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An elementary approach to characterizing Sheffer A-type 0 orthogonal polynomial sequences

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In 1939, Sheffer published "Some properties of polynomial sets of type zero", which has been regarded as an indispensable paper in the theory of orthogonal polynomials. Therein, Sheffer basically proved that every polynomial sequence can be classified as belonging to exactly one type. In addition to various interesting and important relations, Sheffer's most influential results pertained to completely characterizing all of the polynomial sequences of the most basic type, called A-type 0, and subsequently establishing which of these sets were also orthogonal. However, Sheffer's elegant analysis relied heavily on several characterization theorems. In this work, we show all of the Sheffer A-type 0 orthogonal polynomial sequences can be characterized by using only the generating function that defines this class and a monic three-term recurrence relation.

1. Introduction

In his seminal work, I. M. Sheffer [1939] basically showed that every polynomial sequence can be classified as belonging to exactly one *type*. The majority of his paper was dedicated to developing a wealth of aesthetic results regarding the most basic type, entitled *A-type 0*. This included various interesting characterization theorems. Moreover, one of Sheffer's most important results was his classification of the A-type 0 orthogonal sets, which are often simply called the Sheffer sequences. Sheffer attributed these orthogonal sets to J. Meixner, who originally discovered them in [Meixner 1934]. The Sheffer sequences (also called Meixner polynomials) are now known to be the very well-studied and applicable Laguerre, Hermite, Charlier, Meixner, Meixner–Pollaczek and Krawtchouk polynomials — refer to [Koekoek and Swarttouw 1996] for details regarding these polynomials and the references therein for additional theory and applications.

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In this paper, we develop and employ an elementary method for characterizing all of the aforementioned Sheffer A-type 0 orthogonal polynomials (Meixner polynomials) that is entirely different than Sheffer's approach. Furthermore, the analysis herein comprises the *most basic* complete characterization of the Sheffer sequences. We also mention that although a very terse overview of the essence of our methodology (for obtaining necessary conditions) is essentially addressed in [Ismail 2009, pp. 524, 525], the rigorous details of applying our approach do not appear anywhere in the current literature.

Since the publication of [Sheffer 1939], a wealth of papers have been written related to the Sheffer sequences, many of which are quite recent. One such work that also develops a basic-type of characterization is [Di Bucchianico and Loeb 1994]. Other papers include [Al-Salam and Verma 1970; Di Bucchianico 1994; Di Nardo et al. 2011; Dominici 2007; Hofbauer 1981; Popa 1997; 1998; Shukla and Rapeli 2011]. In addition, a very large amount of work has been completed pertaining to the theory and applications of specific A-type 0 orthogonal sets, e.g., [Akleylek et al. 2010; Chen et al. 2011; Coffey 2011; Coulembier et al. 2011; Dueñas and Marcellán 2011; Ferreira et al. 2008; Hutník 2011; Khan et al. 2011; Kuznetsov 2008; Miki et al. 2011; Mouayn 2010; Sheffer 1941; Vignat 2011; Wang et al. 2011; Wang and Wong 2011; Yalçinbaş et al. 2011]. Indeed, research on the Sheffer sequences is an active area and important in its own right. Therefore, our current characterization of such a class is certainly of interest.

In order to sufficiently lay the foreground for our analysis, we first discuss all of the preliminary definitions and terminologies that are utilized throughout this paper. Then, we give a concise overview of Sheffer's method for determining the A-type 0 orthogonal polynomial sequences. We conclude this section by briefly summarizing the sections that follow.

1A. ***Preliminaries.*** Throughout this work, we make use of each of the following definitions and terminology.

Definition 1.1. We always assume that a *set* or *sequence* of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is such that each $P_n(x)$ has degree exactly n.

Definition 1.2. A set of polynomials $\{p_n(x)\}_{n=0}^{\infty}$ is *monic* if $p_n(x)-x^n$ is of degree at most $n-1$, or equivalently if the leading coefficient of each $p_n(x)$ is unitary.

Definition 1.3. The set of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is *orthogonal* if it satisfies one of the two following weighted inner product conditions:

$$\text{Continuous}: \ \langle P_m(x), P_n(x)\rangle = \int_{\Omega_1} P_m(x)P_n(x)w(x)\,dx = \alpha_n\delta_{m,n}, \tag{1-1}$$

$$\text{Discrete}: \ \langle P_m(x), P_n(x)\rangle = \sum_{\Omega_2} P_m(x)P_n(x)w(x) = \beta_n\delta_{m,n}, \tag{1-2}$$

where $\delta_{m,n}$ denotes the Kronecker delta

$$\delta_{m,n} := \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n, \end{cases}$$

with $\Omega_1 \subseteq \mathbb{R}$, $\Omega_2 \subseteq \{0, 1, 2, \ldots\}$ and $w(x) > 0$, called the *weight function*.

The Laguerre, Hermite and Meixner–Pollaczek polynomials satisfy a continuous orthogonality relation of the form (1-1). On the other hand, the Charlier, Meixner and Krawtchouk polynomials satisfy a discrete orthogonality relation of the form (1-2); see [Koekoek and Swarttouw 1996].

Definition 1.4. We write each of our orthogonal polynomials in the *hypergeometric form* (${}_rF_s$) as

$$ {}_rF_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \,\middle|\, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r)_k}{(b_1, \ldots, b_s)_k} \frac{z^k}{k!}, \tag{1-3}$$

where the *Pochhammer symbol* $(a)_k$ is defined as

$$(a)_k := a(a+1)(a+2)\cdots(a+k-1), \quad (a)_0 := 1,$$

with

$$(a_1, \ldots, a_j)_k := (a_1)_k \cdots (a_j)_k.$$

The sum (1-3) terminates if one of the numerator parameters is a negative integer; e.g., if one such parameter is $-n$, then (1-3) is a finite sum over $0 \le k \le n$.

Definition 1.5. We define a *linear generating function* for a polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ by

$$\sum_{\Lambda} \xi_n P_n(x) t^n = F(x, t),$$

where $\{\xi_n\}_{n=0}^{\infty}$ is a sequence in n, independent of x and t, with $\Lambda \subseteq \{0, 1, 2, \ldots\}$. Moreover, we say that the function $F(x, t)$ *generates* the set $\{P_n(x)\}_{n=0}^{\infty}$.

It is important to mention that a linear generating function need not converge, as several relationships can be derived when $F(x, t)$ is divergent. For example, by expanding each of the generating functions used in this paper as a *formal* power series in t, the respective polynomial $P_k(x)$ can be determined by evaluating the coefficient of t^k.

It is well-known that a necessary and sufficient condition for a set of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ to be orthogonal is that it satisfies a three-term recurrence relation (see [Rainville 1960]), which can be written in different forms. In particular, we utilize the following two forms in this work, and adhere to the nomenclature used in [Al-Salam 1990].

