpha(s)g \text{ for all } s\in\mathfrak{s}\right\}.$$

Example 2. The Lie algebra $\mathfrak{g}=sl(2,\mathbb{C})$ introduced in Example 1 is diagonalizable over its subalgebra $\mathfrak{h}=\{h\}$. Let $\alpha\in\mathfrak{h}^*$ be defined by $\alpha(h)=2$. Then

$$\mathfrak{g}_\alpha=\left\{g\in\mathfrak{g}\mid [h,g]=2g \text{ for all } h\in\mathfrak{h}\right\}=\operatorname{span}\{e\}.$$

Similarly, $\mathfrak{g}_0=\operatorname{span}\{h\}=\mathfrak{h}$ and $\mathfrak{g}_{-\alpha}=\operatorname{span}\{f\}$. Therefore $\mathfrak{g}$ is $\mathfrak{h}$-diagonalizable with decomposition $\mathfrak{g}=\mathfrak{g}_\alpha\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{-\alpha}$.

We say a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is *abelian* if $[a,b]=0$ for all $a,b\in\mathfrak{s}$. A *Cartan subalgebra*, $\mathfrak{h}$, of $\mathfrak{g}$ is a maximal abelian subalgebra of $\mathfrak{g}$ such that $\mathfrak{g}$ is diagonalizable over $\mathfrak{h}$. Given a Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ we define the *roots* of $\mathfrak{g}$ as those nonzero $\alpha\in\mathfrak{h}^*$ such that

$$\mathfrak{g}_\alpha=\{g\in\mathfrak{g}\mid [h,g]=\alpha(h)g \text{ for all } h\in\mathfrak{h}\}\neq 0.$$

In this case we call $\mathfrak{g}_\alpha$ the *root-space* associated with the root α and $\dim(\mathfrak{g}_\alpha)$ the *multiplicity* of the root α.

The decomposition of $sl(2, \mathbb{C})$ given in Example 2 is a root-space decomposition for this algebra because the subalgebra $\mathfrak{h} = \text{span}\{h\}$ is a Cartan subalgebra of $sl(2, \mathbb{C})$. The root spaces in this example are simply the eigenspaces for $\mathfrak{g}$ relative to $\mathfrak{h}$. In general, when the Cartan subalgebra $\mathfrak{h}$ is of dimension greater than one the root spaces of $\mathfrak{g}$ are generalized eigenspaces for $\mathfrak{g}$ relative to $\mathfrak{h}$.

2.1. ***Cartan matrices and the Weyl group.*** The algebra $sl(2, \mathbb{C})$ is a member of the family of finite dimensional simple Lie algebras $A_n = sl(n+1, \mathbb{C})$, where each algebra A_n consists of the $(n+1) \times (n+1)$ trace-zero complex matrices under the commutator bracket. This family is one of four families of classical finite dimensional simple Lie algebras, each of which can be modeled as collections of familiar types of matrices.

In the late nineteenth century, William Killing and Élie Cartan showed that these four classical families along with five exceptional families were the only finite-dimensional simple Lie algebras. Given a finite-dimensional simple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h} = \text{span}\{h_i\}_{i=1}^n$ of dimension n, they described the root-system of $\mathfrak{g}$ using a linearly independent set of *simple roots*,

$$\Pi = \{\alpha_i\}_{i=1}^n \subseteq \mathfrak{h}^*,$$

and recorded defining information for the simple roots in a Cartan matrix

$$A_{n\times n} = (a_{ij}) \text{ with } a_{ij} = \alpha_j(h_i). \tag{1}$$

Let $\mathfrak{g}(A)$ be the Lie algebra with Cartan matrix A, Cartan subalgebra $\mathfrak{h}$ and simple roots $\{\alpha_i\}_{i=1}^n$. For each $i \in \{1, 2, \dots, n\}$ define the *simple reflection* r_i on $\mathfrak{h}^*$ by $r_i(\lambda) = \lambda - \lambda(h_i)\alpha_i$. The *Weyl group* associated with A is the group generated by all simple reflections and for finite Lie algebras this group is also finite. If

$$\omega = \prod_{k=1}^{t} r_{i_k},$$

where t is minimal amongst all such expressions we say ω is a reduced expression and call t the length of ω, denoted $\ell(\omega)$. We can recover the set of all roots of the finite-dimensional simple Lie algebra $\mathfrak{g}(A)$ by letting the Weyl group associated with A act on the set of simple roots. Each root space in a finite-dimensional simple Lie algebra, $\mathfrak{g}(A)$, is one-dimensional and therefore understanding the root system is equivalent to understanding the algebra itself. Hence, classifying the finite-dimensional simple Lie algebras amounts to classifying their Cartan matrices. See [Berman and Parshall 2002] for an excellent source on the historical development of Kac–Moody algebras beginning with Killing and Cartan's work on the classification of the finite-dimensional simple Lie algebras.

Each Cartan matrix as given in (1) has the following properties, where the indices range from 1 through n:

$$a_{ii} = 2 \qquad \text{for all } i. \tag{2}$$

$$a_{ij} \in \mathbb{Z}_{\leq 0} \qquad \text{for } i \neq j. \tag{3}$$

$$a_{ij} = 0 \iff a_{ji} = 0 \qquad \text{for all } i, j. \tag{4}$$

$$\det A \neq 0. \tag{5}$$

$$\text{Each proper principal minor of } A \text{ is positive.} \tag{6}$$

Every indecomposable matrix — that is, every matrix whose rows or columns cannot be permuted to block diagonal form — with these properties is the Cartan matrix (in the sense of (1)) for a unique (up to isomorphism) simple Lie algebra. Therefore we call any indecomposable square matrix with properties (2)–(6) an indecomposable *Cartan matrix*. In 1966, Jean-Pierre Serre developed a presentation of the unique (up to isomorphism) simple Lie algebra corresponding to a given indecomposable Cartan matrix $A_{n\times n} = (a_{ij})$ using generators $\{e_i, h_i, f_i\}_{i=1}^n$ and the following relations:

$$\left.\begin{aligned} [h_i, h_j] &= 0 \\ [h_i, e_j] &= a_{ij} e_j \\ [h_i, f_j] &= -a_{ij} f_j \end{aligned}\right\} \qquad \text{for all } i, j,$$

$$[e_i, f_j] = \begin{cases} h_i & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{7}$$

$$\underbrace{[e_i, \cdots [e_i, [e_i}_{1-a_{ij} \text{ times}}, e_j]] \cdots] = \underbrace{[f_i, \cdots [f_i, [f_i}_{1-a_{ij} \text{ times}}, f_j]] \cdots] = 0 \quad \text{if } i \neq j.$$

In fact, Serre's presentation applied to decomposable Cartan matrices as well, in which case the corresponding semisimple Lie algebra is the direct sum of the simple algebras associated with the indecomposable blocks.

