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Computing positive semidefinite minimum rank for small graphs

Steven Osborne and Nathan Warnberg

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The positive semidefinite minimum rank of a simple graph G is defined to be the smallest possible rank over all positive semidefinite real symmetric matrices whose ij-th entry (for $i \neq j$) is nonzero whenever $\{i, j\}$ is an edge in G and is zero otherwise. The computation of this parameter directly is difficult. However, there are a number of known bounding parameters and techniques which can be calculated and performed on a computer. We programmed an implementation of these bounds and techniques in the open-source mathematical software Sage. The program, in conjunction with the orthogonal representation method, establishes the positive semidefinite minimum rank for all graphs of order 7 or less.

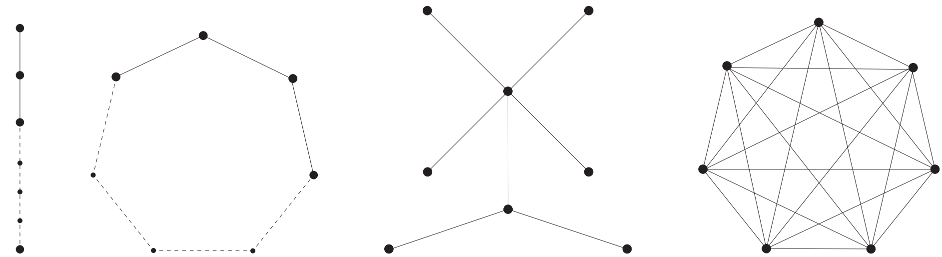
1. Introduction

Define a *graph* $G = (V, E)$ with *vertex set* $V = V(G)$ and *edge set* $E = E(G)$. The graphs discussed herein are simple (no loops or multiple edges) and undirected. The *order* of G, $|G|$, is the cardinality of $V(G)$. Two vertices v and w of a graph G are *neighbors* if $\{v, w\} \in E(G)$. If H is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ we call H a *subgraph* of G. H is an *induced subgraph* of G if H is a subgraph of G and if for all pairs $v, w \in V(H)$, $\{v, w\} \in E(H)$ if $\{v, w\} \in E(G)$. Given a set of vertices $S \subseteq V(G)$, $G - S$ is the induced subgraph of G with vertices $V(G) \setminus S$.

A graph $P = (V, E)$, where $V(P) = \{v_1, v_2, \ldots, v_n\}$, is called a *path* if the edges of the graph are exactly $\{v_i, v_{i+1}\}$ for $i = 1, 2, \ldots, n-1$. A *cycle* is a path that also has the edge $\{v_n, v_1\}$. A graph G is *chordal* if every induced cycle has length no greater than 3. A graph is *connected* if for any two vertices, v_1, v_2, there exists a path with endpoints v_1 and v_2. A connected graph with no cycles is a *tree*. An induced graph that is a tree is an *induced tree*. A graph with n vertices in which there is an edge between every vertex is called a *complete* graph and is denoted K_n. See Figure 1 for examples.

MSC2010: primary 05C50; secondary 15A03.

Keywords: zero forcing number, maximum nullity, minimum rank, positive semidefinite, zero forcing, graph, matrix.



(a) (b) (c) (d)

Figure 1. Examples of graphs: (a) a path; (b) a cycle; (c) a tree; (d) the complete graph on 7 vertices.

Let $S_n(\mathbb{R})$ denote the set of real symmetric $n \times n$ matrices. For $A = [a_{ij}] \in S_n(\mathbb{R})$, the *graph of* A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, 2, \ldots, n\}$ and edges $\big\{\{i, j\} : a_{ij} \neq 0 \text{ and } i \neq j\big\}$.

The *positive semidefinite maximum nullity* of G is

$$\mathrm{M}_+(G) = \max\{\text{null } A : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}$$

and the *positive semidefinite minimum rank* of G is

$$\mathrm{mr}_+(G) = \min\{\text{rank } A : A \in S_n(\mathbb{R}) \text{ is positive semidefinite and } \mathcal{G}(A) = G\}.$$

Clearly $\mathrm{mr}_+(G) + \mathrm{M}_+(G) = |G|$.

The following concept was introduced in [Barioli et al. 2010]: in a graph G where all vertices in some vertex set $S \subseteq V(G)$ are colored black and the remaining vertices are colored white, the *positive semidefinite color change rule* is: If $W_1, W_2, \ldots, W_k$ are the sets of vertices of the k connected components of $G - S$ ($k = 1$ is a possibility), $w \in W_i$, $u \in S$, and w is the only white neighbor of u in the subgraph of G induced by $V(W_i \cup S)$, then change the color of w to black, written as $u \to w$. Given an initial set B of black vertices, the *final coloring* of B is the set of vertices colored black as result of applying the positive semidefinite color change rule iteratively until no more vertices may be colored black. If the final coloring of B is $V(G)$, B is called a *positive semidefinite zero forcing set* of G. The *positive semidefinite zero forcing number* of a graph G, denoted $\mathrm{Z}_+(G)$, is the minimum of $|B|$ for all B positive semidefinite zero forcing sets of G. In [Barioli et al. 2010] it was shown that if G is a graph then $\mathrm{M}_+(G) \leq \mathrm{Z}_+(G)$.

