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On attractors and their basins

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We prove that the map assigning to a given vector field the Lebesgue measure of the union of the basins of its attractors is lower semicontinuous in a residual subset of vector fields. Moreover, we prove that the Lebesgue measure of the union of the basins of attractors of a generic sectional axiom A vector field is total. For this, we also improve a result of Morales about sectional-hyperbolic sets. We also remark that homoclinic classes are topologically ergodic and that for a generic tame diffeomorphism, the union of the stable manifolds of the hyperbolic periodic orbits is dense in the manifold.

1. Introduction

One of the key notions in the theory of dynamical systems is that of attractors. By definition, an attractor captures the asymptotic information of a large set of orbits, called its basin, which always contains an open set. As an example, if an attractor is hyperbolic, then the asymptotic behavior of an orbit in its basin is governed by the dynamics of one orbit inside it (a shadowing property).

Moreover, that essentially every orbit is attracted by one attractor and that the set of attractors is finite (and possibly hyperbolic) implies that the dynamics of the system are nicely described by the attractors. For instance, this led Palis [2005] to conjecture that "there is a dense set D of dynamical systems such that any element of D has finitely many attractors whose union of basins of attraction has total probability".

Mathematicians have made many efforts to understand attractors and their basins, not only for finite-dimensional dynamics, but also for PDEs (infinite-dimensional dynamical systems). See, for instance, [Constantin et al. 1985] or [Hale 2000].

On the other hand, to understand properties of the entire set of dynamical systems is a difficult task, and it is more reasonable to try to understand a large part of the set of dynamical systems. This reasoning leads to the theory of generic dynamics. Since

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the C^r-topology turns the space of diffeomorphisms (or vector fields) into a Baire space, it is natural to show that some properties holds for a residual subset of the space of dynamical systems, i.e., a countable intersection of open and dense subsets, since this will show the presence of this property for a dense subset and this property could be used to show another property in another residual subset. Indeed, the intersection of two residual subsets is also a residual subset. Usually, we say that a property holds for a generic system if it holds in a residual subset of dynamical systems.

The purpose of this article is to give some remarks about attractors, and their basins, of certain classes of dynamical systems, both diffeomorphisms and vector fields. These remarks are the results obtained by Obata [2010], guided by Arbieto, in his undergraduate monograph. We will state the results and refer the reader to the next section for the precise definitions of the more technical objects used in the statements.

Let M be a Riemannian closed manifold. We denote by $\mathrm{Diff}^1(M)$ the space of diffeomorphisms and by $\mathfrak{X}^1(M)$ the space of vector fields, both endowed with the C^1-topology. We denote by m the Lebesgue measure and by d the geodesic distance, both induced by the Riemannian metric. If $X \in \mathfrak{X}^1(M)$, we denote by X_t the flow generated by X.

Results for flows. An attractor is an invariant compact subset Λ of M such that there exists a neighborhood U of Λ with

$$X_t(\bar{U}) \subset U \text{ for } t > 0 \quad \text{and} \quad \bigcap_{t\geq 0} X_t(U) = \Lambda.$$

The set U is called the *local basin* of Λ and $B(\Lambda) := \bigcup_{t\leq 0} X_t(U)$ is the *basin* of Λ. We also define a set R to be a repeller if R is an attractor for $-X$.

Let X be a vector field, and denote by $m(B(X))$ the Lebesgue measure of the union of the basins of the attractors of X. This generates a map $\Phi : \mathfrak{X}^1(M) \to [0, +\infty]$, defined as $\Phi(X) := m(B(X))$ if there exists an attractor and $\Phi(X) := 0$ if not.

Theorem 1. *There exists a residual subset $\mathcal{R}$ such that $\Phi|_{\mathcal{R}}$ is lower semicontinuous.*

The analogous statement holds for diffeomorphisms using the same proof.

Metzger and Morales [2008] extended the notion of axiom A vector fields for flows with singularities, called *sectional axiom A* vector fields. As an intermediate step to studying sectional axiom A vector fields, we have the following result:

Theorem 2. *There exists a residual subset $\mathcal{R}$ such that if X is in $\mathcal{R}$ and $\Gamma = \Lambda_1 \cup \cdots \cup \Lambda_k$, with $\Gamma \subset \Omega(X)$, is a disjoint union of homogeneous sectional-hyperbolic sets for X or $-X$, and Γ is a proper subset of M, then $m(\Gamma) = 0$.*

We remark that it is well known that if M is a closed manifold which is a sectional-hyperbolic set for X, then X has no singularities and X is Anosov [Bautista and Morales 2011].