Definition 1.6 (the three-term recurrence relations). It is a necessary and sufficient condition that an orthogonal set $\{P_n(x)\}_{n=0}^{\infty}$ satisfies an *unrestricted three-term recurrence relation* of the form

$$P_{n+1}(x) = (A_n x + B_n)P_n(x) - C_n P_{n-1}(x), \quad A_n A_{n-1} C_n > 0, \qquad \text{where } P_{-1}(x) = 0 \text{ and } P_0(x) = 1. \tag{1-4}$$

If $p_n(x)$ represents the monic form of $P_n(x)$, then it is a necessary and sufficient condition that $\{p_n(x)\}_{n=0}^{\infty}$ satisfies the following *monic three-term recurrence relation*

$$p_{n+1}(x) = (x + b_n)p_n(x) - c_n p_{n-1}(x), \ c_n > 0, \qquad \text{where } P_{-1}(x) = 0 \text{ and } P_0(x) = 1. \tag{1-5}$$

1B. ***A summary of Sheffer's A type-0 analysis.*** In order to determine each of the A-type 0 orthogonal sets previously discussed, Sheffer first developed a characterization theorem, which gave necessary and sufficient conditions for a polynomial sequence to be A-type 0 via a linear generating function. Meixner [1934] essentially determined which orthogonal sets satisfy the A-type 0 generating function using a different approach than Sheffer. Meixner used the A-type 0 generating function as the *definition* of the A-type 0 class. In our present work, we follow Meixner's convention. The reader can also refer to [Al-Salam 1990] for a concise overview of Meixner's analysis. In addition, for rigorous developments of the methods of Sheffer and Meixner, as well as related results, extensions and applications, see [Galiffa 2013].

Definition 1.7. A polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ is classified as A-type 0 if there exist $\{a_j\}_{j=0}^{\infty}$ and $\{h_j\}_{j=1}^{\infty}$ such that

$$A(t)e^{xH(t)} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1-6}$$

with

$$A(t) := \sum_{n=0}^{\infty} a_n t^n, \quad a_0 = 1 \quad \text{and} \quad H(t) := \sum_{n=1}^{\infty} h_n t^n, \quad h_1 = 1. \tag{1-7}$$

To determine which orthogonal sets satisfy (1-6), Sheffer utilized a monic three-term recurrence relation of the form

$$P_n(x) = (x + \lambda_n)P_{n-1}(x) - \mu_n P_{n-2}(x), \quad n = 1, 2, \ldots. \tag{1-8}$$

Along with several additional results, Sheffer essentially established the following:

Theorem 1.8. *A necessary and sufficient condition for an A-type* 0 *set* $\{P_n(x)\}_{n=0}^{\infty}$ *to satisfy* (1-8) *is that*

$$\lambda_n = \alpha + bn \quad \textit{and} \quad \mu_n = (n-1)(c + dn),$$

with $c + dn \neq 0$ *for* $n > 1$.

In other words, Sheffer proved that in order for an A-type 0 set $\{P_n(x)\}_{n=0}^{\infty}$ defined by (1-6) to be orthogonal, it must be that λ_n is at most linear in n and μ_n is at most quadratic in n.

Since in our present work we make use of a monic three-term recurrence relation of the form (1-5), i.e., the contemporary form, we scale (1-8) via $n \mapsto n+1$, giving

$$P_{n+1}(x) = (x+\lambda_{n+1})P_n(x) - \mu_{n+1}P_{n-1}(x), \quad n = 0, 1, 2, \dots, \tag{1-9}$$

and the recursion coefficients in Theorem 1.8 therefore take on the form

$$\lambda_{n+1} = (\alpha+b)+bn, \qquad \mu_{n+1} = (c+d)n + dn^2. \tag{1-10}$$

Theorem 1.8, again along with additional results, eventually led Sheffer to the following characterizing theorem, which yields all of the general A-type 0 orthogonal sets in terms of their linear generating functions, and which is written below using the same notation as in [Sheffer 1939].

Theorem 1.9. *A polynomial set $\{P_n(x)\}_{n=0}^{\infty}$ is A-type* 0 *and orthogonal if and only if $A(t)e^{xH(t)}$ in* (1-6) *is of one of the following forms*:

$$A(t)e^{xH(t)} = \mu(1-bt)^c \exp\left\{\frac{d+atx}{1-bt}\right\}, \quad abc\mu \neq 0, \tag{1-11}$$

$$A(t)e^{xH(t)} = \mu \exp[t(b+ax)+ct^2], \quad ac\mu \neq 0, \tag{1-12}$$

$$A(t)e^{xH(t)} = \mu e^{ct}(1-bt)^{d+ax}, \quad abc\mu \neq 0, \tag{1-13}$$

$$A(t)e^{xH(t)} = \mu(1-t/c)^{d_1+x/a}(1-t/b)^{d_2-x/a}, \quad abc\mu \neq 0,\ b \neq c. \tag{1-14}$$

By judiciously choosing each of the parameters in (1-11)–(1-14) we can achieve all of the Sheffer A-type 0 orthogonal sets. For emphasis, we write each of these parameter selections below and then display the corresponding generating function as it appears in [Koekoek and Swarttouw 1996]. We also call upon each of these generating relations in Section 4.

The Laguerre polynomials. In (1-11), we select the parameters as $\mu = 1$, $a = -1$, $b = 1$, $c = -(\alpha+1)$ and $d = 0$ to obtain

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (1-t)^{-(\alpha+1)} \exp\left(\frac{xt}{t-1}\right). \tag{1-15}$$

The Hermite polynomials. With the assignments $\mu = 1$, $a = 2$, $b = 0$ and $c = -1$ in (1-12), we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)t^n = \exp(2xt - t^2). \tag{1-16}$$

The Charlier polynomials. If in (1-13) we choose $\mu = 1$, $a = 1$, $b = 1/\alpha$, $c = 1$, and $d = 0$, then we obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} C_n(x;\alpha)t^n = e^t \left(1 - \frac{t}{\alpha}\right)^x. \tag{1-17}$$

The Meixner polynomials. In (1-14), we select $\mu = 1$, $a = 1$, $b = 1$, c arbitrary, $d_1 = 0$ and $d_2 = -\beta$ leading to

$$\sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M(x;\beta,c)t^n = \left(1 - \frac{t}{c}\right)^x (1-t)^{-(x+\beta)}. \tag{1-18}$$

The Meixner–Pollaczek polynomials. Taking $\mu = 1$, $a = -i$, $b = e^{i\phi}$, $c = e^{-i\phi}$ and $d_1 = d_2 = -\lambda$ in (1-14) leads to

$$\sum_{n=0}^{\infty} P_n^{(\lambda)}(x;\phi)t^n = (1 - e^{i\phi}t)^{-\lambda+ix}(1 - e^{-i\phi}t)^{-\lambda-ix}. \tag{1-19}$$

The Krawtchouk polynomials. Lastly, selecting $\mu = 1$, $a = 1$, $b = -1$, $c = p/(1-p)$, $d_1 = 0$ and $d_2 = N$ in (1-14) yields

$$\sum_{n=0}^{N} \mathrm{C}(N,n)K_n(x;p,N)t^n = \left(1 - \frac{1-p}{p}t\right)^x (1+t)^{N-x}, \tag{1-20}$$

for $x = 0, 1, 2, \ldots, N$, where $\mathrm{C}(N,n)$ denotes the binomial coefficient.