2.2. ***Kac–Moody algebras.*** Kac [1968] and Moody [1968] independently extended Serre's construction to a larger class of algebras, now known as Kac–Moody algebras. These algebras are defined in terms of a generalized Cartan matrix (GCM) which must meet only the conditions (2), (3), and (4) associated with a Cartan matrix. The Kac–Moody algebra $\mathfrak{g}(A)$ defined by GCM $A_{n\times n}$ is the algebra with generators $\{e_i, f_i, h_i\}_{i=1}^n$ subject to Serre's relations. We define the Weyl group associated with a GCM in the same way as that associated with a Cartan matrix; however, the Weyl group associated with a GCM may be infinite dimensional. Additionally, while always finite-dimensional, root spaces in a Kac–Moody algebra may be of dimensional greater than one. Roots of the Kac–Moody algebra $\mathfrak{g}(A)$ that are reflections of one another — that is roots α and α' such that $\alpha' = \omega(\alpha)$ for some ω

in the Weyl group associated with A — are called *conjugate* and conjugate roots have identical multiplicities.

The Kac–Moody algebra associated with indecomposable GCM A is of one of three types: finite, affine, or indefinite. If A is nonsingular and each proper principal minor of A is positive, $\mathfrak{g}(A)$ is of finite type. In this case A is not only a generalized Cartan Matrix, but also a Cartan matrix and thus $\mathfrak{g}(A)$ is a finite-dimensional simple Lie algebra whose root spaces are necessarily one-dimensional. If A is singular but each proper principal minor of A is positive, A is of affine type. While affine algebras are infinite dimensional and contain some roots of multiplicity greater than one, root multiplicities in these algebras are well-understood; see [Kac 1990, Chapter 6] for details. If $\mathfrak{g}(A)$ is neither finite nor affine, it is indefinite. In general root-multiplicities for these infinite dimensional algebras are not well-understood.

Each GCM $A_{n\times n} = (a_{ij})$, of rank ℓ is associated with a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ where the Cartan subalgebra $\mathfrak{h}$ is a $2n-\ell$ dimensional complex vector space and the simple roots $\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subseteq \mathfrak{h}^*$ and simple coroots

$$\Pi^\vee = \{h_1, h_2, \dots, h_n\} \subseteq \mathfrak{h}$$

are such that Π and $\Pi^\vee$ are both linearly independent with $\alpha_j(h_i) = a_{ij}$ for all $i, j \in \{1, 2, \dots, n\}$. All roots of the Kac–Moody algebra $\mathfrak{g}(A)$ are either positive, that is the root can be written as a nonnegative integral linear combination of the simple roots, or negative, that is the root can be written as a nonpositive integral linear combination of the simple roots. Let Δ, Δ_+, and Δ_- represent the set of roots, positive roots, and negative roots respectively. Then, $\mathfrak{g}_+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}$ (resp. $\mathfrak{g}_- = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}$) and we have the triangular decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$.

2.3. *Dynkin diagrams.* The generalized Cartan matrix is often presented in an equivalent, graphical form known as a Dynkin diagram. In this paper we will only consider Lie algebras with symmetric GCM. The *Dynkin diagram* for the Lie algebra $\mathfrak{g}(A)$ with symmetric GCM, $A_{n\times n}$, is a graph with n vertices, each associated with a simple root α_i, in which vertex i is connected to vertex j using $a_{ij}^2 = [\alpha_j(h_i)]^2$ edges for $i \neq j$.

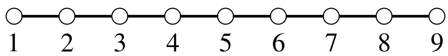
Example 3. The Lie algebra $\mathfrak{g} = A_9 = sl(10, \mathbb{C})$ plays a key role in our work. Let E_{ij} be the 10×10 matrix with (i, j) entry equal to one and all other entries zero. Then $\mathfrak{g}$ is the 99-dimensional algebra generated by

$$\left\{e_i = E_{i,i+1},\ f_i = E_{i+1,i},\ h_i = E_{ii} - E_{i+1,i+1}\right\}_{i=1}^{9},$$

with the nine-dimensional Cartan subalgebra $\mathfrak{h} = \operatorname{span}\{h_i\}_{i=1}^{9}$. Define the linear functionals ϵ_i on $\mathfrak{h}$ by $\epsilon_i(X) = X_{ii}$. Then every root of $\mathfrak{g}$ is a nonnegative or nonpositive integral linear combination of the simple roots $\Pi = \{\alpha_i = \epsilon_i - \epsilon_{i+1}\}_{i=1}^{9}$. Therefore, A_9 has the Cartan matrix $A = (a_{ij})$, where, for $i, j \in \{1, 2, \dots, 8\}$,

$$a_{ij} = \alpha_j(h_i) = \begin{cases} -1 & \text{if } |i-j| = 1, \\ 2 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

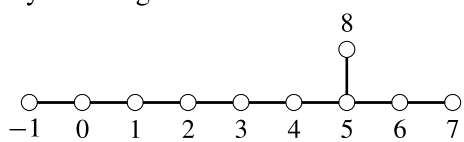
Here is the corresponding Dynkin diagram:



Example 4. We wish to explore root multiplicities in the Kac–Moody algebra E_{10}. Using the standard ordering of the simple roots, E_{10} is associated with the GCM $A = (a_{ij})$, where, for $i, j \in \{-1, 0, 1, \dots, 8\}$,

$$a_{ij} = \alpha_j(h_i) = \begin{cases} -1 & \text{if } |i-j| = 1 \text{ and } i, j \in \{-1, 0, 1, \dots, 7\}, \\ -1 & \text{if } i = 5 \text{ and } j = 8, \\ 2 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

or equivalently with Dynkin diagram



(9)

Note that the Dynkin diagram formed by removing vertex -1 from (9) corresponds to GCM $A' = (a_{ij})$ for $i, j \in \{0, 1, \dots, 8\}$ with a_{ij} given in (8). Since $\det A' = 0$ we see that E_{10} is of indefinite type.

2.4. ***Lie algebra modules.*** A vector space V over $\mathbb{C}$ is a *module* over the Lie algebra $\mathfrak{g}$ if there is a bilinear map from $\mathfrak{g} \times V$ into V given by $(g, v) \to g \cdot v$ such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad \text{for all } x, y \in \mathfrak{g} \text{ and } v \in V. \tag{10}$$

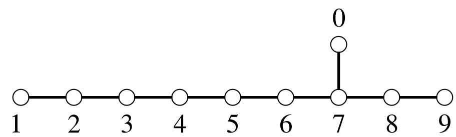
Every Lie algebra $\mathfrak{g}$ is a module over itself via the *adjoint action* $g \cdot v = [g, v]$; in this case (10) is simply the Jacobi identity. In our work we will only deal with modules over finite algebras and the definitions below pertain to such algebras. However, each of these ideas can be extended to all Kac–Moody algebras (see [Kac 1990, Chapter 9]).

