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Infinite cardinalities in the Hausdorff metric geometry

Alexander Zupan

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The Hausdorff metric measures the distance between nonempty compact sets in $\mathbb{R}^n$, the collection of which is denoted $\mathcal{H}(\mathbb{R}^n)$. Betweenness in $\mathcal{H}(\mathbb{R}^n)$ can be defined in the same manner as betweenness in Euclidean geometry. But unlike betweenness in $\mathbb{R}^n$, for some elements A and B of $\mathcal{H}(\mathbb{R}^n)$ there can be many elements between A and B at a fixed distance from A. Blackburn et al. ("A missing prime configuration in the Hausdorff metric geometry", *J. Geom.*, **92**:1–2 (2009), pp. 28–59) demonstrate that there are infinitely many positive integers k such that there exist elements A and B having exactly k different elements between A and B at each distance from A while proving the surprising result that no such A and B exist for $k = 19$. In this vein, we prove that there do not exist elements A and B with exactly a countably infinite number of elements at any location between A and B.

1. Introduction

The Hausdorff metric provides a means to measure distance in the family $\mathcal{H}(\mathbb{R}^n)$ of nonempty compact sets in n-dimensional Euclidean space. There is a natural embedding of $\mathbb{R}^n$ into $\mathcal{H}(\mathbb{R}^n)$ that takes $x \in \mathbb{R}^n$ to $\{x\} \in \mathcal{H}(\mathbb{R}^n)$. The notion of betweenness in $\mathbb{R}^n$ extends naturally to $\mathcal{H}(\mathbb{R}^n)$. However, in Euclidean space, there is a unique point between a and b at a given distance less than $d(a, b)$ from a, while in $\mathcal{H}(\mathbb{R}^n)$ there can be many distinct elements between elements A and B at a given distance from A. For instance, for infinitely many numbers k we can find A and B with exactly k elements between A and B at a given distance from A, and we can also find A and B such that this number of elements between A and B is infinite. Blackburn et al. [2009] proved the surprising result that there exist no two elements A and B in $\mathcal{H}(\mathbb{R}^n)$ with the property that A and B have exactly 19 elements of $\mathcal{H}(\mathbb{R}^n)$ between them at a given distance from A. In this paper, we will prove that there is another cardinality that is missing; namely, there exist no

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two elements $A, B \in \mathcal{H}(\mathbb{R}^n)$ with exactly a countably infinite number of elements between them at any location. The argument uses a different approach than that of [Blackburn et al. 2009]: the proof there is exhaustive, and while it is succinct enough to prove the absence of 19, the method may be too unwieldy to show the existence of larger conjectured missing numbers. It is our hope that the idea of too many removable points forcing a larger cardinality of sets between A and B might be adapted to the finite case.

2. Preliminaries

Let $\mathcal{H}(\mathbb{R}^n)$ denote the collection of nonempty compact subsets of $\mathbb{R}^n$. We will refer to these compact sets as "elements" of $\mathcal{H}(\mathbb{R}^n)$. For any $a \in \mathbb{R}^n$ and $B \in \mathcal{H}(\mathbb{R}^n)$, let $d(a, B) = \min_{b \in B} d_E(a, b)$, where d_E denotes the Euclidean metric on $\mathbb{R}^n$. The Hausdorff metric is then defined as follows:

Definition 2.1. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. The Hausdorff distance between A and B is given by

$$h(A, B) = \max\{d(A, B), d(B, A)\},$$

where $d(A, B) = \max_{a \in A} d(a, B)$.

In other words, the distance from A to B is the maximum of the distances between points in A to the set B, and the Hausdorff distance between A and B is the maximum of the distance from A to B and the distance from B to A. Note the maximum and minimum in the definitions above are well-defined since both A and B are compact sets. To verify that the Hausdorff distance defines a metric on $\mathcal{H}(\mathbb{R}^n)$, see, for instance, [Edgar 1990].

Example 2.2. Let $n = 2$ and consider the sets shown on the left in Figure 1. Let $S^1(r)$ denote the circle of radius r centered at the origin, so that $A = \{(0, 0)\} \cup S^1(4)$, $B = S^1(2)$, and $C = S^1(1) \cup S^1(3)$. Then, for any $a \in A$ and $b \in B$, we have $d_E(a, b) \geq 2$. Further, for such a there exists a point $b \in B$ such that $d(a, b) = 2$, which implies that $d(a, B) = 2$ for all $a \in A$; hence, $d(A, B) = 2$. Similarly, for every $b \in B$, we have $d_E(b, a_0) = 2$ where a_0 is the origin, so $d(b, A) = 2$ for all $b \in B$, which shows $d(B, A) = 2$ as well. It follows that $h(A, B) = 2$. A similar verification shows that $h(A, C) = h(B, C) = 1$.