Interestingly enough, Sheffer only stated (1-15) and (1-16) by their names, i.e., the Laguerre and Hermite polynomials respectively. Moreover, at the time when [Sheffer 1939] was published, the remaining orthogonal polynomials were not yet commonly referred to by the names above; the exception to this being the Charlier polynomials, which were called the Poisson–Charlier polynomials by Meixner [1934] and others.

1C. ***An overview of our present A-type 0 analysis.*** Our current work amounts to determining which A-type 0 polynomial sequences are also orthogonal by utilizing only the generating function (1-6) and the monic three-term recurrence relation (1-9), without calling upon any additional relationships. It is in this regard that our approach is elementary. The remainder of this paper is organized as follows.

In Section 2, we derive necessary conditions for the Sheffer A-type 0 recursion coefficients λ_{n+1} and μ_{n+1} as in (1-10), which in fact comprise only the terms a_1, a_2, h_2 and h_3 in (1-7). In Section 3, we prove that the A-type 0 orthogonal sets are necessarily the monic forms of the Laguerre, Hermite, Charlier, Meixner, Meixner–Pollaczek and Krawtchouk polynomials by appropriately selecting the parameters a_1, a_2, h_2 and h_3. As a supplement to this analysis, in Section 4 we first derive linear generating functions for each of the monic forms of the A-type 0 orthogonal sets using (1-15)–(1-20). From these relations, we obtain the same parameter values as those in Section 3. We conclude this paper in Section 5 by showing that the conditions on the recursion coefficients λ_{n+1} and μ_{n+1} from Section 2 are also sufficient. This provides six additional basic characterizations of our orthogonal sets.

2. Deriving the Sheffer A-type 0 recursion coefficients

In this section, we derive necessary conditions for the recursion coefficients λ_{n+1} and μ_{n+1} to be as in (1-10). In order to do this, we first determine the coefficients of x^n, x^{n-1} and x^{n-2} of the arbitrary Sheffer A-type 0 polynomial $P_n(x)$ in (1-6), which we label as $c_{n,0}$, $c_{n,1}$ and $c_{n,2}$, respectively. To obtain these values, we compare the coefficients of $x^k t^n$ for $k = n, n-1, n-2$ on both sides of (1-6). After these leading coefficients are discovered, we substitute our polynomial $P_n(x) = c_{n,0}x^n + c_{n,1}x^{n-1} + c_{n,2}x^{n-2} + \mathscr{O}(x^{n-3})$ into the three-term recurrence relation (1-4) and derive a system of simultaneous linear equations, the solution of which yields the recursion coefficients A_n, B_n and C_n as in (1-4). We then transform the resulting unrestricted recurrence relation into monic form, which gives λ_{n+1} and μ_{n+1}.

We begin by expanding the left side of (1-6) and accounting for $h_1 = 1$ via (1-7):

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n t^n \cdot \exp(x(t + h_2 t^2 + h_3 t^3 + \cdots)) \\
&\qquad = \sum_{n=0}^{\infty} a_n t^n \cdot \exp(xt) \cdot \exp(h_2 x t^2) \cdot \exp(h_3 x t^3) \cdots \\
&\qquad = \sum_{k_0=0}^{\infty} a_{k_0} t^{k_0} \cdot \sum_{k_1=0}^{\infty} \frac{(xt)^{k_1}}{k_1!} \cdot \sum_{k_2=0}^{\infty} \frac{(h_2 x t^2)^{k_2}}{k_2!} \cdot \sum_{k_3=0}^{\infty} \frac{(h_3 x t^3)^{k_3}}{k_3!} \cdots .
\end{aligned}$$

We next express the general term in each of the products above as

$$a_{k_0} t^{k_0} \cdot \frac{x^{k_1} t^{k_1}}{k_1!} \cdot \frac{h_2^{k_2} x^{k_2} t^{2k_2}}{k_2!} \cdot \frac{h_3^{k_3} x^{k_3} t^{3k_3}}{k_3!} \cdots . \tag{2-1}$$

Thus, discovering the coefficient of $x^r t^s$ is equivalent to determining all of the nonnegative integer solutions $\{k_0, k_1, k_2, \ldots\}$ to the linear Diophantine equations

$$k_1 + k_2 + k_3 + \cdots = r, \tag{2-2}$$

$$k_0 + k_1 + 2k_2 + 3k_3 + \cdots = s, \tag{2-3}$$

where (2-2) represents the x-exponents and (2-3) the t-exponents. We can now discover the coefficients $x^n t^n$, $x^{n-1} t^n$ and $x^{n-2} t^n$, which we partition into the three parts below.

The coefficient of $x^n t^n$. For this case, we subtract (2-2) from (2-3) with $r = n$ and $s = n$, yielding

$$k_0 + k_2 + 2k_3 + \cdots = 0.$$

It is then readily seen that k_1 is a free variable and $k_0 = k_2 = k_3 = \cdots = 0$. Thus, from substituting these values into (2-3) with $s = n$, we see that $k_1 = n$, and after comparing with (2-1) we observe that the coefficient of $x^n t^n$ is $1/n!$.

The coefficient of $x^{n-1} t^n$. Here, we subtract (2-2) from (2-3) with $r = n - 1$ and $s = n$, which gives

$$k_0 + k_2 + 2k_3 + \cdots = 1,$$

yielding two cases:

Case 1. $k_0 = 1$ and $k_2 = k_3 = \cdots = 0$. Substituting these values into (2-3) gives $k_1 = n - 1$, and via (2-1) we achieve

$$\frac{a_1}{(n-1)!}.$$

Case 2. $k_2 = 1$ and $k_0 = k_3 = \cdots = 0$. Now, (2-3) becomes $k_1 = n - 2$, and from (2-1) we have

$$\frac{h_2}{(n-2)!}.$$

Therefore, we know that the coefficient of $x^{n-1} t^n$ is

$$\frac{a_1}{(n-1)!} + \frac{h_2}{(n-2)!}.$$

The coefficient of $x^{n-2} t^n$. Lastly, we subtract (2-2) from (2-3) with $r = n - 2$ and $s = n$, and obtain

$$k_0 + k_2 + 2k_3 = 2,$$

which has four solutions, yielding four cases. In the same way as in the previous cases, we see that the coefficient of $x^{n-2} t^n$ is

$$\frac{a_2}{(n-2)!} + \frac{a_1 h_2 + h_3}{(n-3)!} + \frac{h_2^2}{2!(n-4)!};$$

the details have been omitted for brevity. Thus, we have established the following:

Lemma 2.1. *For the Sheffer A-type* 0 *polynomial* $P_n(x) = c_{n,0} x^n + c_{n,1} x^{n-1} + c_{n,2} x^{n-2} + \mathcal{O}(x^{n-3})$ *as in* (1-6), *we have*

$$\begin{aligned} c_{n,0} &= \frac{1}{n!}, \qquad c_{n,1} = \frac{a_1}{(n-1)!} + \frac{h_2}{(n-2)!}, \\ c_{n,2} &= \frac{a_2}{(n-2)!} + \frac{a_1 h_2 + h_3}{(n-3)!} + \frac{h_2^2}{2!(n-4)!}. \end{aligned} \tag{2-4}$$