Example 1.1. Consider the graph G in Figure 2(a) with the set $B = \{v_4\}$ initially colored black. When the positive semidefinite color change rule is applied, the connected component W_1 of $G - B$ is the induced subgraph of G on the vertices $\{v_1, v_2, v_3\}$. Since v_3 is the only white neighbor of v_4 in the subgraph of G induced by $W_1 \cup B$ (this is actually all of G), $v_4 \to v_3$ as demonstrated in Figure 2(b). For the

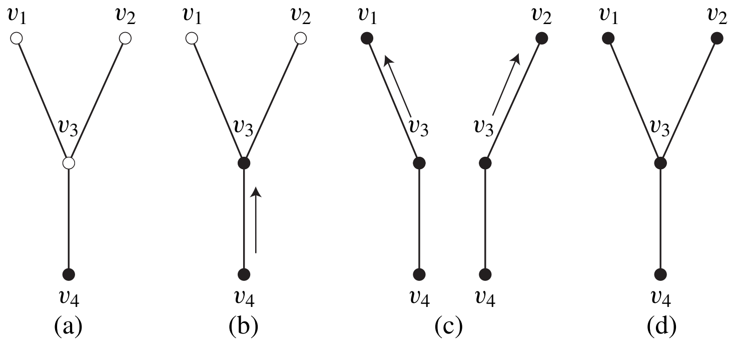


Figure 2. Illustrating Example 1.1.

next iteration, the set of black vertices is $B' = \{v_3, v_4\}$. The connected components of $G - B'$ are W_1', induced by $\{v_1\}$, and W_2', induced by $\{v_2\}$. Vertex v_1 is the only white neighbor of vertex v_3 in the subgraph of G induced by $W_1' \cup B'$ and v_2 is the only white neighbor of vertex v_3 in the subgraph of G induced by $W_2' \cup B'$. Therefore, $v_3 \to v_1$ and $v_3 \to v_2$; see Figure 2(c). Now, the entire graph has been forced black, as shown in Figure 2(d), and since the process was started by a single black vertex, $\mathrm{Z}_+(G) \le 1$. However, at least one vertex must be colored to begin the zero forcing process. Therefore, $\mathrm{Z}_+(G) = 1$.

Let G be a graph and S the smallest subset of $V(G)$ such that $G - S$ is disconnected. Then $|S| = \kappa(G)$ is called the *vertex connectivity* of G. A *clique covering* of G is a set of induced subgraphs $\{S_i\}$ of G such that each S_i is complete and $E(G) = \bigcup E(S_i)$. The *clique cover number* of a graph G, denoted $\mathrm{cc}(G)$, is the minimum of $|\{S_i\}|$ over all $\{S_i\}$ clique coverings of G.

In [Booth et al. 2008] $\mathrm{M}_+(G)$ was determined for every graph G of order at most 6. Use of published software (Zq.py; see [Butler and Grout 2011]) for computing $\mathrm{Z}_+(G)$ establishes $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$ for $|G| \le 6$. We developed a program (see [Osborne and Warnberg 2011a]) in the open-source computer mathematics software system Sage (sagemath.org) to compute bounds for positive semidefinite maximum nullity. The program uses Zq.py [Butler and Grout 2011] and known results for computing positive semidefinite maximum nullity. These results are summarized in Section 2. A detailed description of the program may be found in Appendix A. Sections 2 and 3 provide a survey of techniques for computing positive semidefinite minimum rank.

In Section 3 we determine $\mathrm{M}_+(G)$ for $|G| \le 7$ and show $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$ for all such graphs. For all but 13 graphs of order 7, $\mathrm{M}_+(G)$ can be computed by the program. We then established $\mathrm{M}_+(G)$ for the remaining 13 graphs by utilizing orthogonal representation to find a positive semidefinite matrix A with $\mathscr{G}(A) = G$ and nullity of $A = \mathrm{Z}_+(G)$. This establishes that $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$ for each graph G of order at most 7. These matrices are listed in Appendix B.

2. Known results used by the program to establish positive semidefinite minimum rank/maximum nullity

Note that all of our parameters sum over the connected components of a disconnected graph. Given its relation to the positive semidefinite zero forcing number, the following results are given in terms of positive semidefinite maximum nullity. However, given a graph G, $\mathrm{M}_+(G)+\mathrm{mr}_+(G)=|G|$, so all of the following results may easily be translated to positive semidefinite minimum rank.