Let $\mathfrak{g}$ be a finite Lie algebra with Cartan subalgebra $\mathfrak{h}$. Given any $\lambda \in \mathfrak{h}^*$ the λ *weight space*, V_λ, is defined as $V_\lambda = \{v \in V \mid h \cdot v = \lambda(h) \cdot v \text{ for all } h \in \mathfrak{h}\}$. If $V_\lambda \neq 0$, we call λ a *weight* of V and $\dim(V_\lambda)$ the *weight multiplicity* of λ in V. The $\mathfrak{g}$-module V is a *highest weight module* if there exists a $\lambda \in \mathfrak{h}^*$ and a $v_\lambda \in V$, $v_\lambda \neq 0$, such that $e_i \cdot v_\lambda = 0$ for all $i \in \{1, 2, \dots, n\}$, $h_i \cdot v_\lambda = \lambda(h_i) v_\lambda$ for all $i \in \{1, 2, \dots, n\}$, and V is generated by the images of v_λ under successive applications of the elements

$f_i \in \mathfrak{g}$ where the e_i, f_i, and h_i are the generators of $\mathfrak{g}$ subject to Serre's relations (7). In such a case, we call v_λ a *highest weight vector* and λ the *highest weight* of V.

3. The construction

If we remove vertex 8 from the Dynkin diagram for E_{10} given in (9) we have, up to the labeling of the vertices, the Dynkin diagram for $\mathfrak{g} = A_9$ (with Cartan subalgebra $\mathfrak{h}$). For convenience, we relabel the simple roots of $\tilde{\mathfrak{g}} = E_{10}$ (with Cartan subalgebra $\tilde{\mathfrak{h}}$) according to the following Dynkin diagram:



(11)

With this choice of ordering, restricting the domain of the simple roots α_i $(i = 1, \dots, 9)$ of $\tilde{\mathfrak{g}}$ to $\mathfrak{h}$ gives the corresponding simple roots for $\mathfrak{g}$. Thus we use the same notation for the simple roots in both algebras.

Kang [1993a] introduced a construction for certain indefinite Kac–Moody algebras in which he builds the larger algebra, $\tilde{\mathfrak{g}}$, from a smaller algebra, $\tilde{\mathfrak{g}}_0$, a suitable $\mathfrak{g}_0$-module V, and V^*. In this section we specify Kang's construction to the algebra $\tilde{\mathfrak{g}} = E_{10}$.

As one would expect, $\mathfrak{g} = A_9$ plays an important role in our version of Kang's construction. More specifically, we let $\tilde{\mathfrak{g}}_0 = A_9 + \tilde{\mathfrak{h}}$. and choose the highest weight $\tilde{\mathfrak{g}}_0$-module $V = V(-\alpha_0)$ where $\alpha_0 \in \tilde{\mathfrak{h}}^*$ is given by

$$\alpha_0(h_i) = \begin{cases} 2 & \text{if } i = 0, \\ -1 & \text{if } i = 7, \\ 0 & \text{if } i = 1, 2, 3, 4, 5, 6, 8, 9. \end{cases} \tag{12}$$

The Cartan matrix for $\tilde{\mathfrak{g}}$ is nonsingular and thus the simple coroots $\{h_i\}_{i=0}^9$ form a basis for $\tilde{\mathfrak{h}}$. Since $\{h_i\}_{i=1}^9$ is a basis for the Cartan subalgebra of A_9 we have $\tilde{\mathfrak{g}}_0 = A_9 + \tilde{\mathfrak{h}} = A_9 \oplus \operatorname{span}\{h_0\}$ and any $\tilde{\mathfrak{g}}_0$-module, specifically $V(-\alpha_0)$, will be an A_9-module as well via the restricted module action.

We can realize $\tilde{\mathfrak{g}}$ as

$$\Big(\bigoplus_{i\geq 1} \tilde{\mathfrak{g}}_{-i}\Big) \oplus \tilde{\mathfrak{g}}_0 \oplus \Big(\bigoplus_{i\geq 1} \tilde{\mathfrak{g}}_i\Big), \tag{13}$$

where $\tilde{\mathfrak{g}}_{-1} = V(-\alpha_0)$, $\tilde{\mathfrak{g}}_1 = V(-\alpha_0)^*$, and each subspace $\tilde{g}_{-j}$ (resp. $\tilde{g}_j$) with $j > 1$ is a quotient of the space consisting of all brackets (in the free sense) of j vectors from $V(-\alpha_0)$ (resp. $V(-\alpha_0)^*$). Furthermore, each subspace $\tilde{g}_{\pm j}$ with $j > 1$ is completely reducible as a sum of highest-weight A_9-modules. See [Kang 1993a] for details regarding the construction including the bracket structure for (13).

4. Combinatorial representation theory of A_9

The construction of the previous section allows us to use the combinatorial representation theory of A_9 to study root multiplicities in E_{10}. We say that an A_9-weight $\mu_i = \sum_{i=1}^{10} k_i \epsilon_i$ is *dominant* if and only if $k_1 \geq k_2 \geq \cdots \geq k_{10} \geq 0$. For example, restricting the domain of the weight $-\alpha_0$ as defined in (12) to $\mathfrak{h}$, the Cartan subalgebra of A_9, we find $-\alpha_0 \mid_{\mathfrak{h}} = \epsilon_1 + \epsilon_2 + \cdots + \epsilon_7$, a dominant A_9-weight. We can express the dominant weights, λ, of A_9 using certain ordered sets of positive integers, known as partitions and study weight multiplicities in $V(\lambda)$ using related combinatorial objects, known as Young tableaux.

A *partition of the positive integer* n is a set $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ of positive integers written in weakly decreasing order such that $\lambda_1 + \lambda_2 + \cdots + \lambda_t = n$. We call $\ell(\lambda) = t$ the length of the partition λ and say that $|\lambda| = n$. We identify the dominant A_9-weight $\lambda = \sum_{i=1}^{10} \lambda_i \epsilon_i$ with the partition $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_t\}$ where t is the largest integer such that $\lambda_t \neq 0$. We compare partitions λ and μ using the *dominance order on partitions*, in which we fill either λ or μ with trailing zeros so that each partition is of the same length and say $\lambda \geq \mu$ if and only if

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} \mu_i \quad \text{for } m \text{ from 1 to } \max\{\ell(\lambda), \ell(\mu)\}.$$

A *Young diagram* is a collection of boxes arranged in left-justified rows with a weakly decreasing number of boxes in each row and a *Young tableau* is a filling of the Young diagram with positive integers in such a way that the entries are weakly increasing across each row and strictly increasing down each column. For a given Young tableau, Y, let λ_i give the number of boxes in row i of the tableaux and μ_i gives the number of $i's$ that appear in the filling of the tableaux. We say Y is of *shape* λ and *weight* μ. The shape of a Young tableau is necessarily a partition while the weight of the tableau may or may not be.