Proof. See the above analysis. ☐

We now have the following result:

Theorem 2.2. *The Sheffer A-type* 0 *recursion coefficients* A_n, B_n *and* C_n *satisfying* (1-4) *are given by*

$$A_n = \frac{1}{n+1}, \qquad B_n = \frac{a_1 + 2h_2 n}{n+1},$$
$$C_n = \frac{a_1^2 - 2a_2 + 2a_1 h_2 - 4h_2^2 + 3h_3 + (4h_2^2 - 3h_3)n}{n+1}. \tag{2-5}$$

Proof. We see that upon substituting $P_n(x) = c_{n,0}x^n + c_{n,1}x^{n-1} + c_{n,2}x^{n-2} + \mathscr{O}(x^{n-3})$ into the three-term recurrence relation (1-4), we obtain

$$\begin{aligned} c_{n+1,0}x^{n+1} + c_{n+1,1}x^n + c_{n+1,2}x^{n-1} + \mathscr{O}(x^{n-2}) \\ = A_n c_{n,0}x^{n+1} + A_n c_{n,1}x^n + A_n c_{n,2}x^{n-1} + \mathscr{O}(x^{n-2}) \\ + B_n c_{n,0}x^n + B_n c_{n,1}x^{n-1} + B_n c_{n,2}x^{n-2} + \mathscr{O}(x^{n-3}) \\ - C_n c_{n-1,0}x^{n-1} - C_n c_{n-1,1}x^{n-2} - C_n c_{n-1,2}x^{n-3} + \mathscr{O}(x^{n-4}). \end{aligned}$$

Thus, comparing the coefficients of x^{n+1}, x^n and x^{n-1} above results in the lower-triangular simultaneous system of linear equations

$$\begin{bmatrix} c_{n,0} & 0 & 0 \\ c_{n,1} & c_{n,0} & 0 \\ c_{n,2} & c_{n,1} & -c_{n-1,0} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = \begin{bmatrix} c_{n+1,0} \\ c_{n+1,1} \\ c_{n+1,2} \end{bmatrix}.$$

Since the diagonal terms $c_{n,0}$ and $c_{n-1,0}$ are nonzero by Definition 1.1, the solution to the above system is unique and determined via Gauss–Jordan Elimination to be

$$A_n = \frac{c_{n+1,0}}{c_{n,0}}, \qquad B_n = \frac{c_{n+1,1}c_{n,0} - c_{n+1,0}c_{n,1}}{c_{n,0}^2},$$
$$C_n = \frac{c_{n+1,0}(c_{n,0}c_{n,2} - c_{n,1}^2) + c_{n,0}(c_{n+1,1}c_{n,1} - c_{n+1,2}c_{n,0})}{c_{n-1,0}c_{n,0}^2}.$$

Substituting (2-4) accordingly yields our desired result. ☐

We now determine λ_{n+1} and μ_{n+1}. To accomplish this, we must derive a monic three-term recurrence relation of the form (1-9) from the recursion coefficients (2-5). Thus, we replace $P_n(x)$ with $d_n Q_n(x)$ in (1-4), resulting in

$$Q_{n+1}(x) = \frac{d_n}{d_{n+1}} A_n x Q_n(x) + \frac{d_n}{d_{n+1}} B_n Q_n(x) - \frac{d_{n-1}}{d_{n+1}} C_n Q_{n-1}(x).$$

Therefore, we require

$$\frac{d_n}{d_{n+1}} A_n = 1,$$

which is a first-order linear difference equation readily solved via iterations to be

$$d_n = \frac{1}{n!}.$$

Then, we have

$$\lambda_{n+1} = \frac{d_n}{d_{n+1}} B_n = a_1 + 2h_2 n, \tag{2-6}$$

$$\mu_{n+1} = \frac{d_{n-1}}{d_{n+1}} C_n = (a_1^2 - 2a_2 + 2a_1 h_2 - 4h_2^2 + 3h_3)n + (4h_2^2 - 3h_3)n^2. \tag{2-7}$$

Thus, we have shown that λ_{n+1} is at most linear in n and that μ_{n+1} is at most quadratic in n. Hence, we have the following statement:

Theorem 2.3. *For a polynomial sequence $\{P_n(x)\}_{n=0}^{\infty}$ to be A-type 0 and orthogonal, the recursion coefficients λ_{n+1} and μ_{n+1} in*

$$P_{n+1}(x) = (x + \lambda_{n+1}) P_n(x) - \mu_{n+1} P_{n-1}(x), \quad n = 0, 1, 2, \ldots$$

must necessarily be of the form

$$\lambda_{n+1} = c_1 + c_2 n \quad \text{and} \quad \mu_{n+1} = c_3 n + c_4 n^2, \qquad c_1, \ldots, c_4 \in \mathbb{R},$$

with $\mu_{n+1} > 0$.

Interestingly enough, the parameters $c_1, \ldots, c_4$ above are only in terms of the first two nonunitary coefficients of t in $A(t)$ and $H(t)$ of (1-7), i.e., a_1, a_2, h_2 and h_3. Furthermore, in regard to Sheffer's analysis, we can readily write λ_{n+1} and μ_{n+1} in Theorem 2.3 as in (1-10) and uniquely determine the parameters α, b, c and d.

Corollary 2.4. *The parameters α, b, c and d in the Sheffer A-type 0 monic recursion coefficients λ_{n+1} and μ_{n+1} of (1-10) are*

$$\alpha = a_1 - 2h_2, \quad b = 2h_2, \quad c = a_1^2 - 2a_2 + 2a_1 h_2 - 8h_2^2 + 6h_3 \quad \text{and} \quad d = 4h_2^2 - 3h_3.$$

3. The Sheffer A-type 0 orthogonal polynomials

In this section, we prove the following theorem, which relies on the analysis conducted in Section 2:

Theorem 3.1. *The following orthogonal polynomial sequences all necessarily belong to the Sheffer A-type 0 class:*

$$\{(-1)^n n! L_n^{(\alpha)}(x)\}, \quad \{2^{-n} H_n(x)\}, \quad \{(-a)^n C_n(x; a)\}, \quad \left\{ \frac{c^n (\beta)_n}{(c-1)^n} M_n(x; \beta, c) \right\},$$
$$\{(2 \sin\phi)^{-n} n! P_n^{(\lambda)}(x; \phi)\}, \quad \{(-N)_n p^n K_n(x; p, N)\};$$

these are respectively the monic forms of the Laguerre, Hermite, Charlier, Meixner, Meixner–Pollaczek and Krawtchouk polynomials, as defined in (1-15)–(1-20).

Proof. We first substitute λ_{n+1} and μ_{n+1} as in (2-6) and (2-7), respectively, into (1-9). We therefore see that every A-type 0 orthogonal set must necessarily satisfy a monic three-term recurrence of the form

$$\begin{aligned} P_{n+1}(x) &= [x + a_1 + 2h_2 n] P_n(x) \\ &\quad - \big[(a_1^2 - 2a_2 + 2a_1 h_2 - 4h_2^2 + 3h_3)n + (4h_2^2 - 3h_3)n^2\big] P_{n-1}(x). \end{aligned} \tag{3-1}$$

We now separately consider each of the monic three-term recurrence relations for the Laguerre, Hermite, Charlier, Meixner, Meixner–Pollaczek and Krawtchouk polynomials, and then uniquely determine the values that the parameters a_1, a_2, h_2 and h_3 must take in each case.