The set C' pictured on the right in Figure 1 is a compact subset of C. Here we see that for every $a \in A$ and $c \in C'$, $d_E(a, c) \geq 1$. Additionally, for every $a \in A$, there exists $c \in C'$ such that $d_E(a, c) = 1$, so $d(a, C') = 1$ for all such A and $d(A, C') = 1$. Likewise, for all $c \in C'$, there is some $a \in A$ such that $d_E(c, a) = 1$, so $d(C', A) = 1$ and $h(A, C') = 1$. A similar computation shows that $d(B, C') = 1$.

In $\mathbb{R}^n$ we say that c is between a and b at a distance $t \in \mathbb{R}$ from a (where $0 < t < d_E(a, b)$) if $d_E(a, b) = d_E(a, c) + d_E(c, b)$ and $d_E(a, c) = t$. If $\{a\}, \{b\} \in \mathcal{H}(\mathbb{R}^n)$

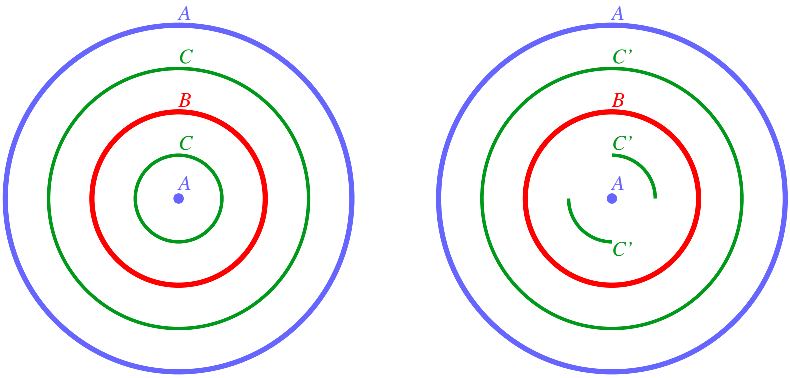

Figure 1. Two elements C and C' between sets A and B.

are single point sets, it is easy to see that $d_E(a, b) = h(\{a\}, \{b\})$. Thus, we can naturally extend betweenness in $\mathbb{R}^n$ to $\mathscr{H}(\mathbb{R}^n)$.

Definition 2.3. Let $A, B, C \in \mathscr{H}(\mathbb{R}^n)$ and $0 < t < h(A, B)$. We say that C is between A and B at a distance t from A if

$$h(A, B) = h(A, C) + h(C, B) \quad \text{and} \quad h(A, C) = t.$$

Thus, betweenness in $\mathbb{R}^n$ is preserved under the natural embedding of $\mathbb{R}^n$ into $\mathscr{H}(\mathbb{R}^n)$. To see that there can be multiple elements at some location between compact sets A and B, recall the previous example:

Example 2.4. As computed above, we have

$$h(A, B) = 2 = 1 + 1 = h(A, C) + h(C, B).$$

However, we also have

$$h(A, B) = 2 = 1 + 1 = h(A, C') + h(C', B)$$

for the same sets A and B, so C' and C lie between A and B one unit from A. In fact, if C'' is the union of $S^1(3)$ and any nonempty compact subset of $S^1(1)$, C'' is also between A and B one unit from A, so for this example there are uncountably many elements of $\mathscr{H}(\mathbb{R}^2)$ between A and B at a distance one from A.

Lemma 2.5 of [Blackburn et al. 2009] says that if $A, B \in \mathscr{H}(\mathbb{R}^n)$ and there exists some $a \in A$ or $b \in B$ such that $d(a, B) \neq h(A, B)$ or $d(b, A) \neq h(A, B)$, then there are infinitely many elements $C \in \mathscr{H}(\mathbb{R}^n)$ between A and B at any location. We can improve this result. Under the hypotheses, the authors find an injective map from the open interval $(0, \epsilon)$ to the collection of elements in $\mathscr{H}(\mathbb{R}^n)$ that lie between A and B at a given location to conclude that there are infinitely many such elements, but this implies further that under the assumptions of the lemma, then there are, in fact, uncountably many elements of $\mathcal{H}(\mathbb{R}^n)$ between A and B at any location, since $(0, \epsilon)$ is uncountable.