Theorem 2.1 [Ekstrand et al. 2013]. *Let G be a graph.*

(i) $\mathrm{Z}_+(G)=1$ *if and only if* $\mathrm{M}_+(G)=1$.

(ii) $\mathrm{Z}_+(G)=2$ *if and only if* $\mathrm{M}_+(G)=2$.

(iii) $\mathrm{Z}_+(G)=3$ *implies* $\mathrm{M}_+(G)=3$.

Corollary 2.2. *If* $\mathrm{Z}_+(G)\geq 3$, *then* $\mathrm{M}_+(G)\geq 3$.

Observation 2.3 [Ekstrand et al. 2013]. $\mathrm{Z}_+(G)=|G|-1$ *if and only if* $\mathrm{M}_+(G)=|G|-1$.

Note that the only graph G having $\mathrm{Z}_+(G)=|G|-1$ is K_n, the complete graph on n vertices.

For a chordal graph G, it was shown in [Booth et al. 2008] that $\mathrm{cc}(G)=\mathrm{mr}_+(G)$, in [Hackney et al. 2009] it was shown that $OS(G)=\mathrm{cc}(G)$, and in [Barioli et al. 2010] it was shown that $\mathrm{Z}_+(G)+OS(G)=|G|$, where $OS(G)$ is the ordered subgraph number of G (see [Mitchell et al. 2010] for the definition of $OS(G)$). Thus $\mathrm{Z}_+(G)=\mathrm{M}_+(G)$, which gives the next theorem.

Theorem 2.4 [Barioli et al. 2010; Booth et al. 2008; Hackney et al. 2009]. *If G is chordal*, *then* $\mathrm{M}_+(G)=\mathrm{Z}_+(G)$.

Example 2.5. Consider graph $G551$ in Figure 3, left. Sets of vertices of size 1 or 2 are clearly not positive semidefinite zero forcing sets, so $\mathrm{Z}_+(G551)\geq 3$. Notice that choosing an initial set of 3 black vertices that are all nonadjacent does not force anything. By symmetry this reduces to two cases. In the first case we choose $\{1,2\}$ as our adjacent black vertices and as our third we choose any of the remaining vertices and notice that the graph will not be forced. Similarly, choosing $\{1,3\}$ as our adjacent black vertices and any of the remaining vertices as our third also fails to force the graph. Thus, $\mathrm{Z}_+(G551)\geq 4$. Observe that $\{1,3,4,5\}$ forms a positive semidefinite zero forcing set meaning $\mathrm{Z}_+(G551)\leq 4$, hence $\mathrm{Z}_+(G551)=4$. However, $G551$ is chordal as its largest cycle is size 3. Therefore, by Theorem 2.4 $\mathrm{M}_+(G551)=4$.

Theorem 2.6 [Lovász et al. 1989; 2000]. *For every graph* G, $\kappa(G)\leq\mathrm{M}_+(G)$.

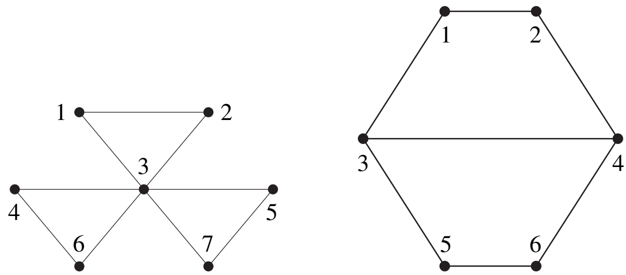


Figure 3. Graphs $G551$ (left) and $G128$ (right).

Example 2.7. By inspection, removing any one vertex from graph $G128$ (see Figure 3, right) will not result in a disconnected graph. Therefore, $\kappa(G) \geq 2$. Further, $\{3, 4\}$ forms a positive semidefinite zero forcing set for $G128$. Thus, $\mathrm{Z}_+(G) \leq 2$. This gives $2 \leq \kappa(G) \leq \mathrm{M}_+(G) \leq \mathrm{Z}_+(G) \leq 2$.

For a graph G the *neighborhood* of $v \in V(G)$ is

$$N_G(v) = \{w \in V(G) \mid v \text{ is adjacent to } w\}.$$

Vertices v and w are called *duplicate vertices* if $N_G(v) \cup \{v\} = N_G(w) \cup \{w\}$.

Proposition 2.8 [Ekstrand et al. 2013]. *If v and w are duplicate vertices in a connected graph G with $|G| \geq 3$, then* $\mathrm{Z}_+(G - v) = \mathrm{Z}_+(G) - 1$.

