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# Growth functions of finitely generated algebras

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We study the growth of finitely presented two-generator monomial algebras. In particular, we seek to improve an upper bound found by the last author. Our search lead us to a connection to de Bruijn graphs and a drastically improved bound.

The growth of algebras has been long studied by algebraists; it goes hand-in-hand with the Gelfand–Kirillov dimension of algebras. An excellent source is [Krause and Lenagan 2000]. Throughout this paper $F$ denotes a field and $0 \in \mathbb{N}$. We focus our work on the growth of algebras of the form $F\langle x, y\rangle/I$, where $I$ is an ideal of the free algebra $F\langle x, y\rangle$ generated by finitely many monomials. Such an algebra is called a *finitely presented two-generator monomial algebra*. It is customary to refer to monomials as words. Let $A$ be one of these algebras. We consider the set $\mathcal{B}$ of all words in $x$ and $y$ that do not have any of the words in the generators for $I$ as factors or subwords. It is standard to show that the image of $\mathcal{B}$ is a basis for $A$. Instead of referring to images of words, we will view the multiplication on $A$ as follows. For any words $u$ and $v$ in $\mathcal{B}$, $uv$ is simply $uv$ if $uv$ has no generator of $I$ as a subword, and $uv = 0$ otherwise. We define the *length* of a word to be the number of letters in it, counting repetitions. Now we can define a function $g : \mathbb{N} \to \mathbb{N}$ by setting $g(n)$ to be the number of words in $\mathcal{B}$ of length at most $n$. This function $g$ is called a *growth function* for $A$ and the *growth* of $A$ is essentially the type of function $g$ is, such as a polynomial of some degree or an exponential. Let's consider a couple of examples.

**Example 1** (Determine a growth function for $A = F\langle x, y\rangle$, the free algebra in two variables). Then the set $\mathcal{B}$ consists of all of the words in $x$ and $y$, such as 1 (the word of length zero), $x$, $y$, $x^2$, $xy$, $yx$, and $y^2$. Now, given an $n \in \mathbb{N}$, we see that there are two choices for each of the $n$ letters in a word of length $n$, and so there are $2^n$ words of length $n$ in $\mathcal{B}$. Thus $g(n) = \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$. In this case the growth of $A$ is exponential.

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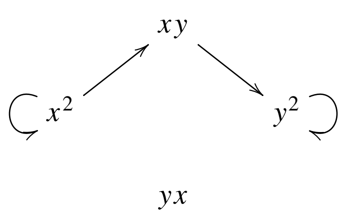
**Example 2** (Determine a growth function for $A = F\langle x, y\rangle/(xy)$). Now $\mathcal{B}$ consists of all of the words in $x$ and $y$ that do not have $xy$ as a subword. A few of them are $1, x, y, x^2, yx$, and $y^2$. Since $xy$ is a subword of $x^2y$, $x^2y \notin \mathcal{B}$. Let $n \in \mathbb{N}$. Since no word having $xy$ as a subword is in $\mathcal{B}$, the words of length $n$ in $\mathcal{B}$ are of the form $y^k x^{n-k}$ for $k = 0, 1, \dots, n$. We see that there are $n+1$ of these and thus

$$g(n) = \sum_{i=0}^{n}(i+1) = \frac{n^2+3n+2}{2}.$$

The growth function is a quadratic polynomial, so we say that $A$ has quadratic growth.

These two examples were fairly straightforward as there were very few generators for the ideals. We can only imagine how complicated the counting could get when there are several generators. It could easily become a combinatorial nightmare. However we were fortunate that Ufnarovskiĭ [1982] came up a very nice way to overcome this. He considered the cycle structure of a particular directed graph, which is constructed as follows. Consider one of our algebras, with $d+1$ being the maximum length of the words that generate the ideal, where $d \geq 2$. The set of vertices of the directed graph is the set of all words in $x$ and $y$ of length $d$ in $\mathcal{B}$. We draw an arrow from a vertex $u$ to a vertex $v$ provided $ua = bv \in \mathcal{B}$, where $a, b \in \{x, y\}$. This graph is called the *overlap graph* for $A$ and will be denoted $\Gamma_A$.

**Example 3** (Construct $\Gamma_A$ for $A = F\langle x, y\rangle/I$ where $I = (yx^2, y^2x, xyx, yxy)$). Since the maximum length of generators for $I$ is 3, $d = 2$. Since all of the generators for $I$ have length 3, the vertices for $\Gamma_A$ are the words in $\mathcal{B}$ of length 2: $x^2, y^2, xy, yx$. Notice that the words in $\mathcal{B}$ of length 3 are $x^3, y^3, x^2y$, and $xy^2$. Here is $\Gamma_A$:




We have an arrow from $x^2$ to $x^2$ because $x^3 \in \mathcal{B}$ and $x^3 = (x^2)x = x(x^2)$. Also we have an arrow from $x^2$ to $xy$ as $x^2y \in \mathcal{B}$ and $x^2y = (x^2)y = x(xy)$. Even though $(y^2)x = y(yx)$, there is no arrow from $y^2$ to $yx$, as $y^2x \notin \mathcal{B}$.