In light of this observation, we employ the following definition:

Definition 2.5 [Blackburn et al. 2009]. A configuration $[A, B]$ is a pair of sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $A \neq B$ such that

$$h(A, B) = d(b, A) = d(a, B) \quad \text{for all } a \in A \text{ and } b \in B.$$

It follows that if the pair $A, B \in \mathcal{H}(\mathbb{R}^n)$ is not a configuration, then the number of elements at any location between A and B is uncountable. Hence, if there are countably many elements at each location between A and B, then the pair must be a configuration $[A, B]$. Note that the elements A and B described in the example above constitute a configuration, and so being a configuration is necessary but not sufficient for there to be countably many elements of $\mathcal{H}(\mathbb{R}^n)$ at a given location between A and B.

We adopt the notation $\#([A, B])_t$ to represent the cardinality of the collection of elements between A and B in a configuration at a distance $0 < t < h(A, B)$ from A. Blackburn et al. [2009] demonstrated that when A and B are finite sets, $\#([A, B])_t$ is finite and $\#([A, B])_s = \#([A, B])_t$ for every s, t satisfying $0 < s, t < h(A, B)$. In this case, $\#([A, B])_t$ is simply denoted $\#([A, B])$.

3. An alternative characterization of $\#([A, B])_t$

Let $(A)_t$ denote the dilation of $A \in \mathcal{H}(\mathbb{R}^n)$ by t; that is, $(A)_t = \{x \in \mathbb{R}^n : d(x, A) \leq t\}$. In addition, for elements $A, B \in \mathcal{H}(\mathbb{R}^n)$ with $t + s = h(A, B)$, $t, s > 0$, define $C(t)$ to be the set $(A)_t \cap (B)_s$. Lemma 3.6 of [Bogdewicz 2000] shows that for such A, B, t, and s, the set $C(t)$ is between A and B at a distance t from A. In [Braun et al. 2005] it is shown that any element $C \in \mathcal{H}(\mathbb{R}^n)$ with $h(A, C) = t$ satisfies $C \subset (A)_t$. It follows that if C is any element between A and B at a distance t from A, then $C \subset C(t)$. Thus, we can think of $C(t)$ as the largest element between A and B at a distance t from A. From this point forward, we set the convention that for any configuration $[A, B]$ with $0 < t < h(A, B)$, we have $s = h(A, B) - t$ and $C(t) = (A)_t \cap (B)_s$.

Example 3.1. Using our previous example with $t = s = 1$, we can see that $(A)_t$ is the union of the unit disk with an annulus with inner and outer radii of three and five, while $(B)_s$ is an annulus with inner and outer radii of one and three, so that $C(t) = (A)_t \cap (B)_s = S^1(1) \cup S^1(3) = C$, where C is the set pictured on the left side of Figure 1.

In general, one way to determine $\#([A, B])_t$ is to count the number of elements of $\mathcal{H}(\mathbb{R}^n)$ at a location t from A between A and B. Alternatively, we can count the number of ways to remove subsets $U \subset \mathbb{R}^n$ from the largest set $C(t)$ between A and B to get another element $C(t) \setminus U \in \mathcal{H}(\mathbb{R}^n)$ between A and B at a distance t from A. We note immediately that if $C(t) \setminus U$ is to be compact, U must be open in $C(t)$.

Recall that in a configuration $[A, B]$, we have that $h(A, B) = d(a, B) = d(b, A)$ for every $a \in A$ and $b \in B$. Thus, by the compactness of B, for every $a \in A$ there must be at least one $b \in B$ such that $d_E(a, b) = h(A, B)$, and likewise for each $b \in B$. This relation between pairs of points in A and B proves to be especially important, and so we make the following definition:

Definition 3.2. Let $A, B \in \mathcal{H}(\mathbb{R}^n)$. We say that $a \in A$ and $b \in B$ are *adjacent*, and write $a \leftrightharpoons b$, if $d_E(a, b) = h(A, B)$. The adjacency set of a in B, $[a]_B$, is defined to be $[a]_B = \{b \in B : a \leftrightharpoons b\}$.