Proposition 2.9 [Booth et al. 2008]. *If v and w are duplicate vertices in a connected graph G with $|G| \geq 3$, then* $\mathrm{mr}_+(G - v) = \mathrm{mr}_+(G)$.

Recall that for any graph G, $\mathrm{mr}_+(G) + \mathrm{M}_+(G) = |G|$, which gives the following corollary.

Corollary 2.10. *If v and w are duplicate vertices in a connected graph G with $|G| \geq 3$, then* $\mathrm{M}_+(G - v) = \mathrm{M}_+(G) - 1$.

Example 2.11. In graph $G1196$ (see Figure 4, left) notice that 2 and 4 are duplicate vertices, as are vertices 3 and 5. Removal of vertices 2 and 3 results in a graph that is isomorphic to graph $G43$ (see Figure 4, right). $\mathrm{Z}_+(G43) = 2$ thus $\mathrm{M}_+(G43) = 2$ by Theorem 2.1. Therefore, $\mathrm{M}_+(G1196) = 4$ by Corollary 2.10.

Cut-vertex reduction is a standard technique in the study of minimum rank. A vertex v of a connected graph G is a *cut-vertex* if $G - v$ is disconnected. Suppose G_i, $i = 1, \ldots, h$, are graphs of order at least two, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \cup_{i=1}^{h} G_i$ (if $h \geq 2$, then clearly v is a cut-vertex of G). Then it is observed in [van der Holst 2009] that

$$\mathrm{mr}_+(G) = \sum_{i=1}^{h} \mathrm{mr}_+(G_i).$$

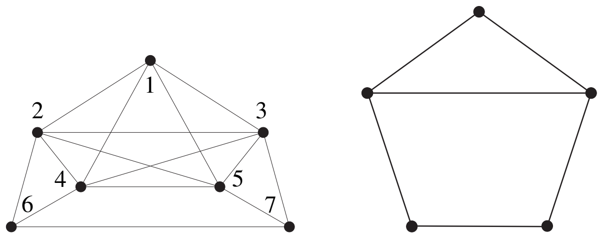


Figure 4. Graphs $G1196$ (left) and $G43$ (right).

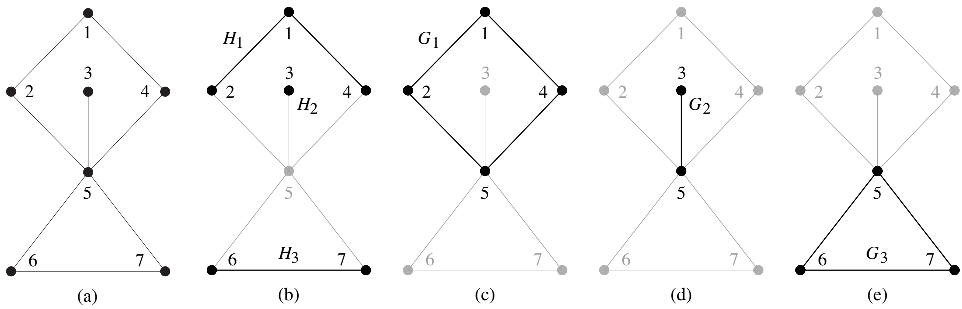


Figure 5. Graph $G419$.

Because $\mathrm{mr}_+(G) + \mathrm{M}_+(G) = |G|$, this is equivalent to

$$\mathrm{M}_+(G) = \left(\sum_{i=1}^{h} \mathrm{M}_+(G_i)\right) - h + 1. \tag{1}$$

It is shown in [Mitchell et al. 2010] that

$$OS(G) = \sum_{i=1}^{h} OS(G_i).$$

Since $OS(G) + \mathrm{Z}_+(G) = |G|$ (shown in [Barioli et al. 2010]), this is equivalent to

$$\mathrm{Z}_+(G) = \left(\sum_{i=1}^{h} \mathrm{Z}_+(G_i)\right) - h + 1. \tag{2}$$

Example 2.12. Equation (2) can be used to compute $\mathrm{Z}_+(G419)$ and $\mathrm{M}_+(G419)$ (see Figure 5(a)). Notice that vertex 5 is a cut vertex of the graph since removing it results in a disconnected graph with 3 components, namely H_1, H_2 and H_3. When vertex 5 is reconnected to each of our components it is easy to see that $G_i \cap G_j = \{5\}$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$, as illustrated by Figures 5(c)–(e). It is also clear that $\cup_{i=1}^{3} G_i = G419$, $\mathrm{Z}_+(G_1) = 2$, $\mathrm{Z}_+(G_2) = 1$, and $\mathrm{Z}_+(G_3) = 2$. Thus, by

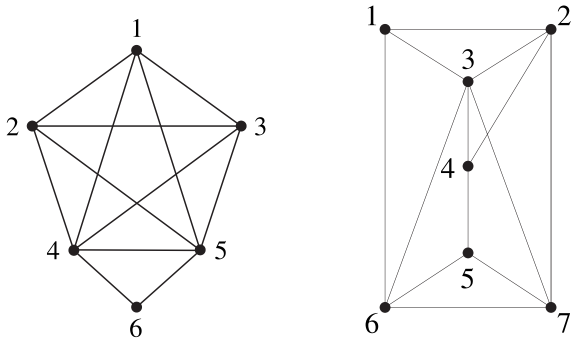


Figure 6. Graphs $G200$ (left) and $G1090$ (right).