Example 5. The tableaux in Figure 1 are the only tableaux of shape

$$\{2, 2, 2, 2, 2, 2, 1, 1\} = \{2^6, 1^2\}$$

and weight

$$\{2, 2, 2, 2, 1, 1, 1, 1, 1, 1\} = \{2^4, 1^6\}.$$

The basis vectors for the highest weight A_9-module with dominant highest-weight λ can be parameterized by the set of all Young tableaux of shape λ. If $v \in V(\lambda)$ corresponds to the Young tableau Y then the weight of the vector v in $V(\lambda)$ is the same as the weight of the Young tableau Y, leading to the following proposition.

1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
5	6	5	6	5	6	5	7	5	7	5	7	5	8	5	8	5	9
7	8	7	9	7	10	6	8	6	9	6	10	6	9	6	10	6	10
9		8		8		9		8		8		7		7		7	
10		10		9		10		10		9		10		9		8	

Figure 1. Young diagrams for Example 5.

Proposition 1 [Fulton 1997]. *Let λ be a dominant weight of A_9. The weight multiplicity of μ in $V(\lambda)$, denoted* $\dim V(\lambda)_\mu$ *is the number of Young tableaux of shape λ and weight μ.*

Furthermore, every weight $\mu = \sum_{i=1}^{10} \mu_i \epsilon_i$ of the highest-weight A_9-module with dominant highest-weight λ is conjugate to the dominant weight formed by rearranging the coefficients of the ϵ_i in weakly decreasing order. If μ is dominant then μ may be identified with a partition and the number of Young tableaux of shape λ and weight μ is known as the *Kostka number* $\mathcal{K}_{\lambda,\mu}$.

5. Root multiplicity calculations in E_{10}

Each positive root of E_{10} is conjugate to its negative which will be of the same multiplicity. Therefore we may restrict our studies to the negative roots of E_{10}; let $\alpha = -\sum_{i=0}^{9} k_i \alpha_i$ be such a root. We call $j = k_0$ the degree of the root α. In this section we determine the multiplicities of roots of degree $-5 \le j \le 0$.

Viewing E_{10} as presented in (13), roots of degree zero appear as roots of $\tilde{\mathfrak{g}}_0 = A_9 + \tilde{\mathfrak{h}}$ and hence as roots of the finite-dimensional simple Lie algebra A_9. These roots are of multiplicity one.

Roots of negative degree appear as weights of certain A_9-modules and each weight is conjugate to a dominant weight of the same multiplicity. Therefore, we will consider only dominant, negative roots in E_{10}. Let α be any dominant E_{10} root of degree $-j$ with $j > 0$. There exist positive integers k_i such that

$$\begin{aligned}
\alpha &= -j\alpha_0 + \sum_{i=1}^{9} k_i \alpha_i \\
&= j(\epsilon_1+\epsilon_2+\cdots+\epsilon_7) + (j-k_1)\epsilon_1 + \sum_{i=2}^{7}(j-k_i+k_{i+1})\epsilon_i + \sum_{i=8}^{9}(k_{i-1}-k_i)\epsilon_i + k_9\epsilon_i \\
&= \big\{ j-k_1,\ j-k_2+k_1,\ j-k_3+k_2,\ j-k_4+k_3,\ j-k_5+k_4, \\
&\qquad j-k_6+k_5,\ j-k_7+k_6,\ k_7-k_8,\ k_8-k_9,\ k_9 \big\}.
\end{aligned}$$

The (necessarily positive) sums of the first t terms in α for $t = 1, 2, \ldots, 10$ are

$j-k_1, 2j-k_2, 3j-k_3, 4j-k_4, 5j-k_5, 6j-k_6, 7j-k_7, 7j-k_8, 7j-k_9$, and $7j$, respectively, with each k_i a positive integer, leading to the following proposition.

Proposition 2. *Every dominant degree $-j$ root of E_{10} is a partition α of $7j$ such that $\ell(\alpha) \leq 10$ and $\alpha \leq \{j^7\}$ where $\leq$ is the dominance order introduced on page 536.*

5.1. ***Roots of degree negative one.*** Using the construction of E_{10} presented in (13), roots of degree -1 appear as weights of the A_9-module $V(-\alpha_0) = V(\{1^7\})$. By Proposition 2 any dominant weight of $V(\{1^7\})$, μ must be a partition of seven such that $\mu < \{1^7\}$. However any $\mu < \{1^7\}$ can be a partition of at most six and thus $\{1^7\}$ is the only dominant weight of $V(-\alpha_0)$. Therefore the roots of degree -1 are of the form $\alpha = k_1\epsilon_1 + k_2\epsilon_2 + \cdots + k_{10}\epsilon_{10}$ where $\{k_i\}_{i=1}^{10}$ is a permutation of $\{1, 1, 1, 1, 1, 1, 1, 0, 0, 0\}$. By Proposition 1, each of these roots is of multiplicity

$$\mathcal{K}_{\{1^7\},\{1^7\}} = 1$$

5.2. ***Roots of degree less than negative one — Kang's formula.*** Again viewing E_{10} as presented in (13), root spaces for roots of degree less than -1 appear as weights of the A_9-module $\bigoplus_{i\geq 1} \tilde{\mathfrak{g}}_{-i}$. Kang [1994b] used the specific structure of $\bigoplus_{i\geq 1} \tilde{\mathfrak{g}}_{-i}$ (see [Kang 1993a] for details regarding this structure), the Euler–Poincaré principle, and Kostant's formula to develop both recursive and closed form multiplicity formulas for algebras with realizations such as the one given in (13). Theorem 3 gives Kang's recursive formula as it is summarized in [Hontz and Misra 2002a] and as it pertains to E_{10}. One can find similar applications of this formula in [Benkart et al. 1995].

Theorem 3. *Let $\alpha = \sum_{i=0}^{9} k_i\alpha_i$ be a dominant root of E_{10} with $\deg(\alpha) = k_0 = -j$. Then for $j \geq 2$,*

$$\operatorname{mult}(\alpha) = \sum_{k=2}^{j} (-1)^k X_k(\alpha) - \sum_{k=2}^{j} (-1)^k Y_k(\alpha),$$

with

$$X_k(\alpha) = \sum_{\substack{\beta_1 > \cdots > \beta_r \\ k_1 + \cdots + k_r = k \\ k_1\beta_1 + \cdots + k_r\beta_r = \alpha}} \binom{\operatorname{mult}(\beta_1)}{k_1} \cdots \binom{\operatorname{mult}(\beta_r)}{k_r}, \text{ and}$$

$$Y_k(\alpha) = \sum_{\substack{\omega \in W(S) \\ \ell(\omega) = k \\ \deg(\omega\rho - \rho) = -j}} \dim V(\omega\rho - \rho)_\alpha,$$

where $\operatorname{mult}(\beta_i)$ *is the multiplicity of β_i as a root of E_{10}, $\rho \in \tilde{\mathfrak{h}}^*$ such that $\rho(h_i) = 1$ for all $i \in \{0, 1, \ldots, 9\}$, $V(\omega\rho - \rho)$ is the highest-weight A_9-module with highest* *weight $\omega\rho - \rho$, and the reflections $\omega \in W(S)$ can be built from simple reflections using the recursive procedure defined in Lemma 4.*

For a more precise definition of the set $W(S)$ along with a proof of Lemma 4 see [Kang 1993a].