To prove this theorem, we extend a result of [Morales 2007]; see Theorem 13.

As a corollary, we obtain the following result, which improves Corollary D of [Alves et al. 2007] in two ways. We do not require that the vector field be $C^{1+\varepsilon}$ or that the dimension of the manifold be 3. Indeed, in [Alves et al. 2007] it is proved that if a sectional axiom A vector field X over M^3 is $C^{1+\varepsilon}$, then the Lebesgue measure of the union of the basins of its hyperbolic or sectional-hyperbolic attractors is total. We remark also that the union of the sets of $C^{1+\varepsilon}$ vector fields, over any $\varepsilon > 0$, is a meager subset of vector fields.

Theorem 3. *Let X be a generic sectional axiom A vector field. Then either X is Anosov, or the Lebesgue measure of the nonwandering set of X is zero and the Lebesgue measure of the union of the basins of its attractors is total.*

A difficulty in proving this theorem is that it is not known whether the set of sectional axiom A vector fields (without cycles) is open. This is an interesting question. Even so, there are open sets of vector fields formed by sectional axiom A sets [Bautista and Morales 2011]. Moreover, [Morales and Pacifico 2003] shows that in dimension 3, generically, either a vector field has infinitely many sinks or sources or it is sectional axiom A. So, we obtain the following corollary:

Corollary 4. *If $\dim(M) = 3$, a generic vector field either has infinitely many sinks or sources, or the Lebesgue measure of the union of the basins of its attractors is total.*

Results for diffeomorphisms. Abdenur [2003] proved that attractors for generic diffeomorphisms are homoclinic classes. These classes are always transitive. However it can be proved that they have another property called topological ergodicity.[1]

Proposition 5. *Any homoclinic class of a periodic point p, with period k, of a diffeomorphism f is topologically ergodic. Moreover, for any two open sets U and V, the density of $N(U, V) = \{i \geq 1 : f^i(U) \cap V \neq \varnothing\}$ is bounded by below $1/k$.*

Finally, the techniques used in the proof of the results above can be used to prove a folklore result. Since, as far as the authors know, it was never written, we include here a proof of this result:

Proposition 6. *If f is a C^1-generic tame diffeomorphism, then the union of the stable manifolds of the hyperbolic periodic orbits is dense in M.*

We observe that this result was proved in a more general setting (partially hyperbolic diffeomorphisms with one-dimensional central bundle) by Bonatti, Gan and Wen [Bonatti et al. 2007]. In particular, they obtain this corollary using stronger

[1] Recently Abdenur and Crovisier [2012] investigated the mixing property for isolated sets.

methods. However, the short proof given here only uses the connecting lemma. This is a particular case of Bonatti's conjecture; see [Bonatti et al. 2007].

Conjecture 7. *There exists a residual subset* $\mathcal{R} \subset \mathrm{Diff}^1(M)$ *such that for any* $f \in \mathcal{R}$, *the union of the stable manifolds of the hyperbolic periodic orbits is dense in* M.

This paper is organized as follows. In Section 2, we give precise definitions of terms used in the introduction. In Section 3, we prove Theorem 1. In Section 4, we prove Theorems 2 and 3 and also prove an extension of a theorem by Morales. In Section 5, we give a proof of Proposition 5. Finally, in Section 6, we give a proof of Proposition 6.

2. Preliminaries

In this section, we give precise definitions of terms used in the introduction and collect some useful results.

2.1. ***Topology.*** As remarked before, both $\mathrm{Diff}^1(M)$ and $\mathfrak{X}^1(M)$ are Baire spaces. We will say that a property P is generic if it holds for a residual subset of these spaces. If the residual subset is fixed, we also say that an element of it is generic.

Let $F(M)$ denote the space of compact subsets of M; it is a metric space under the *Hausdorff metric*, given by

$$d_H(A, B) = \max\{d_A(B), d_B(A)\} \quad \text{for all } A, B \in F(M),$$

where $d_A(B) = \max_{b \in B}\{\min_{a \in A}(d(a, b))\}$.