Equation (2), $Z_+(G419) = 2+1+2-3+1 = 3$. A similar argument shows that $M_+(G419) = 3$.

Observe that if $\kappa(G) = 1$, there exists a cut vertex. The next result is an immediate consequence of the cut-vertex reduction Equations (1) and (2).

Observation 2.13 [Ekstrand et al. 2013]. *Suppose G_i, $i = 1, \dots, h$ are graphs, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^{h} G_i$. If $M_+(G_i) = Z_+(G_i)$ for all $i = 1, \dots, h$, then $M_+(G) = Z_+(G)$.*

Observation 2.14 [Hackney et al. 2009]. *If G is a graph then $\mathrm{cc}(G) \geq \mathrm{mr}_+(G)$.*

Corollary 2.15. $|G| - \mathrm{cc}(G) \leq M_+(G)$.

Example 2.16. In Figure 6, left, notice that graph $G200$ is not complete so

$$\mathrm{mr}_+(G200) \geq 2.$$

Also, note that the subgraphs induced by $S_1 = \{1, 2, 3, 4, 5\}$ and $S_2 = \{4, 5, 6\}$ are complete and $E(G200) = E(S_1) \cup E(S_2)$ so $\mathrm{cc}(G200) \leq 2$, hence $\mathrm{mr}_+(G200) = 2$.

In [Booth et al. 2008] the *tree size* of a graph G, denoted $\mathrm{ts}(G)$, is defined to be the number of vertices in a maximum induced tree of G. Also from [Booth et al. 2008], if T is a maximum induced tree and w is a vertex not belonging to T, denote by $\mathscr{E}(w)$ the set of all edges of all paths in T between every pair of vertices of T that are adjacent to w. The following theorem was established by Booth et al. [2008].

Theorem 2.17 [Booth et al. 2008]. *For a connected graph G,*

$$\mathrm{mr}_+(G) = \mathrm{ts}(G) - 1 \tag{3}$$

if the following condition holds: there exists a maximum induced tree T such that for u and w not on T, $\mathscr{E}(u) \cap \mathscr{E}(w) \neq \varnothing$ if and only if u and w are adjacent in G.

Note that Equation (3) may be rewritten as $M_+(G) = |G| - \mathrm{ts}(G) + 1$.

Example 2.18. To illustrate the previous theorem we consider graph $G1090$ (see Figure 6, right). To find $\mathrm{ts}(G1090)$ notice that $G1090$ has two disjoint, induced K_3's, namely the graphs induced by vertex sets $\{1, 2, 3\}$ and $\{5, 6, 7\}$. This means in order to find an induced tree, removal of one vertex from each K_3 is required. By inspection, removal of any of the nine pairs $\big\{\{1, 5\}, \{1, 6\}, \{1, 7\}, \{2, 5\}, \dots, \{3, 7\}\big\}$ results in a graph with a cycle, thus $\mathrm{ts}(G1090) \le 4$. However, the subgraph induced by $\{1, 4, 5, 6\}$ is a tree (call it T), hence $\mathrm{ts}(G1090) = 4$. We show T satisfies the condition of Theorem 2.17. The vertices not in $G1090 - T$ are 2, 3, and 7, which are all adjacent in $G1090$.

$$\mathscr{E}(2) = \big\{(1, 6), (5, 6), (4, 5)\big\} = \mathscr{E}(3) \quad \text{and} \quad \mathscr{E}(7) = \big\{(5, 6)\big\}.$$

Therefore, $\mathscr{E}(2) \cap \mathscr{E}(3) \cap \mathscr{E}(7) \ne \varnothing$ and the condition holds because $\{2, 3, 7\}$ are pairwise adjacent. Thus $\mathrm{M}_+(G1090) = 4$.