The following theorem yields the connection between the overlap graph and the growth of the algebra.

**Theorem 4** [Ufnarovskiĭ 1982]. *Let $A = F\langle x, y\rangle/I$, where $I$ is generated by finitely many monomials of maximum length $d+1$ for some $d \geq 2$, and let $\Gamma_A$ be the overlap graph for $A$. Then*:

(1) *There is a one-to-one correspondence between words in* $\mathcal{B}$ *of length* $d+j$ *and paths in* $\Gamma_A$ *of length* $j$ *for each* $j \in \mathbb{N}$. (*We define the length of a path to be the number of arrows in it, counting repetitions*).

(2) *If* $\Gamma_A$ *has two intersecting cycles, then the growth of* $A$ *is exponential.*

(3) *If* $\Gamma_A$ *has no intersecting cycles, then the growth of* $A$ *is polynomial of degree* $s$, *where* $s$ *is the maximal number of distinct cycles on a path in* $\Gamma_A$.

Referring to Example 3 above, we see that $\Gamma_A$ has no intersecting cycles, but does have two distinct cycles on a path. So its growth is degree two, or quadratic, as we have already seen. Given $d \geq 2$ in Theorem 4, we wish to determine the highest-possible-degree polynomial that bounds the growth for $A$. In [Ellingsen Jr. 1993] it was shown that $2^d - d + 1$ is an upper bound for this degree.

Now we come to the connection to de Bruijn graphs. We are very grateful to Dr. Jo Ellis-Monaghan of St. Michael's College in Vermont for making us aware of them. It turns out that the overlap graphs for our algebras can be considered as subgraphs of de Bruijn graphs, with the only difference being that de Bruijn used 0 and 1 instead of $x$ and $y$. For a given $d \geq 2$, the vertices of the *de Bruijn graph* $B_d$ are all of the binary $d$-tuples, and there is an arrow from the binary $d$-tuple $u = u_1u_2\cdots u_d$ to the binary $d$-tuple $v = v_1v_2\cdots v_d$ if and only if $u_2u_3\cdots u_d = v_1v_2\cdots v_{d-1}$, that is, $u_1u_2\cdots u_dv_d = u_1v_1v_2\cdots v_d$. Replacing 0 and 1 with $x$ and $y$ yields the overlap graph using all the words in $x$ and $y$ of length $d$ with all possible arrows. After some online searching, the student authors found that much work has been done on de Bruijn graphs, the most remarkable of which is the following theorem proven by Mykkeltveit [1972], but originally conjectured by Golomb.

**Theorem 5.** *For any* $d \geq 2$, *the maximum number of simultaneous disjoint cycles in* $B_d$ *is* $Z(d) = (1/d)\sum_{k|d}\phi(k)2^{d/k}$, *where* $\phi$ *is Euler's phi function.*

Our main theorem follows.

**Theorem 6.** *Let* $d \geq 2$ *and let* $I$ *be an ideal of* $F\langle x, y\rangle$ *generated by finitely many words of maximum length* $d+1$. *If the growth function for* $A = F\langle x, y\rangle/I$ *is not exponential, then the maximum possible polynomial degree for the growth of* $A$ *is* $Z(d)$.

*Proof.* Let $d \geq 2$, let $I$ be an ideal of $F\langle x, y\rangle$ generated by finitely many words of maximum length $d+1$ and let $A = F\langle x, y\rangle/I$. Assume that the growth of $A$ is not exponential. Let $\Gamma$ be the overlap graph for the words of length $d$ with all possible arrows and $\Gamma_A$ the overlap graph for $A$. By the previous theorem we know that there are at most $Z(d)$ disjoint cycles in $B_d$, which is identical to $\Gamma$. Thus there can be at most $Z(d)$ distinct cycles on any path in $\Gamma$. Since $\Gamma_A$ is a subgraph of $\Gamma$, $Z(d)$ is also the maximum possible number of distinct cycles in $\Gamma_A$. Hence by Ufnarovskiĭ's theorem the maximum possible polynomial degree for the growth of $A$ is $Z(d)$. □

The following table illustrates the drastic improvement of the new upper bound:

| $d$ | $2^d - d + 1$ | $Z(d)$ |
|---|---|---|
| 2 | 3 | 3 |
| 3 | 6 | 4 |
| 4 | 13 | 6 |
| 5 | 28 | 8 |
| 6 | 59 | 14 |
| 7 | 122 | 20 |
| 8 | 249 | 36 |
| 9 | 504 | 60 |

We have found explicitly that this bound is sharp for $d \in \{2, 3, 4, 5, 6, 7\}$ [Flores et al. 2009; Hunt 2002], and are working on the conjecture that is it sharp for all $d \geq 2$.

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