Lemma 4 [Kang 1993a]. *The only length one element of $W(S)$ is $\omega' = r_0$. Suppose $\omega = \omega' r_j$ and $l(\omega) = l(\omega') + 1$. Then $\omega \in W(S)$ if and only if $\omega' \in W(S)$ and $\omega'(\alpha_j) = \sum_{i=0}^{9} k_i \alpha_i$ where each $k_i \geq 0$ and $k_0 \neq 0$.*

In the following examples we apply Theorem 3 to determine the multiplicities of specific degree -2 and degree -3 roots. Recall, we are using the ordering of the simple roots given in (11).

Example 6. In this example, we find the multiplicity in E_{10} of the degree -2 root $\alpha = -2\alpha_0 - \alpha_5 - 2\alpha_6 - 3\alpha_7 - 2\alpha_8 - \alpha_9$. The root α can be expressed by

$$\begin{aligned}\alpha &= 2\alpha_0 - \alpha_5 - 2\alpha_6 - 3\alpha_7 - 2\alpha_8 - \alpha_9\\ &= 2(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7) - (\epsilon_5 - \epsilon_6) - 2(\epsilon_6 - \epsilon_7)\\ &\qquad - 3(\epsilon_7 - \epsilon_8) - 2(\epsilon_8 - \epsilon_9) - (\epsilon_9 - \epsilon_{10})\\ &= 2\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 + 2\epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8 + \epsilon_9 + \epsilon_{10}\\ &= \{2^4, 1^6\}\end{aligned}$$

and $\text{mult}(\alpha) = X_2(\alpha) - Y_2(\alpha)$, where X_2 and Y_2 are defined in Theorem 3.

Given that all degree -1 roots are of multiplicity one, we have

$$\begin{aligned}X_2(\alpha) &= \sum_{\substack{\beta_1\\ 2\beta_1=\alpha}} \underbrace{\binom{1}{2}}_{0} + \sum_{\substack{\beta_1>\beta_2\\ \beta_1+\beta_2=\alpha}} \underbrace{\binom{1}{1}\binom{1}{1}}_{1}\\ &= \text{the number of pairs } (\beta_1, \beta_2) \text{ of roots of degree } -1\\ &\qquad\qquad \text{such that } \beta_1 > \beta_2 \text{ and } \beta_1 + \beta_2 = \alpha. \qquad (14)\end{aligned}$$

Let β_1 and β_2 be roots of degree -1 such that $\beta_1 + \beta_2 = \{2^4, 1^6\}$. Then, β_1 and β_2 can each be viewed as ordered sets whose terms are permutations of $\{1, 1, 1, 1, 1, 1, 1, 0, 0, 0\}$. The first term in both β_1 and β_2 must be one as this is the only way for their sum to be two. The same statement holds for the second, third, and fourth terms of β_1 and β_2. The remaining six terms of β_1 could then be any of the $C(6, 3)$ permutations of $\{1, 1, 1, 0, 0, 0\}$. Once we have determined β_1, $\beta_2 = \{2^4, 1^6\} - \beta_1$ is fixed. Therefore, we have $C(6, 3)$ pairs (β_1, β_2) of distinct degree -2 roots with $\beta_1 + \beta_2 = \{2^4, 1^6\}$, exactly half which will be such that $\beta_1 > \beta_2$. Hence, $X_2(\{2^6, 1^4\}) = C(6, 3)/2 = 10$.

Next we turn our attention to the calculation of $Y_2(\alpha)$. To do this we must first find all $\omega \in W(S)$ of length two such that $\deg(\omega\rho - \rho) = -2$. Lemma 4 implies that any $\omega \in W(S)$ of length two will be of the form $\omega = r_0 r_j$ for $j \in \{0, 1, \dots, 9\}$ where $r_0(\alpha_j) = \sum_{i=0}^{9} k_i \alpha_i$ for some k_i with each $k_i \geq 0$ and $k_0 \neq 0$. For any simple root α_j, $r_0(\alpha_j) = \alpha_j - \alpha_j(h_0)\alpha_0$. Referring to the Dynkin diagram for E_{10} given in (11), we see

$$\alpha_j(h_0) = \begin{cases} 2 & \text{if } j = 0, \\ -1 & \text{if } j = 7, \\ 0 & \text{otherwise.} \end{cases}$$

and thus

$$r_0(\alpha_j) = \begin{cases} \alpha_0 - 2\alpha_0 = -\alpha_0 & \text{if } j = 0, \\ \alpha_7 + \alpha_0 & \text{if } j = 7, \\ \alpha_j & \text{otherwise.} \end{cases}$$

Observe that the only $\omega \in W(S)$ of length two is $\omega = r_0 r_7$ and for this choice of ω, $\omega\rho - \rho$ is of degree -2. Specifically,

$$\begin{aligned} \omega\rho - \rho &= r_o r_7(\rho) - \rho \\ &= r_0\,[\rho - \rho(h_7)\alpha_7] - \rho \\ &= r_0\,(\rho - \alpha_7) - \rho \quad \text{(since } \rho(h_i) = 1 \text{ for all } i) \\ &= -2\alpha_0 - \alpha_7 \\ &= 2\epsilon_1 + 2\epsilon_2 + 2\epsilon_3 + 2\epsilon_4 + 2\epsilon_5 + 2\epsilon_6 + \epsilon_7 + \epsilon_8. \end{aligned}$$

Therefore,

$$\begin{aligned} Y_2(\alpha) &= \sum_{\substack{\omega \in W(S) \\ \ell(\omega)=2 \\ \deg(\omega\rho-\rho)=-2}} \dim V(\omega\rho - \rho)_\alpha \\ &= \dim V(\{2^6, 1^2\})_\alpha \\ &= \mathcal{K}_{\{2^6,1^2\},\alpha} && \text{(by Proposition 1)} \\ &= \mathcal{K}_{\{2^6,1^2\},\{2^4,1^6\}} \\ &= 9 && \text{(by Example 5)} \end{aligned} \tag{15}$$

and $\text{mult}(\alpha) = \text{mult}(\{2^4, 1^6\}) = X_2(\{2^4, 1^6\}) - Y_2(\{2^4, 1^6\}) = 10 - 9 = 1$.