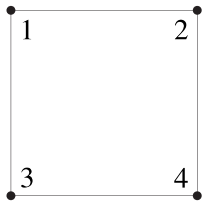
3. Computation of positive semidefinite maximum nullity of graphs of order 7 or less

The program developed by Osborne and Warnberg [2011a] implements the results from Section 2. Running the program on all graphs of order 7 or less yielded positive semidefinite maximum nullity for 1239 of 1252 graphs. It may be noted that the positive semidefinite maximum nullity was already known for the 208 graphs of order 6 or less (see [Booth et al. 2008]). However, the program was able to successfully compute the positive semidefinite maximum nullity for these graphs without referencing this information. For the remaining 13 graphs, the method of orthogonal representations was used to construct a matrix representation exhibiting nullity equal to the positive semidefinite zero forcing number. These matrices are shown in Appendix B.

A set $\vec{V} = \{\vec{v}_1, \dots, \vec{v}_n\}$ in $\mathbb{R}^d$ is an *orthogonal representation* of the graph G if for $i \ne j$, the dot product of $\vec{v}_i$ with $\vec{v}_j$ is nonzero if the vertices i and j are adjacent, and zero otherwise. If $\vec{V} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal representation of the graph G in $\mathbb{R}^d$ and $B = [\vec{v}_1 \dots \vec{v}_n]$, then $B^T B \in \mathscr{S}_+(G)$ and rank $B^T B \le d$. Thus, if a representation is found in $\mathbb{R}^d$ then $\mathrm{mr}_+(G) \le d$ and $\mathrm{M}_+(G) \ge |G| - d$.

Example 3.1. Consider graph $G17$ in Figure 7, left. Note that when we refer to a graph in the form $G17$ we are using notation from [Read and Wilson 1998]. To start constructing an orthogonal representation for $G17$ let $v_1, v_2, v_3, v_4 \in \mathbb{R}^2$ correspond to vertices 1, 2, 3 and 4 respectively. Choose as many disjoint vertices as possible, say 1 and 4. By definition $v_1 \cdot v_4 = 0$ so let $v_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $v_4 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. To find v_2 and v_3, set

$$v_2 = \begin{bmatrix} ca_2 \\ b_2 \end{bmatrix} \quad \text{and} \quad v_3 = \begin{bmatrix} a_3 \\ b_3 \end{bmatrix}.$$



$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix}$$

Figure 7. Graph $G17$ (left); A, a matrix representation of $G17$ (right).

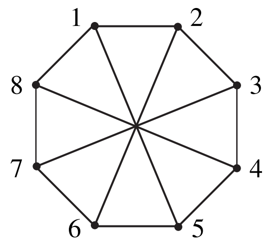


Figure 8. Möbius ladder on 8 vertices.

Now, v_2 is adjacent to v_1 and v_4 so $v_1 \cdot v_2 \neq 0$ and $v_2 \cdot v_4 \neq 0$. Thus $a_2 \neq 0 \neq b_2$. Similarly, $a_3 \neq 0 \neq b_3$. Since v_2 and v_3 are not adjacent, we know $v_2 \cdot v_3 = a_2 a_3 + b_2 b_3 = 0$. With these restrictions it is clear that $a_2 = a_3 = b_2 = 1$ and $b_3 = -1$ is a solution and an orthogonal representation construction is complete. This gives

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B^T B = A$$

(see Figure 7, right). By construction, $\operatorname{rank}(A) = 2$. Thus $\operatorname{mr}_+(G17) \leq 2$ and $\operatorname{M}_+(G17) \geq |G| - 2 = 2$. Observe that $\{1, 2\}$ forms a positive semidefinite zero forcing set for graph $G17$ hence $\operatorname{Z}_+(G17) \leq 2$. Finally, $2 \leq \operatorname{M}_+(G17) \leq \operatorname{Z}_+(G17) \leq 2$.

In every case, positive semidefinite maximum nullity was found to equal the positive semidefinite zero forcing number. This has established the next result.

Theorem 3.2. *If G is a graph with* 7 *or fewer vertices*, *then* $\operatorname{M}_+(G) = \operatorname{Z}_+(G)$.

See [Osborne and Warnberg 2011b] for a complete spreadsheet containing positive semidefinite maximum nullity and zero forcing number for all graphs with 7 or fewer vertices.

Corollary 3.3. *Suppose G_i, $i = 1, \ldots, h$, are graphs with $|G_i| \leq 7$, there is a vertex v such that for all $i \neq j$, $G_i \cap G_j = \{v\}$, and $G = \bigcup_{i=1}^{h} G_i$. Then* $\operatorname{M}_+(G) = \operatorname{Z}_+(G)$.

Proof. Apply Theorem 3.2 to Observation 2.13. □

Note that Theorem 3.2 cannot be extended to graphs with more than 7 vertices as $\operatorname{Z}_+(V_8) = 4$ and $\operatorname{M}_+(V_8) = 3$ (shown in [Mitchell et al. 2010]), where V_8 is the Möbius ladder on 8 vertices (see Figure 8).