Note that under the definition of adjacency, it is not necessary for the sets A and B to form a configuration, but for a configuration $[A, B]$, we have that for any $a \in A$ and $b \in B$, both $[a]_B$ and $[b]_A$ are nonempty. Referring back to our original example, we have for the origin $a_0 \in A$ that $[a_0]_B = B$, since every point $b \in B$ satisfies $d_E(a_0, b) = 2$. On the other hand, for any $b \in B$, we can write $b = 2e^{i\theta}$, and $[b]_A$ consists of the origin a_0 and $4e^{i\theta}$.

Suppose a configuration $[A, B]$ has largest set $C(t)$ between A and B. Lemma 3.1 of [Blackburn et al. 2009] says that for every $c \in C(t)$, there is precisely one $a \in A$ and $b \in B$ such that $c \leftrightharpoons a$ and $c \leftrightharpoons b$. Also, $[a]_{C(t)}$ and $[b]_{C(t)}$ are nonempty. Thus, the functions $q_A : C(t) \to A$ and $q_B : C(t) \to B$ that map c to these unique points a and b, respectively, are both well-defined and onto.

Now, we return to the idea of deciding which sets U we can remove from $C(t)$ to get some element $C(t) \setminus U$ between A and B at the same location. Observe that if we remove every point in $[a]_{C(t)}$ for some $a \in A$, then $d(a, C(t) \setminus U) > d(a, C(t)) = t$, and thus $C(t) \setminus U$ cannot be between A and B at the same location. Similarly, we cannot remove every point in $C(t)$ adjacent to some $b \in B$. Thus, we define a new collection of sets Υ_t, which will turn out to be the collection of removable sets described above:

Given $[A, B]$ and $0 < t < h(A, B)$, define Υ_t to be the collection of sets U open in $C(t)$ such that

- for every $a \in q_A(U)$, $[a]_{C(t)} \setminus U \neq \varnothing$, and
- for every $b \in q_B(U)$, $[b]_{C(t)} \setminus U \neq \varnothing$.

These two conditions ensure that for any $U \in \Upsilon_t$, we have $C(t) \setminus U$ is between A and B at a distance t from A. Note that $\varnothing$ is always an element of Υ_t, and $C(t)$ is never such an element. We set one more convention, that if $[A, B]$ is a configuration and $0 < t < h(A, B)$, then $\mathcal{H}_t$ is the collection of all elements of $\mathcal{H}(\mathbb{R}^n)$ between A and B at a distance t from A. More precisely, we have:

Theorem 3.3 [Blackburn et al. 2009]. *For any configuration* $[A, B]$ *and any* t *satisfying* $0 < t < h(A, B)$, *the function* $f : \Upsilon_t \to \mathcal{K}_t$ *defined by* $f(U) = C(t) \setminus U$ *is a bijection.*

From the theorem it follows that $\#([A, B])_t = |\mathcal{K}_t| = |\Upsilon_t|$. This is the exact tool we will need to show that no configuration $[A, B]$ and $0 < t < h(A, B)$ satisfies $\#([A, B])_t = |\mathbb{Z}|$. In our example, with $t = 1$, Υ_1 is the collection of all sets $U \neq S^1(1)$ that are open in $S^1(1)$. We observe that in this case Υ_1 is uncountable, verifying that there are uncountably many elements of $\mathcal{H}(\mathbb{R}^n)$ between A and B at a distance one from A.

4. Orders of infinity between sets in a configuration

Recall from above that if two elements A, B do not form a configuration, then there are uncountably many elements between A and B at every location. Thus, we may restrict our search for pairs of sets $A, B \in \mathcal{H}(\mathbb{R}^n)$ with countably many such elements to configurations. We use the fact that if $[A, B]$ is a configuration with $0 < t < h(A, B)$, then as stated above, $\#([A, B])_t = |\Upsilon_t|$.

We will need a definition and two lemmas to prove our main result.

Definition 4.1. A point w contained in a set W is a cluster point of W if for every $\epsilon > 0$, $B_\epsilon(w) \cap (W \setminus \{w\}) \neq \varnothing$. If w is not a cluster point, it is isolated.

The first lemma follows directly from our definition of Υ_t.

Lemma 4.2. *For a configuration* $[A, B]$ *with* $0 < t < h(A, B)$, *let* $U \in \Upsilon_t$. *If* $V \subset U$ *such that* V *is open in* C_t, *then* $V \in \Upsilon_t$ *as well.*

Proof. By definition of Υ_t, we have that

- for all $a \in q_A(U)$ there exists $c \in [a]_{C(t)}$ such that $c \notin U$ and
- for all $b \in q_B(U)$ there exists $c \in [b]_{C(t)}$ such that $c \notin U$.