Example 7. In this example we find the multiplicity of the E_{10} root

$$\alpha = -3\alpha_0 - \alpha_2 - 2\alpha_3 - 3\alpha_4 - 4\alpha_5 - 5\alpha_6 - 6\alpha_7 - 4\alpha_8 - 2\alpha_9.$$

The root α is of degree -3 and thus $\text{mult}(\alpha) = X_2(\alpha) - X_3(\alpha) - Y_2(\alpha) + Y_3(\alpha)$ where X_2, X_3, Y_2, and Y_3 are defined in Theorem 3.

Given that all degree -1 and degree -2 roots are of multiplicity one, we have

$$X_2(\alpha) = \sum_{\substack{\beta_1 > \beta_2 \\ \beta_1+\beta_2=\alpha}} \underbrace{\binom{1}{1}\binom{1}{1}}_{1}$$

$$= \text{The number of pairs } (\beta_1, \beta_2) \text{ with } \deg(\beta_1) = -2, \quad \deg(\beta_2) = -1, \text{ and } \beta_1 + \beta_2 = \alpha. \quad (16)$$

Let β_1 be a root of degree -2 and β_2 be a root of degree -1 such that

$$\begin{aligned}\beta_1+\beta_2 &= \alpha \\ &= -3\alpha_0-\alpha_2-2\alpha_3-3\alpha_4-4\alpha_5-5\alpha_6-6\alpha_7-4\alpha_8-2\alpha_9 \\ &= 3(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4+\epsilon_5+\epsilon_6+\epsilon_7)-(\epsilon_2-\epsilon_3)-2(\epsilon_3-\epsilon_4)-3(\epsilon_4-\epsilon_5) \\ &\qquad -4(\epsilon_5-\epsilon_6)-5(\epsilon_6-\epsilon_7)-6(\epsilon_7-\epsilon_8)-4(\epsilon_8-\epsilon_9)-2(\epsilon_9-\epsilon_{10}) \\ &= 3\epsilon_1+2\epsilon_2+2\epsilon_3+2\epsilon_4+2\epsilon_5+2\epsilon_6+2\epsilon_7+2\epsilon_8+2\epsilon_9+2\epsilon_{10} \\ &= \{3, 2^9\}.\end{aligned}$$

Then, β_1 and β_2 can be viewed as ordered sets whose terms are permutations of $\{2, 2, 2, 2, 1, 1, 1, 1, 1, 1\}$ and $\{1, 1, 1, 1, 1, 1, 1, 0, 0, 0\}$ respectively. The first term in β_1 must be two with the first term of β_2 being one, as this is the only way for their sum to be three. The remaining nine terms of β_1 could then be any of the $C(9, 3)$ permutations of $\{2, 2, 2, 1, 1, 1, 1, 1, 1\}$. Once we have determined β_1, $\beta_2 = \{3, 2^2\} - \beta_1$ is fixed. Therefore $X_2(\{3, 2^9\}) = C(9, 3) = 84$.

We can also simplify $X_3(\alpha) = X_3(\{3, 2^9\})$ using the fact that all degree -1 and degree -2 roots are of multiplicity one.

$$X_3(\alpha) = \sum_{\substack{\beta_1 \\ 3\beta_1=\alpha}} \underbrace{\binom{1}{3}}_{0} + \sum_{\substack{\beta_1\neq\beta_2 \\ 2\beta_1+\beta_2=\alpha}} \underbrace{\binom{1}{2}\binom{1}{1}}_{0} + \sum_{\substack{\beta_1>\beta_2>\beta_3 \\ \beta_1+\beta_2+\beta_3=\alpha}} = \underbrace{\binom{1}{1}\binom{1}{1}\binom{1}{1}}_{1}$$

$$= \text{The number of triples } (\beta_1, \beta_2, \beta_3) \text{ with } \beta_1 > \beta_2 > \beta_3 \text{ and } \beta_1+\beta_2+\beta_3 = \alpha. \quad (17)$$

Let β_1, β_2 and β_3 be degree -1 roots such that $\beta_1+\beta_2+\beta_3 = \alpha = \{3, 2^9\}$. Then, β_1, β_2, and β_3 can each be viewed as ordered sets whose terms are permutations of $\{1, 1, 1, 1, 1, 1, 1, 0, 0, 0\}$. The first term in β_1, β_2, and β_3 must each be one as this is the only way for their sum to be three. The remaining nine terms of β_1 could then be any of the $C(9, 3)$ permutations of $\{1, 1, 1, 1, 1, 1, 0, 0, 0\}$. Three of the nine remaining terms in β_1 will be zero. The corresponding terms in β_2 and β_3 must both be one as this is the only way for the terms to sum to two. The remaining six terms of β_2 could then be any of the $C(6, 3)$ permutations of $\{1, 1, 1, 0, 0, 0\}$. Once we have determined β_1 and β_2, $\beta_3 = \{3, 2^9\} - \beta_1 - \beta_2$ is fixed. Therefore, we have

$C(9,3)\cdot C(6,3)$ pairs (β_1,β_2) of distinct degree -2 roots with $\beta_1+\beta_2+\beta_3=\{3,2^9\}$, exactly $1/3!$ of which will be such that $\beta_1>\beta_2>\beta_3$. Hence,

$$X_3(\{3,2^9\})=(C(9,3)\cdot C(6,2))/3!=280.$$

Next we turn our attention to the calculation of $Y_2(\alpha)$ and $Y_3(\alpha)$. Recall that

$$Y_2(\alpha)=\sum_{\substack{\omega\in W(S)\\ \ell(\omega)=2\\ \deg(\omega\rho-\rho)=-3}}\dim V(\omega\rho-\rho)_\alpha.$$