Appendix A: Method used by the program

The program uses the following general method:

(1) Separate the graph into its connected components and work on each component separately. Results will be summed before reporting.

(2) Compute $Z_+(G)$.

(a) If $Z_+(G) \leq 3$, apply the results of Theorem 2.1.

(b) Else, use Corollary 2.2 to establish a lower bound for $M_+(G)$.

(3) If $Z_+(G) = |G| - 1$, apply the results of Observation 2.3.

(4) If G is chordal, apply Theorem 2.4.

(5) Compute the vertex connectivity of G ($\kappa(G)$).

(a) If $\kappa(G) = Z_+(G)$, apply Theorem 2.6.

(b) Else, if $\kappa(G)$ is a tighter bound for $M_+(G)$, improve the lower bound.

(6) If there are duplicate vertices in the graph, discard all but one copy by applying Corollary 2.10 and returning to step 2.

(7) Apply the cut-vertex formula iteratively by applying Equation (1) and returning to step 2 for each component.

(8) Compute the clique cover number of G.

(a) If $|G| - \mathrm{cc}(G) = Z_+(G)$, apply Corollary 2.15.

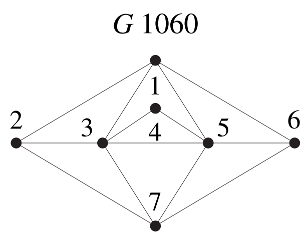
(b) Else, if $\mathrm{cc}(G)$ is a tighter bound for $M_+(G)$, improve the lower bound.

(9) Apply Theorem 2.17 to determine if $M_+(G) = |G| - \mathrm{ts}(G) + 1$.

Appendix B: Matrix representations

Each of the following thirteen matrices satisfies $\mathrm{null}(A) = 4 = Z_+(G)$.

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 2 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 2 \end{bmatrix}$$



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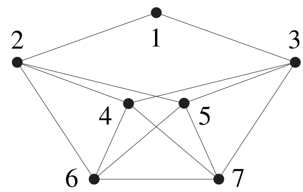
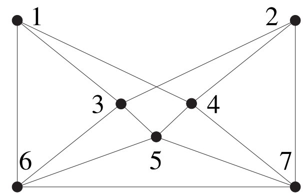
$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & 3 & 1 & 0 \\ 1 & 0 & 2 & -2 & 1 & 0 & -1 \\ 0 & -1 & -2 & 5 & 0 & 1 & 3 \\ 0 & 3 & 1 & 0 & 5 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & -1 & 3 & 1 & 1 & 2 \end{bmatrix}$$

G 1075



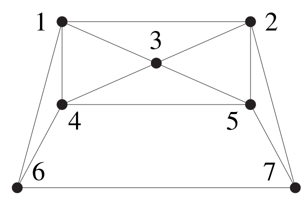
$$\begin{bmatrix} 1 & 0 & -1 & 4 & 0 & -1 & 0 \\ 0 & 1 & 4 & 2 & 0 & 0 & 1 \\ -1 & 4 & 33 & 0 & -4 & -15 & 0 \\ 4 & 2 & 0 & 21 & 1 & 0 & 3 \\ 0 & 0 & -4 & 1 & 1 & 4 & 1 \\ -1 & 0 & -15 & 0 & 4 & 17 & 4 \\ 0 & 1 & 0 & 3 & 1 & 4 & 2 \end{bmatrix}$$

G 1100



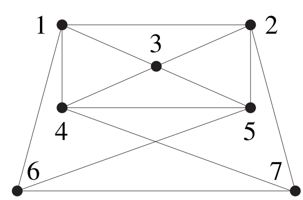
$$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 3 & 0 \\ 1 & 6 & 7 & 0 & -1 & 0 & 1 \\ 1 & 7 & 10 & -1 & -3 & 0 & 0 \\ 2 & 0 & -1 & 5 & 1 & 7 & 0 \\ 0 & -1 & -3 & 1 & 2 & 0 & 1 \\ 3 & 0 & 0 & 7 & 0 & 11 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

G 1104



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & -1 & 0 \\ 1 & 3 & 2 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 6 & 1 & 0 & -2 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}$$

G 1105



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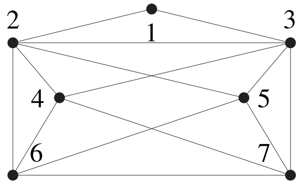
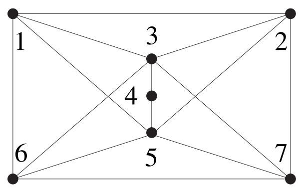
$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 3 & -2 & 1 & 1 & 3 & 0 \\ -1 & -2 & 6 & -2 & 1 & 0 & 3 \\ 0 & 1 & -2 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 3 & 0 & 1 & 2 & 5 & 1 \\ 0 & 0 & 3 & -1 & 1 & 1 & 2 \end{bmatrix}$$