The Laguerre polynomials. The Laguerre polynomials satisfy a monic three-term recurrence relation of the form

$$\mathscr{L}_{n+1}^{(\alpha)}(x) = (x - (\alpha+1) - 2n)\mathscr{L}_n^{(\alpha)}(x) - (\alpha n + n^2)\mathscr{L}_{n-1}^{(\alpha)}(x), \tag{3-2}$$

where

$$\mathscr{L}_n^{(\alpha)}(x) := (-1)^n n! L_n^{(\alpha)}(x) \tag{3-3}$$

with

$$L_n^{(\alpha)}(x) := \frac{(\alpha+1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \;\Big|\; x \right).$$

Therefore, comparing (3-2) with (3-1), we see that

$$a_1 = -(\alpha+1), \quad a_2 = \tfrac{1}{2}(\alpha+1)(\alpha+2), \quad h_2 = -1, \quad h_3 = 1. \tag{3-4}$$

Thus, $\big\{(-1)^n n! L_n^{(\alpha)}(x)\big\}_{n=0}^{\infty}$ is a Sheffer A-type 0 orthogonal set.

The Hermite polynomials. The monic recurrence relation for the Hermite polynomials is

$$\mathscr{H}_{n+1}(x) = x\mathscr{H}_n(x) - \tfrac{1}{2} n \mathscr{H}_{n-1}(x), \tag{3-5}$$

where

$$\mathscr{H}_n(x) := 2^{-n} H_n(x) \tag{3-6}$$

with

$$H_n(x) := 2^n x^n {}_2F_0\left(\begin{matrix} -n/2,\, (1-n)/2 \\ - \end{matrix} \;\Big|\; -\frac{1}{x^2} \right).$$

From comparing (3-5) with (3-1), we obtain

$$a_1 = 0, \quad a_2 = -1/4, \quad h_2 = 0, \quad h_3 = 0. \tag{3-7}$$

Thus, $\{2^{-n} H_n(x)\}_{n=0}^{\infty}$ is a Sheffer A-type 0 orthogonal set.

The Charlier polynomials. The Charier polynomials satisfy a monic three-term recurrence relation of the form

$$\mathscr{C}_{n+1}(x) = (x - a - n)\mathscr{C}_n(x) - an\mathscr{C}_{n-1}(x) \tag{3-8}$$

where

$$\mathscr{C}_n(x) := (-1)^n a^n C_n(x; a) \tag{3-9}$$

with

$$C_n(x; a) := {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix} \,\middle|\, -\frac{1}{a}\right).$$

Therefore, weighing (3-8) against (3-1), we see that

$$a_1 = -a, \quad a_2 = \frac{a^2}{2!}, \quad h_2 = -1/2, \quad h_3 = 1/3, \tag{3-10}$$

and we conclude that $\{(-1)^n a^n C_n(x; a)\}_{n=0}^{\infty}$ is a Sheffer A-type 0 orthogonal set.

The Meixner polynomials. The monic three-term recurrence relation for the Meixner polynomials is

$$\mathscr{M}_{n+1}(x) = \left(x + \frac{c\beta}{c-1} + \frac{c+1}{c-1}n\right)\mathscr{M}_n(x) - \left(\frac{\beta-1}{(c-1)^2}cn + \frac{c}{(c-1)^2}n^2\right)\mathscr{M}_{n-1}(x), \tag{3-11}$$

where

$$\mathscr{M}_n(x) := (\beta)_n \left(\frac{c}{c-1}\right)_n^{nM}(x; \beta, c), \tag{3-12}$$

with

$$M_n(x; \beta, c) := {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix} \,\middle|\, 1 - \frac{1}{c}\right).$$

Then, from comparing (3-11) with (3-1) we arrive at

$$a_1 = \frac{c\beta}{c-1}, \quad a_2 = \frac{c^2\beta(\beta+1)}{2(c-1)^2}, \quad h_2 = \frac{c+1}{2(c-1)}, \quad h_3 = \frac{1+c+c^2}{3(c-1)^2}. \tag{3-13}$$

Hence, we have shown that $\{c^n(\beta)_n/(c-1)^n M_n(x; \beta, c)\}_{n=0}^{\infty}$ is a Sheffer A-type 0 orthogonal set.

The Meixner–Pollaczek polynomials. The Meixner–Pollaczek polynomials have the monic three-term recurrence relation

$$\mathscr{P}_{n+1}(x) = \left(x + \frac{\lambda}{\tan\phi} + \frac{n}{\tan\phi}\right)\mathscr{P}_n(x) - \left(\frac{2\lambda - 1}{4\sin^2\phi}n + \frac{n^2}{4\sin^2\phi}\right)\mathscr{P}_{n-1}(x), \tag{3-14}$$

where

$$\mathscr{P}_n(x) := \frac{n!}{(2\sin\phi)^n} P_n^{(\lambda)}(x; \phi), \tag{3-15}$$

with

$$P_n^{(\lambda)}(x;\phi) := \frac{(2\lambda)_n}{n!} e^{in\phi} {}_2F_1\left(\begin{matrix} -n, \lambda+ix \\ 2\lambda \end{matrix} \,\middle|\, 1-e^{-2i\phi}\right), \quad \lambda > 0 \quad \text{and} \quad \phi \in (0,\pi).$$

After comparing (3-14) with (3-1), we obtain

$$a_1 = \lambda\cot\phi, \qquad a_2 = \frac{4\cos^2\phi\lambda(\lambda+1)-2\lambda}{8\sin^2\phi},$$
$$h_2 = \frac{1}{2}\cot\phi, h_3 = \frac{1}{4}\cot^2\phi - \frac{1}{12}. \tag{3-16}$$

Thus, $\{(2\sin\phi)^{-n} n! P_n^{(\lambda)}(x;\phi)\}_{n=0}^{\infty}$ is a Sheffer A-type 0 orthogonal set.

The Krawtchouk polynomials. The Krawtchouk polynomials have

$$\begin{aligned}\mathscr{K}_{n+1}(x) = {} & [x - pN + (2p-1)n]\mathscr{K}_n(x) \\ & -[p(1-p)(N+1)n - p(1-p)n^2]\mathscr{K}_{n-1}(x)\end{aligned} \tag{3-17}$$

as a monic recurrence relation, where

$$\mathscr{K}_n(x) := (-N)_n p^n K_n(x; p, N) \tag{3-18}$$

with

$$K_n(x; p, N) := {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix} \,\middle|\, \frac{1}{p}\right), \quad n = 0, 1, 2, \ldots, N.$$

After equating the recursion coefficients in (3-17) with those in (3-1) it follows that

$$a_1 = -Np, \quad a_2 = \tfrac{1}{2}(N-1)Np^2, \quad h_2 = p - \tfrac{1}{2}, \quad h_3 = p^2 - p + \tfrac{1}{3} \tag{3-19}$$

and therefore $\{(-N)_n p^n K_n(x; p, N)\}_{n=0}^{\infty}$ is a Sheffer A-type 0 orthogonal set.