However, in Example 6 we found all $\omega\in W(S)$ of length two and none of these were of degree -3. Therefore,

$$Y_2(\alpha)=0. \tag{18}$$

To evaluate $Y_3(\alpha)$ we must first determine all $\omega\in W(S)$ of length three such that $\deg(\omega\rho-\rho)=-3$. Since the only $\omega\in W(S)$ of length two is r_0r_7, Lemma 4 implies that any $\omega\in W(S)$ of length three will be of the form $\omega=r_0r_7r_j$ for $j\in\{0,1,\dots,9\}$ where $r_0r_7(\alpha_j)=\sum_{i=0}^{9}k_i\alpha_i$ for some k_i with each $k_i\ge 0$ and $k_0\ne 0$. But then,

$$r_0r_7(\alpha_j)=\begin{cases}-\alpha_7 & \text{if } j=0,\\ \alpha_0+\alpha_6+\alpha_7 & \text{if } j=6,\\ \alpha_0+\alpha_7+\alpha_8 & \text{if } j=8,\\ \alpha_j & \text{otherwise.}\end{cases}$$

and so the only $\omega\in W(S)$ of length 3 are $\omega_1=r_0r_7r_6$ and $\omega_2=r_0r_7r_8$. Since

$$\omega_1\rho-\rho=3\epsilon_1+3\epsilon_2+3\epsilon_3+3\epsilon_4+3\epsilon_5+2\epsilon_6+2\epsilon_7+2\epsilon_8$$

and

$$\omega_2\rho-\rho=3\epsilon_1+3\epsilon_2+3\epsilon_3+3\epsilon_4+3\epsilon_5+3\epsilon_6+\epsilon_7+\epsilon_8+\epsilon_9,$$

each of which is of degree -3, we have

$$\begin{aligned}Y_3(\alpha)&=\sum_{\substack{\omega\in W(S)\\ \ell(\omega)=3\\ \deg(\omega\rho-\rho)=-3}}\dim V(\omega\rho-\rho)_\alpha\\ &=\dim V(3\epsilon_1+3\epsilon_2+3\epsilon_3+3\epsilon_4+3\epsilon_5+2\epsilon_6+2\epsilon_7+2\epsilon_8)_\alpha\\ &\qquad+\dim V(3\epsilon_1+3\epsilon_2+3\epsilon_3+3\epsilon_4+3\epsilon_5+3\epsilon_6+\epsilon_7+\epsilon_8+\epsilon_9)_\alpha\\ &=\mathcal{K}_{\{3^5,2^3\},\{3,2^9\}}+\mathcal{K}_{\{3^6,1^3\},\{3,2^9\}}\\ &=120+84=204.\end{aligned} \tag{19}$$

Therefore, $\operatorname{mult}(\alpha)=\operatorname{mult}(\{3,2^9\})=X_2-X_3-Y_2+Y_3=84-280-0+204=8.$

Theorem 5. *The only dominant degree -2 root of E_{10} is $\{2^4, 1^6\}$ and this root is of multiplicity one.*

Proof. If α is a dominant degree -2 root of E_{10} then α meets the conditions given in Proposition 2 for $j = 2$ with $0 \neq \text{mult}(\alpha) = X_2(\alpha) - Y_2(\alpha)$. Using the counting methods demonstrated in Example 6, we have found $X_2(\alpha)$ as stated in (14), $Y_2(\alpha)$ as stated in (15), and mult(α) for each potential dominant degree -2 root, as follows:

α	X_2	Y_2	mult(α)
$\{2^7\}$	0	0	0
$\{2^6, 1^2\}$	1	1	0
$\{2^5, 1^4\}$	3	3	0
$\{2^4, 1^6\}$	10	9	1

The table shows that $\{2^4, 1^6\}$ is the only dominant E_{10} root of degree -2. ☐

Theorem 6. *The only dominant degree -3 roots of E_{10} are $\{3, 2^9\}$ and $\{3^2, 2^7, 1\}$ which are of multiplicities eight and one respectively.*

Proof. If α is a dominant degree -3 root of E_{10} then α meets the conditions given in Proposition 2 for $j = 3$ and $0 \neq \text{mult}(\alpha) = X_2(\alpha) - X_3(\alpha) - Y_2(\alpha) + Y_3(\alpha)$. Using the counting methods demonstrated in Example 7, we have found $X_2(\alpha)$ as stated in (16), $X_3(\alpha)$ as stated in (17), $Y_2(\alpha)$ as stated in (18), $Y_3(\alpha)$ as stated in (19), and mult(α) for each potential dominant degree -3 root, as follows:

α	X_2	X_3	Y_2	$Y_3 = \mathcal{K}_{\{3^5,2^3\},\alpha} + \mathcal{K}_{\{3^6,1^3\},\alpha}$	mult(α)
$\{3^7\}$	0	*0	0	0	0
$\{3^6, 2, 1\}$	0	*0	0	0	0
$\{3^6, 1^3\}$	0	*1	0	$0+1$	0
$\{3^5, 2^3\}$	0	*1	0	$1+0$	0
$\{3^5, 2^2, 1^2\}$	0	*2	0	$1+1$	0
$\{3^5, 2, 1^4\}$	0	*6	0	$2+4$	0
$\{3^4, 2^4, 1\}$	0	6	0	$4+2$	0
$\{3^4, 2^3, 1^3\}$	1	15	0	$7+7$	0
$\{3^3, 2^6\}$	0	15	0	$10+5$	0
$\{3^3, 2^5, 1^2\}$	5	40	0	$20+15$	0
$\{3^2, 2^7, 1\}$	21	105	0	$50+35$	1
$\{3, 2^9\}$	82	280	0	$120+84$	8

The table shows that $\{3, 2^9\}$ and $\{3^2, 2^7, 1\}$ are the only dominant roots of degree -3. ☐

We developed Maple worksheets to automate the multiplicity calculations. (For examples see mathsci.appstate.edu/∼vlw/E10mult.html.) These worksheets apply Kang's multiplicity formula using Maple packages by John Stembridge [2004; 2005] to do the combinatorial calculations. Using the worksheets we have found all dominant E_{10} roots of degree up to -5.

Theorem 7. *The dominant degree* -4 *roots of* $\tilde{\mathfrak{g}} = E_{10}$ *are* $\{4, 3^6, 2^3\}$, $\{3^9, 1\}$, *and* $\{3^8, 2^2\}$, *with multiplicities of* 1, 1, *and* 8 *respectively.*

Theorem 8. *The dominant degree* -5 *roots of* $\tilde{\mathfrak{g}} = E_{10}$ *are* $\{4^6, 3^3, 2\}$, $\{5, 4^3, 3^6\}$, *and* $\{4^5, 3^5\}$ *with multiplicities* 1, 1, *and* 8 *respectively.*

6. Conclusions

As in [Kac et al. 1988] we have studied root multiplicities in E_{10}. We have worked in the basis $\mathcal{B}_\epsilon = \{\epsilon_i\}_{i=1}^{10}$ whereas the authors of [Kac et al. 1988] use the basis $\mathcal{B}_{\alpha'} = \left\{\alpha'_i\right\}_{i=-1}^{8}$ ordered according to the Dynkin diagram given in (9). Using the transition matrix from the basis B_ϵ to the basis $\mathcal{B}_{\alpha'}$,

$$\begin{bmatrix} 6/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 \\ 5/7 & 5/7 & -2/7 & -2/7 & -2/7 & -2/7 & -2/7 & -2/7 & -2/7 & -2/7 \\ 4/7 & 4/7 & 4/7 & -3/7 & -3/7 & -3/7 & -3/7 & -3/7 & -3/7 & -3/7 \\ 3/7 & 3/7 & 3/7 & 3/7 & -4/7 & -4/7 & -4/7 & -4/7 & -4/7 & -4/7 \\ 2/7 & 2/7 & 2/7 & 2/7 & 2/7 & -5/7 & -5/7 & -5/7 & -5/7 & -5/7 \\ 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & 1/7 & -6/7 & -6/7 & -6/7 & -6/7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 & -1/7 \end{bmatrix}$$

and remembering that any permutation of a dominant E_{10} root is again a root of the same multiplicity we were able to compare our results with those of [Kac et al. 1988].