G 1135



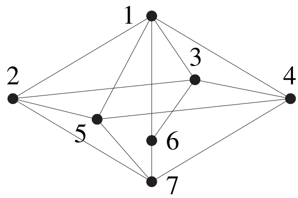
$$\begin{bmatrix} 2 & 1 & -3 & 0 & 3 & -1 & 0 \\ 1 & 1 & -2 & 0 & 2 & 0 & 1 \\ -3 & -2 & 30 & 5 & 0 & 1 & -1 \\ 0 & 0 & 5 & 1 & 1 & 0 & 0 \\ 3 & 2 & 0 & 1 & 6 & -1 & 1 \\ -1 & 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 & 2 \end{bmatrix}$$

G 1137



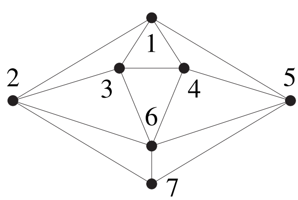
$$\begin{bmatrix} 3 & 1 & -3 & 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 0 & 2 & 0 & 1 \\ -3 & 2 & 21 & -4 & 0 & -1 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 & 1 \\ 3 & 2 & 0 & 1 & 5 & 0 & 3 \\ 1 & 0 & -1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 1 & 3 & -2 & 6 \end{bmatrix}$$

G 1165



$$\begin{bmatrix} 1 & 2 & 1 & 1 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 0 & 2 & 1 \\ 1 & 1 & 2 & 3 & 0 & -1 & 0 \\ 1 & 0 & 3 & 5 & -1 & -2 & 0 \\ 1 & 0 & 0 & -1 & 11 & -2 & -3 \\ 0 & 2 & -1 & -2 & -2 & 2 & 1 \\ 0 & 1 & 0 & 0 & -3 & 1 & 1 \end{bmatrix}$$

G 1167



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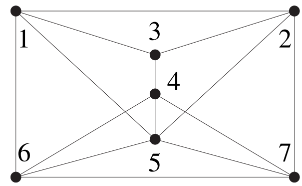
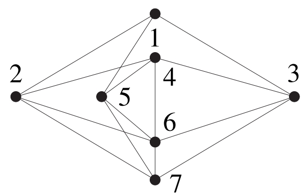
$$\begin{bmatrix} 2 & -3 & 1 & 0 & 1 & 1 & 0 \\ -3 & 6 & -1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & 2 & 3 & 1 \\ 1 & -1 & 0 & 2 & 2 & 3 & 1 \\ 1 & 0 & 0 & 3 & 3 & 5 & 2 \\ 0 & 1 & 0 & 1 & 1 & 2 & 1 \end{bmatrix}$$

G 1168



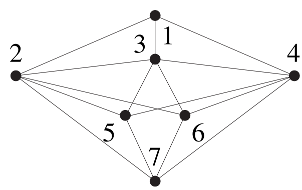
$$\begin{bmatrix} 1 & 1 & 3 & 0 & 2 & 0 & 0 \\ 1 & 6 & 0 & -2 & 0 & -1 & 1 \\ 3 & 0 & 14 & 2 & 0 & 3 & 1 \\ 0 & -2 & 2 & 1 & -1 & 1 & 0 \\ 2 & 0 & 0 & -1 & 21 & -5 & -4 \\ 0 & -1 & 3 & 1 & -5 & 2 & 1 \\ 0 & 1 & 1 & 0 & -4 & 1 & 1 \end{bmatrix}$$

G 1169



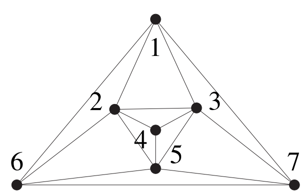
$$\begin{bmatrix} 1 & -4 & 1 & 1 & 0 & 0 & 0 \\ -4 & 21 & -2 & 0 & 1 & -3 & -1 \\ 1 & -2 & 2 & 2 & 1 & -1 & 0 \\ 1 & 0 & 2 & 6 & -1 & -3 & -2 \\ 0 & 1 & 1 & -1 & 2 & 0 & 1 \\ 0 & -3 & -1 & -3 & 0 & 2 & 1 \\ 0 & -1 & 0 & -2 & 1 & 1 & 1 \end{bmatrix}$$

G 1202



$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & -3 \\ 1 & 3 & 1 & 1 & 1 & 4 & 0 \\ 1 & 1 & 3 & 3 & 1 & 0 & -4 \\ 0 & 1 & 3 & 5 & 2 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 4 & 0 & 0 & 1 & 6 & 2 \\ -3 & 0 & -4 & 0 & 1 & 2 & 14 \end{bmatrix}$$

G 1205



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