This means that for all $a \in q_A(V)$, we must have that $a \in q_A(U)$ as $V \subset U$. Thus, there exists $c \in [a]_{C(t)}$ such that $c \notin U$, and so $c \notin V$. Similarly, for every $b \in q_B(V)$, there exists $c \in [b]_{C(t)}$ such that $b \notin V$. It follows that $V \in \Upsilon_t$. □

In other words, if we can remove some set of points U from $C(t)$ to get an element of $\mathcal{K}_t$, then certainly we can remove some relatively open subset V of U from $C(t)$ to get another element of $\mathcal{K}_t$. The next lemma takes a bit more work and lies at the core of our argument.

Lemma 4.3. *For a configuration* $[A, B]$ *with* $0 < t < h(A, B)$, *if* $\#([A, B])_t = \infty$, *then there exists* $W \in \Upsilon_t$ *such that* $|W| = \infty$.

Proof. Suppose by way of contradiction that $|U|$ is finite for every $U \in \Upsilon_t$. Let $U \in \Upsilon_t$, and choose a point $x \in U$. As $|U|$ is finite, we can find some $\epsilon_x > 0$ such that $B_{\epsilon_x}(x) \cap U = \{x\}$. Further, since U is relatively open in $C(t)$, we can choose ϵ_x to be small enough so that $B_{\epsilon_x}(x) \cap C \subset U$. Hence, if $V_x = B_{\epsilon_x}(x) \cap C$, then $V_x = \{x\}$ and certainly V_x is open in C and $V_x \subset U$, so by Lemma 4.2, $V_x \in \Upsilon_t$. Since $|\Upsilon_t| = \infty$ and every element $U \in \Upsilon_t$ can be written as a union of sets V_x, we must have infinitely many such singleton point sets as well.

Define

$$V = \bigcup_{\substack{x \in U \\ U \in \Upsilon_t}} V_x.$$

We split the proof into two cases. Suppose first that there exists some $a \in q_A(V)$ such that $|[a]_V| = \infty$. This means that a is adjacent to infinitely many points in V. Note that if $v_1, v_2 \in V$ satisfy $v_1 \Bumpeq a_0$, $v_2 \Bumpeq a_0$, $v_1 \Bumpeq b_0$, and $v_2 \Bumpeq b_0$ for some $a_0 \in A$ and $b_0 \in B$, then $v_1 = v_2$ by the uniqueness of betweenness in Euclidean geometry. Thus, every pair of distinct points v_1 and v_2 in $[a]_V$ must be adjacent to distinct points in B, or equivalently, q_B is injective on $[a]_V$. Fix a point $v^* \in [a]_V$, and let $W = [a]_V \setminus \{v^*\}$.

It is clear that $|W| = \infty$. We claim that $W \in \Upsilon_t$. First, note that W is the union of singleton point sets, each of which is open in $C(t)$, so W is open in C. We have established that $q_A(W) = \{a\}$, where $v^* \in [a]_{C(t)}$ but $v^* \notin W$. Now, let $b \in q_B(W)$. By the argument above, we have that b is adjacent to exactly one point $w \in W$. Further, $\{w\} \in \Upsilon$, so there exists some $c \in [b]_C$ such that $c \notin \{w\}$; that is, $c \neq w$. Since b is adjacent to no other points in W, it follows that $c \notin W$, as desired. We conclude that $W \in \Upsilon$, which is a contradiction to the assumption that every set in Υ_t is finite. A similar proof holds if there exists some $b \in q_B(V)$ such that $|[b]_V| = \infty$.

In the second case suppose that $[a]_V$ and $[b]_V$ are finite for every $a \in q_A(V)$ and $b \in q_B(V)$. Choose a point $v_1 \in V$ and let $a_1 = q_A(v_1)$ and $b_1 = q_B(v_1)$. Since $[a_1]_V$ and $[b_1]_V$ are finite while V is infinite, we can choose $v_2 \neq v_1 \in V$ such that v_2 is adjacent to neither a_1 nor b_1. Continuing in this manner, we can construct three infinite sequences of distinct points $\{a_i\}_i$, $\{b_i\}_i$, and $\{v_i\}_i$ such that v_m is adjacent to a_m and b_m but v_m is not adjacent to any of the points $a_1, \ldots, a_{m-1}, b_1, \ldots, b_{m-1}$.