Hence, we have now established the theorem. □

4. Verification of parameters via generating function expansion

Here, we supplement the analysis of the previous two sections by implementing a procedure for discovering the a_1, a_2, h_2 and h_3 parameters for each of the Sheffer A-type 0 orthogonal polynomials obtained in Section 3 by using their corresponding generating functions. This analysis yields explicit power series expansions for $A(t)$ and $H(t)$ in (1-7) for each of the A-type 0 orthogonal sets.

The method used throughout this section is as follows. Momentarily, let us assume that $P_n(x)$ is a Sheffer A-type 0 orthogonal polynomial and $p_n(x)$ is its corresponding monic form. Then notice via the proof of Theorem 3.1 that these polynomials must be related in the following way

$$p_n(x) = a_n b^n P_n(x), \tag{4-1}$$

where b is a polynomial parameter, a function of a polynomial parameter, or a constant and a_n is a sequence in n. Furthermore, let us assume that $\{P_n(x)\}_{n=0}^{\infty}$ has a linear generating function of the form

$$\sum_{\Lambda} c_n P_n(x) t^n = F(x,t).$$

Then, we uniquely determine d_n such that $a_n d_n = c_n$, multiply (4-1) by $d_n t^n$ and sum over Λ to obtain

$$\sum_{\Lambda} d_n p_n(x) t^n = \sum_{\Lambda} c_n P_n(x)(bt)^n = F(x,bt). \tag{4-2}$$

The relation (4-2) is a generating function for $\{p_n(x)\}_{n=0}^{\infty}$. Simply stated, we see that it was achieved via the transformation $t \mapsto bt$ of the generating function for $\{P_n(x)\}_{n=0}^{\infty}$. After deriving a relation of the form (4-2), we then can determine $A(t)$ and $H(t)$ and construct their Maclaurin series expansions, from which we can deduce a_1, a_2, h_2 and h_3 and compare them accordingly with those of Section 3.

The Laguerre polynomials. Multiplying relation (3-3) by $t^n/n!$ and summing for $n = 0, 1, 2, \ldots$ gives

$$\sum_{n=0}^{\infty} \frac{1}{n!}\mathscr{L}_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)(-t)^n = (1+t)^{-(\alpha+1)} \exp\left(\frac{xt}{1+t}\right)$$

via (1-15). This yields the following relations for $A(t)$ and $H(t)$:

$$\begin{aligned} A(t) = (1+t)^{-(\alpha+1)} &= \sum_{n=0}^{\infty} \frac{(-1)^n(\alpha+1)_n}{n!} t^n \\ &= 1 - (\alpha+1)t + \tfrac{1}{2}(\alpha+1)(\alpha+2)t^2 + \cdots, \end{aligned}$$

and

$$H(t) = \frac{t}{1+t} = \sum_{n=1}^{\infty} (-1)^{n+1} t^n = t - t^2 + t^3 + \cdots.$$

Thus, we see that a_1, a_2, h_2 and h_3 above correspond exactly with those in (3-4).

The Hermite polynomials. We multiply the relation (3-6) by $t^n/n!$, sum for $n = 0, 1, 2, \ldots$ and then utilize (1-16) to obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!}\mathscr{H}_n(x)t^n = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\big(\tfrac{1}{2}t\big)^n = \exp\big(xt - \tfrac{1}{4}t^2\big).$$

Upon writing this relation in the form $A(t)e^{xH(t)}$, we see that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{H}_n(x) t^n = \exp\bigl(-\tfrac{1}{4}t^2\bigr) \exp(xt),$$

which gives the following expressions for $A(t)$ and $H(t)$:

$$A(t) = \exp\bigl(-\tfrac{1}{4}t^2\bigr) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n} n!} = 1 - \tfrac{1}{4}t^2 + \cdots , \qquad H(t) = t.$$

Hence, we realize that a_1, a_2, h_2 and h_3 are exactly the same as those in (3-7).

The Charlier polynomials. By multiplying the relation (3-9) by $t^n/n!$, summing for $n = 0, 1, 2, \ldots$ and then using (1-17), we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{C}_n(x; a) t^n = \sum_{n=0}^{\infty} \frac{1}{n!} C_n(x; a)(-at)^n = e^{-at}(1+t)^x.$$

We then put this result in the form $A(t)e^{xH(t)}$:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{C}_n(x; a) t^n = e^{-at} e^{x \ln(1+t)},$$

which leads to the relations for $A(t)$ and $H(t)$

$$A(t) = e^{-at} = \sum_{n=0}^{\infty} \frac{(-a)^n t^n}{n!} = 1 - at + \frac{a^2}{2!} t^2 + \cdots ,$$

$$H(t) = \ln(1+t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} t^n}{n} = t - \tfrac{1}{2}t^2 + \tfrac{1}{3}t^3 + \cdots ,$$

and we observe that a_1, a_2, h_2 and h_3 above are indiscernible from those in (3-10).

The Meixner polynomials. We multiply (3-12) by $t^n/n!$, sum for $n = 0, 1, 2, \ldots$ and then use (1-18), which gives

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{M}(x; \beta, c) t^n &= \sum_{n=0}^{\infty} \frac{(\beta)_n}{n!} M(x; \beta, c) \left(\frac{ct}{c-1}\right)^n \\ &= \left(1 - \frac{t}{c-1}\right)^x \left(1 - \frac{ct}{c-1}\right)^{-(x+\beta)}.\end{aligned}$$

Rewriting this result in the form $A(t)e^{xH(t)}$, we see that

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{M}(x; \beta, c) t^n = \bigl(1 - ct/(c-1)\bigr)^{-\beta} \exp\left(x \ln\left(\frac{c-1-t}{c-1-ct}\right)\right),$$

which in turn gives the following relations for $A(t)$ and $H(t)$:

$$\begin{aligned} A(t) &= \bigl(1-ct/(c-1)\bigr)^{-\beta} \\ &= \sum_{n=0}^{\infty} \frac{(\beta)_n c^n}{n!(c-1)^n} t^n = 1 + \frac{c\beta}{c-1} t + \frac{c^2\beta(\beta+1)}{2(c-1)^2} t^2 + \cdots, \\ H(t) &= \ln(c-1-t) - \ln(c-1-ct) \\ &= \sum_{n=1}^{\infty} \frac{c^n-1}{n(c-1)^n} t^n = t + \frac{c+1}{2(c-1)} t^2 + \frac{1+c+c^2}{3(c-1)^2} t^3 + \cdots, \end{aligned}$$

and hence a_1, a_2, h_2 and h_3 are identical to those in (3-13).