We have found the multiplicity of 3527 roots in E_{10} with negative degree. The majority (3442) of these roots have coefficients for α'_{-1} of either -1 or -2 and their multiplicities agree with those stated in [Kac et al. 1988]. The root

$$\{1, 3^9\}_{\mathcal{B}_\epsilon} = \{-3, -4, -5, -6, -7, -8, -9, -6, -3, -4\}_{\mathcal{B}_{\alpha'}}$$

was not addressed in [Kac et al. 1988]. This root is conjugate to the dominant root $\{3^9, 1\}_{\mathcal{B}_\epsilon}$ and thus is of multiplicity one. The remaining 84 new E_{10} roots are conjugate to $\{4^6, 3^3, 2\}_{\mathcal{B}_\epsilon}$ and are also of multiplicity one.

References

[Benkart et al. 1995] G. Benkart, S.-J. Kang, and K. C. Misra, "Indefinite Kac–Moody algebras of special linear type", *Pacific J. Math.* **170**:2 (1995), 379–404. MR 96k:17037 Zbl 0857.17020

[Berman and Parshall 2002] S. Berman and K. H. Parshall, "Victor Kac and Robert Moody: their paths to Kac–Moody Lie algebras", *Math. Intelligencer* **24**:1 (2002), 50–60. MR 2003b:17029 Zbl 1048.17001

[Feingold and Frenkel 1983] A. J. Feingold and I. B. Frenkel, "A hyperbolic Kac–Moody algebra and the theory of Siegel modular forms of genus 2", *Math. Ann.* **263**:1 (1983), 87–144. MR 86a:17006 Zbl 0489.17008

[Fulton 1997] W. Fulton, *Young tableaux*, London Mathematical Society Student Texts **35**, Cambridge University Press, 1997. MR 99f:05119 Zbl 0878.14034

[Hontz and Misra 2002a] J. Hontz and K. C. Misra, "On root multiplicities of $HA_n^{(1)}$", *Internat. J. Algebra Comput.* **12**:3 (2002), 477–508. MR 2003d:17029 Zbl 1035.17035

[Hontz and Misra 2002b] J. Hontz and K. C. Misra, "Root multiplicities of the indefinite Kac–Moody Lie algebras $HD_4^{(3)}$ and $HG_2^{(1)}$", *Comm. Algebra* **30**:6 (2002), 2941–2959. MR 2003f:17028 Zbl 1058.17014

[Kac 1968] V. G. Kac, "Simple irreducible graded Lie algebras of finite growth", *Izv. Akad. Nauk SSSR Ser. Mat.* **32** (1968), 1323–1367. In Russian; translated in *Math. USSR, Isv.* **2**(1968), 1271–1311. MR 41 #4590 Zbl 0222.17007

[Kac 1990] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, 1990. MR 92k:17038 Zbl 0716.17022

[Kac et al. 1988] V. G. Kac, R. V. Moody, and M. Wakimoto, "On E_{10}", pp. 109–128 in *Differential geometrical methods in theoretical physics* (Como, 1987), edited by K. Bleuler and M. Werner, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **250**, Kluwer Acad. Publ., Dordrecht, 1988. MR 90e:17031 Zbl 0674.17007

[Kang 1993a] S.-J. Kang, "Kac–Moody Lie algebras, spectral sequences, and the Witt formula", *Trans. Amer. Math. Soc.* **339**:2 (1993), 463–493. MR 93m:17013 Zbl 0794.17014

[Kang 1993b] S.-J. Kang, "Root multiplicities of the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$", *J. Algebra* **160**:2 (1993), 492–523. MR 94i:17031 Zbl 0828.17027

[Kang 1994a] S.-J. Kang, "On the hyperbolic Kac–Moody Lie algebra $HA_1^{(1)}$", *Trans. Amer. Math. Soc.* **341**:2 (1994), 623–638. MR 94d:17033 Zbl 0828.17026

[Kang 1994b] S.-J. Kang, "Root multiplicities of Kac–Moody algebras", *Duke Math. J.* **74**:3 (1994), 635–666. MR 95c:17036 Zbl 0823.17031

[Kang and Melville 1994] S.-J. Kang and D. J. Melville, "Root multiplicities of the Kac–Moody algebras $HA_n^{(1)}$", *J. Algebra* **170**:1 (1994), 277–299. MR 95m:17018 Zbl 0828.17028

[Klima and Misra 2008] V. W. Klima and K. C. Misra, "Root multiplicities of the indefinite Kac–Moody algebras of symplectic type", *Comm. Algebra* **36**:2 (2008), 764–782. MR 2009c:17036 Zbl 1132.17014

[Moody 1968] R. V. Moody, "A new class of Lie algebras", *J. Algebra* **10** (1968), 211–230. MR 37 #5261

[Stembridge 2004] J. Stembridge, "The coxter package, a Maple package for working with root systems and finite coxeter groups", website, 2004, http://www.math.lsa.umich.edu/~jrs/maple.html.

[Stembridge 2005] J. Stembridge, "The SF package, a Maple package for computations symmetric functions", website, 2005, http://www.math.lsa.umich.edu/~jrs/maple.html.

[Sthanumoorthy and Uma Maheswari 2012] N. Sthanumoorthy and A. Uma Maheswari, "Structure and root multiplicities for two classes of extended hyperbolic Kac–Moody algebras $EHA_1^{(1)}$ and $EHA_2^{(2)}$ for all cases", *Comm. Algebra* **40**:2 (2012), 632–665. MR 2889486 Zbl 1267.17030

Received: 2013-01-17 Accepted: 2013-06-02

klimavw@appstate.edu *Department of Mathematical Sciences, Appalachian State University, 121 Bodenheimer Drive, Boone, NC 28607, United States*

tshatl1@lsu.edu *Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, United States*

thomaska1@appstate.edu *Department of Mathematical Sciences, Appalachian State University, 121 Bodenheimer Drive, Boone, NC 28607, United States*

wilsonat@appstate.edu *Department of Mathematical Sciences, Appalachian State University, 121 Bodenheimer Drive, Boone, NC 28607, United States*

involve

msp.org/involve

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2014 is US $120/year for the electronic version, and $165/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2014 vol. 7 no. 4

1944-4176(2014)7:4;1-5