Let $W = \bigcup v_i$. Then certainly $|W| = \infty$ and W is open in C as the union of open singleton sets. We claim that $W \in \Upsilon_t$. For any $a \in q_A(W)$, we know that $a = a_m$ for some integer m. Now $a_m \Bumpeq v_m$, and since $\{v_m\} \in \Upsilon_t$, we know that there exists some $c \in [a]_C$ such that $c \neq v_m$. But by definition of the sequence $\{v_i\}_i$, $c \neq v_i$ for any $i \neq m$, and thus $c \notin W$. A similar argument shows that for every $b \in q_B(W)$, there exists $c \in [b]_C$ such that $c \notin W$. We conclude that $W \in \Upsilon_t$, which is a contradiction, completing the proof. $\square$

Finally, we are in a position to prove our main theorem.

Theorem 4.4. *There exist no two sets A and B that have a countably infinite number of elements at any given location between A and B.*

Proof. Suppose by way of contradiction there exists a configuration $[A, B]$ and some t, $0 < t < h(A, B)$, such that $\#([A, B])_t = |\Upsilon_t| = |\mathbb{Z}|$. Thus, by Lemma 4.3 there exists some element $W \in \Upsilon_t$ such that $|W| = \infty$. We will find an infinite family of nonempty disjoint open subsets of C contained in W. There are two cases to consider. Suppose first that W contains infinitely many points which are isolated in W, and call a countably infinite subset of these points $\{w_i\}_i$. By definition $w_m \in W$ is isolated if there exists a ball $B_{\epsilon_m}(w_m)$ such that $B_{\epsilon_m}(w_m) \cap W = \{w_m\}$, and by choosing ϵ small enough, we can guarantee that $B_{\epsilon_m}(w_m) \cap C(t) \subset W$. Thus, if $W_m = B_{\epsilon_m}(w_m) \cap C = \{w_m\}$, then $\{W_i\}_i$ is a family of infinite disjoint open subsets of $C(t)$ contained in W.

In the second case, suppose that W contains finitely many isolated points, so that W contains infinitely many cluster points. Choose some cluster point $w_1 \in W$. Since there are finitely many isolated points in W, we can choose $\epsilon_1 > 0$ such that $B_{\epsilon_1}(w_1) \cap W$ contains only cluster points. Since w_1 is itself a cluster point in W, we know $|B_{\epsilon_1}(w_1) \cap W|$ must be infinite, and so if we shrink ϵ_1 further we can find a cluster point $w_2 \in W$ such that $w_2 \notin \overline{B_{\epsilon_1}(w_1)}$. Now, we choose ϵ_2 such that $B_{\epsilon_2}(w_2) \cap W$ consists of infinitely many cluster points and $B_{\epsilon_2}(w_2) \cap B_{\epsilon_1}(w_1) = \varnothing$. Shrinking ϵ_2 slightly yields a cluster point $w_3 \in W$ such that w_3 is in neither $\overline{B_{\epsilon_1}(w_1)}$ nor $\overline{B_{\epsilon_2}(w_2)}$. Continuing in this fashion, we can find an infinite sequence $\{w_i\}_i$ in W with corresponding radii $\{\epsilon_i\}_i$. Let $W_m = B_{\epsilon_m}(w_m) \cap W$. Then $\{W_i\}_i$ is a family of infinite disjoint open subsets of C contained in W.

In either case, we find a family $\{W_i\}_i$ of infinite pairwise disjoint open subsets of C contained in W. Let $2^{\mathbb{Z}}$ be the power set of $\mathbb{Z}$ and define a map $g : 2^{\mathbb{Z}} \to \Upsilon_t$ by

$$g(S) = \bigcup_{i \in S} W_i.$$

First, we note that for any $S \subset \mathbb{Z}$, we have $g(S) \in \Upsilon$ by Lemma 4.2, using the fact that each W_i is an open subset of $W \in \Upsilon$, so $\bigcup_{i \in S} W_i$ is also an open subset of W. Next, we claim that g is injective. But this is clear from the fact that the sets in $\{W_i\}_i$ are disjoint: if $S \neq S'$, then without loss of generality there is some $m \in S$ such that $n \notin S'$, so $W_m \subset g(S)$ whereas $W_m \cap g(S') = \varnothing$, and thus $g(S) \neq g(S')$. It follows directly that $|\Upsilon| \geq |2^{\mathbb{Z}}| = |\mathbb{R}|$. We conclude that $\#([A, B])_t \geq |\mathbb{R}|$, proving the theorem. $\square$

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