The Meixner–Pollaczek polynomials. We multiply the relation (3-15) by $t^n/n!$ and sum for $n = 0, 1, 2, \ldots$ We then use (1-19), and obtain

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{P}_n(x;\phi) t^n = \sum_{n=0}^{\infty} P_n(x;\phi) \left(\frac{t}{2\sin\phi}\right)^n = \left(1 - \frac{e^{i\phi}t}{2\sin\phi}\right)^{-\lambda+ix} \left(1 - \frac{e^{-i\phi}t}{2\sin\phi}\right)^{-\lambda-ix}.$$

Rewriting this result in the form $A(t)e^{xH(t)}$, we have

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{P}_n(x;a) t^n = \left[\left(1 - \frac{e^{i\phi}t}{2\sin\phi}\right)\left(1 - \frac{e^{-i\phi}t}{2\sin\phi}\right)\right]^{-\lambda} \exp\left[x \ln\left(\left(\frac{1-e^{i\phi}t/(2\sin\phi)}{1-e^{-i\phi}t/(2\sin\phi)}\right)^i\right)\right],$$

which leads to $A(t)$ and $H(t)$ below:

$$\begin{aligned} A(t) &= \left[\left(1 - \frac{e^{i\phi}t}{2\sin\phi}\right)\left(1 - \frac{e^{-i\phi}t}{2\sin\phi}\right)\right]^{-\lambda} = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{(\lambda)_k(\lambda)_{n-k} e^{i(n-2k)\phi}}{2^n k!(n-k)!\sin^n\phi}\right] t^n \\ &= 1 + \lambda\cot\phi t + \left(\frac{4\cos^2\phi\lambda(\lambda+1) - 2\lambda}{8\sin^2\phi}\right) t^2 + \cdots, \\ H(t) &= i\Bigl[\ln(1-e^{i\phi}t/(2\sin\phi)) - \ln(1-e^{-i\phi}t/(2\sin\phi))\Bigr] \\ &= \sum_{n=1}^{\infty} \frac{\sin(n\phi)}{2^{n-1} n \sin^n\phi} t^n = t + \tfrac{1}{2}\cot\phi t^2 + \bigl(\tfrac{1}{4}\cot^2\phi - \tfrac{1}{12}\bigr) t^3 + \cdots, \end{aligned}$$

and the a_1, a_2, h_2 and h_3 above are the same as those in (3-16).

The Krawtchouk polynomials. Finally, we multiply (3-18) by $t^n/n!$, sum for $n = 0, 1, 2, \ldots$, use the fact that

$$\frac{(-N)_n}{n!} = \frac{(-1)^n N!}{n!(N-n)!} = (-1)^n \mathrm{C}(N, n),$$

and (1-20) in order to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{K}_n(x;\, p, N) t^n &= \sum_{n=0}^{\infty} \frac{(-N)_n p^n}{n!} K_n(x;\, p, N) t^n \\ &= \sum_{n=0}^{\infty} \mathrm{C}(N, n) K_n(x;\, p, N)(-pt)^n \\ &= (1 + (1-p)t)^x (1-pt)^{N-x}.\end{aligned}$$

We write this result in the form $A(t)e^{xH(t)}$:

$$\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{K}_n(x;\, p, N) t^n = (1-pt)^N \exp\left(x \ln\left(\frac{1+(1-p)t}{1-pt}\right)\right).$$

Then, $A(t)$ and $H(t)$ are

$$A(t) = (1-pt)^N = \sum_{n=0}^{\infty} \frac{(-N)_n p^n}{n!} t^n = 1 + -Npt + \frac{1}{2}(N-1)Np^2t^2 + \cdots,$$

$$\begin{aligned}H(t) &= \ln(1+(1-p)t) - \ln(1-pt) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(1-p)^n + p^n}{n} t^n \\ &= t + \left(p - \tfrac{1}{2}\right)t^2 + \left(p^2 - p + \tfrac{1}{3}\right)t^3 + \cdots\end{aligned}$$

and our a_1, a_2, h_2 and h_3 above correspond exactly to those in (3-19).

5. A proof for sufficiency: the "inverse method"

We have thus far established necessary conditions for the A-type 0 recursion coefficients (2-6) and (2-7). We next show that these conditions are also sufficient and thus achieve a complete characterization of all of the A-type 0 orthogonal sets. Namely, we prove that given (3-1), (1-6) must follow. We call our approach the "inverse method", which is a procedure for obtaining a linear generating function from a three-term recurrence relation and therefore reverses the analysis conducted in Sections 2 and 3. The method is as follows.

Assume $\{P_n(x)\}_{n=0}^{\infty}$ is a polynomial set that satisfies a three-term recurrence relation of the form (1-4). We first multiply this relation by $c_n t^n$, where c_n is a certain function in n that is independent of x and t, and sum for $n = 0, 1, 2, \ldots$. Then, from the assignment $F(t; x) := \sum_{n=0}^{\infty} c_n P_n(x) t^n$, we obtain a first-order differential equation in t, with x regarded as a parameter. The initial condition for this equation is $F(0; x) = 1$, via the initial condition $P_0(x) = 1$ in (1-4). The existence and uniqueness of the solution to this differential equation are guaranteed, and the solution will be a generating function for the set $\{P_n(x)\}_{n=0}^{\infty}$.

We now apply the inverse method to each of the unrestricted three-term recurrence relations of our A-type 0 orthogonal sets — as a byproduct, additional fundamental characterizations (differential equations) are obtained for our generating functions (1-15)–(1-20). To derive each of these relations, we first substitute (1-10) into (1-9), which leads to

$$P_{n+1}(x) = xP_n(x) + (\alpha + b + bn)P_n(x) - ((c+d)n + dn^2)P_{n-1}(x).$$

Here, we use (1-10) as opposed to (2-6) and (2-7) for ease of notation. Now define $P_n(x) := e_n Q_n(x)$, and note that our relation directly above becomes

$$e_{n+1}Q_{n+1}(x) = xe_nQ_n(x) + (\alpha + b + bn)e_nQ_n(x) - ((c+d)n + dn^2)e_{n-1}Q_{n-1}.$$

Taking $e_n := n!$ and dividing both sides by $n!$, we have

$$(n+1)Q_{n+1}(x) = xQ_n(x) + (\alpha + b + bn)Q_n(x) - (c + d + dn)Q_{n-1}(x), \quad (5\text{-}1)$$

which is the unrestricted three-term recurrence relation for the Sheffer A-type 0 orthogonal polynomials. We apply Corollary 2.4 accordingly to determine the recurrence coefficients for each case.

We begin by writing out the rigorous details for the Laguerre case. For the subsequent cases, we outline only the salient details. In these cases, we first display the unrestricted three-term recurrence relation, which we henceforth call UTTRR. Then, we display the c_n and the corresponding definition of F. Finally, we write the resulting differential equation (labeled DE) and its unique solution, which will be the corresponding Sheffer A-type 0 generating function.

The Laguerre polynomials. Using (3-4), Corollary 2.4 and (5-1), we obtain

$$(n+1)L_{n+1}^{(\alpha)}(x) - (2n + \alpha + 1 - x)L_n^{(\alpha)}(x) + (n+\alpha)L_{n-1}^{(\alpha)}(x) = 0.$$

We next multiply both sides of this relation by t^n ($c_n \equiv 1$) and sum for $n = 0, 1, 2, \ldots,$ which yields

$$\sum_{n=0}^{\infty}(n+1)L_{n+1}^{(\alpha)}(x)t^n - 2\sum_{n=1}^{\infty}nL_n^{(\alpha)}(x)t^n - (\alpha+1-x)\sum_{n=0}^{\infty}L_n^{(\alpha)}(x)t^n + \sum_{n=1}^{\infty}nL_{n-1}^{(\alpha)}(x)t^n + \alpha\sum_{n=0}^{\infty}L_{n-1}^{(\alpha)}(x)t^n = 0. \quad (5\text{-}2)$$

We next assign $F := F(t;x) := \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n$, accounting for the fact that $\dot{F}(t;x) = \sum_{n=1}^{\infty} nL_n^{(\alpha)}(x)t^{n-1}$ by $\dot{F} := (\partial/\partial t)F(t;x)$. Recalling that $L_{-1}^{(\alpha)}(x) = 0$ from (1-4), we see that (5-2) becomes

$$\dot{F} - 2t\dot{F} - (\alpha+1-x)F + \sum_{n=2}^{\infty} nL_{n-1}^{(\alpha)}(x)t^n + \alpha tF = 0 \tag{5-3}$$

and also observe that

$$\sum_{n=1}^{\infty} nL_{n-1}^{(\alpha)}(x)t^n = \sum_{n=2}^{\infty}(n-1)L_{n-1}^{(\alpha)}(x)t^n + \sum_{n=1}^{\infty} L_{n-1}^{(\alpha)}(x)t^n = t^2\dot{F} + tF.$$

Then, we can put (5-3) in standard form:

$$\dot{F} + \left[\frac{x+(\alpha+1)(t-1)}{1-2t+t^2}\right]F = 0; \qquad F(0;x) = 1. \tag{5-4}$$

The integrating factor in (5-4) turns out to be

$$\mu = \exp\left[\int \frac{x+(\alpha+1)(t-1)}{1-2t+t^2}\,dt\right]$$

and, through partial fraction decomposition, we attain the general solution

$$F(t;x) = c(x,\alpha)(t-1)^{-(\alpha+1)}\exp\left(\frac{x}{t-1}\right).$$

Therefore, using our initial condition in (5-4) to determine $c(x,\alpha)$, we establish the solution

$$F(t;x) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = (t-1)^{-(\alpha+1)}\exp\left(\frac{xt}{t-1}\right), \tag{5-5}$$

which is the Sheffer A-type 0 generating function for the Laguerre polynomials.

The Hermite polynomials.

UTTRR:	$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$
c_n:	$1/n!$
F:	$F(t;x) := \sum_{n=0}^{\infty}(1/n!)H_n(x)t^n$
DE:	$\dot{F} - 2(x-t)F = 0; \quad F(0;x) = 1$
Solution:	$F(t;x) = \sum_{n=0}^{\infty}(1/n!)H_n(x)t^n = \exp(2xt - t^2)$

The Charlier polynomials.

UTTRR: $-xC_n(x;a) = aC_{n+1}(x;a) - (n+a)C_n(x;a) + nC_{n-1}(x;a)$

c_n: $1/n!$

F: $F(t;x,a) := \sum_{n=0}^{\infty} (1/n!)C_n(x;a)t^n$

DE: $\dot{F} - (1 + x/(t-a))F = 0; \quad F(0;x,a) = 1$

Solution: $F(t;x,a) = \sum_{n=0}^{\infty} (1/n!)C_n(x;a)t^n = e^t(1-t/a)^x$

The Meixner polynomials.

UTTRR: $(c-1)xM_n(x;\beta,c) =$
$c(\beta+n)M_{n+1}(x;\beta,c) - [n+c(\beta+n)]M_n(x;\beta,c) + nM_{n-1}(x;\beta,c)$

c_n: $(\beta)_n 1/n!$

F: $F(t;x,\beta,c) := \sum_{n=0}^{\infty} (\beta)_n/(n!)M_n(x,\beta,c)t^n$

DE: $\dot{F} + \left(\dfrac{(c-1)x+(c-t)\beta}{(1+c-t)t-c}\right)F = 0; \quad F(0;x,\beta,c) = 1$

Solution: $F(t;x,\beta,c) = (1-t/c)^x(1-t)^{-(x+\beta)}$

Remark 5.1. For establishing this differential equation, we made use of the identity $(\beta)_n = (\beta)_{n-1}(\beta+n-1)$.

The Meixner–Pollaczek polynomials.

UTTRR: $(n+1)P_{n+1}^{(\lambda)}(x;\phi) - 2[x\sin\phi + (n+\lambda)\cos\phi]P_n^{(\lambda)}(x;\phi)$
$+ (n+2\lambda-1)P_{n-1}^{(\lambda)}(x;\phi) = 0$

c_n: 1

F: $F(t;x,\lambda,\phi) := \sum_{n=0}^{\infty} P_n^{(\lambda)}(x;\phi)t^n$

DE: $\dot{F} + 2\left(\dfrac{\lambda(t-\cos\phi) - x\sin\phi}{1-2\cos\phi t+t^2}\right)F = 0; \quad F(0;x,\lambda,\phi) = 1$

Solution: $F(t;x,\lambda,\phi) = \sum_{n=0}^{\infty} P_n^{(\lambda)}(x;\phi)t^n = (1-e^{i\phi}t)^{-\lambda+ix}(1-e^{-i\phi}t)^{-\lambda-ix}$

The Krawtchouk polynomials.

UTTRR: $-xK_n(x;P,N) = p(N-n)K_{n+1}(x;P,N)$
$-[p(N-n)+n(1-p)]K_n(x;P,N) + n(1-p)K_{n-1}(x;P,N)$

c_n: $\binom{N}{n}$

F: $F(t;x,p,N) := \sum_{n=0}^{N} \binom{N}{n} K_n(x,p,N)t^n$

DE: $\dot{F} + \left(\dfrac{x/(p+tp-t)-N}{1+t}\right)F = 0; \quad F(0;x,p,N) = 1$

Solution: $F(t;x,p,N) = \sum_{n=0}^{N} \binom{N}{n} K_n(x,p,N)t^n$
$= (1-((1-p)/p)t)^x(1+t)^{N-x}$

We now have the following statement:

Theorem 5.2. *Given the monic recursion coefficients corresponding to each of the A-type* 0 *orthogonal sets of Laguerre*, *Hermite*, *Charlier*, *Meixner*, *Meixner–Pollaczek and Krawtchouk*, *there exists a generating function of the form* (1-6).

Hence, Theorem 2.3 in conjunction with Theorem 5.2 establishes the following culminating statement:

Theorem 5.3. *A necessary and sufficient condition for* $\{P_n(x)\}_{n=0}^{\infty}$ *to be a Sheffer A-type* 0 *orthogonal set is that the monic recursion coefficients* λ_{n+1} *and* μ_{n+1}, *as respectively in* (2-6) *and* (2-7), *have the form*

$$\lambda_{n+1} = c_1 + c_2 n \quad \textit{and} \quad \mu_{n+1} = c_3 n + c_4 n^2, \qquad c_1, \dots, c_4 \in \mathbb{R},$$

with $\mu_{n+1} > 0$.

Finally, we mention that this paper solves Problem 1 in Section 3.9 of [Galiffa 2013].

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