Let (N, d) be a metric space. A map $\varphi : N \to F(M)$ is *lower semicontinuous* at $y \in N$ if $y_n \to y$ implies $d_H(\varphi(y_n), \varphi(y)) \to 0$. Analogously, a map $\varphi : N \to \mathbb{R}$ is lower semicontinuous at $x_0 \in X$ if

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

It is well known that if (N, d) is a Baire space, then the set of continuity points of a lower semicontinuous map, in either definition above, is a residual subset of its domain; see [Kelley 1955].

2.2. ***Flows.*** Let $X \in \mathfrak{X}^1(M)$. The orbit of a point p is the set $\{X_t(p)\}_{t \in \mathbb{R}}$. A *periodic orbit* of X is an orbit $\{X_t(p) : t \in \mathbb{R}\}$ of a point $p \in M$ satisfying $X_T(p) = p$ for some minimal $T > 0$. A singularity σ is a zero of X. By a *closed orbit* we mean a periodic orbit or a singularity. The nonwandering set of X is the set $\Omega(X)$ of points x such that for every neighborhood U of x and $N > 0$, there exists some $T > N$ such that $X_T(U) \cap U \neq \varnothing$.

A subset $\Lambda \subset M$ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$; *transitive* if there exists $p \in \Lambda$ such that its orbit is dense in Λ; *isolated* if there exists a neighborhood U

of Λ such that $\bigcap_{t\in\mathbb{R}} X_t(U)=\Lambda$; and Ω-*isolated* if there exists a neighborhood V of Λ such that $\Omega(X)\cap V=\Lambda$. We remark that any attractor is Ω-isolated.

We say that a subset $\Lambda\subset M$ is sectional-hyperbolic if every singularity in Λ is hyperbolic and it has a nontrivial partially hyperbolic splitting $T_\Lambda M=E\oplus F$ such that E is uniformly contracting and F is sectionally expanding; i.e.,

$$\dim(E^c_x)\geq 2 \quad \text{and} \quad |\det(DX_t(x)/L_x)|\geq K^{-1}e^{\lambda t}$$

for all $x\in\Lambda$, $t\geq 0$, and L_x a two-dimensional subspace of E^c_x.

We say that Λ is *hyperbolic* if there is a continuous invariant tangent bundle decomposition

$$T_\Lambda M=\hat{E}^s_\Lambda\oplus\hat{E}^X_\Lambda\oplus\hat{E}^u_\Lambda,$$

and positive constants K,λ, where $\hat{E}^X_\Lambda$ is the subbundle generated by X and

$$\|DX_t(x)/\hat{E}^s_x\|\leq Ke^{-\lambda t} \quad \text{and} \quad \|DX_{-t}(x)/\hat{E}^u_{X_t(x)}\|\leq Ke^{-\lambda t}$$

for all $x\in\Lambda$ and $t\geq 0$.

A closed orbit is hyperbolic if it is a hyperbolic compact invariant set. A hyperbolic set is a basic set if it is isolated and transitive. Similar notions hold for diffeomorphisms.

Given an invariant splitting $T_\Lambda M=E_\Lambda\oplus F_\Lambda$ over an invariant set Λ of a vector field X, we say that the subbundle E_Λ *dominates* F_Λ if there are positive constants K,λ such that

$$\|DX_t(x)/E_x\|\|DX_{-t}(x)/F_{X_t(x)}\|\leq Ke^{-\lambda t} \quad \text{for all } x\in\Lambda \text{ and } t\geq 0.$$

In such a case we say that $T_\Lambda M=E_\Lambda\oplus F_\Lambda$ is a *dominated splitting*.

We say that Λ is *partially hyperbolic* if it has a dominated splitting $T_\Lambda M=E^s_\Lambda\oplus E^c_\Lambda$ whose dominating subbundle E^s_Λ is *contracting*, that is,

$$\|DX_t(x)/E^s_x\|\leq Ke^{-\lambda t} \quad \text{for all } x\in\Lambda \text{ and } t\geq 0.$$

Moreover, we call the central subbundle E^c_Λ *sectionally expanding* if

$$\dim(E^c_x)\geq 2 \quad \text{and} \quad |\det(DX_t(x)/L_x)|\geq K^{-1}e^{\lambda t}$$

for all $x\in\Lambda$, $t\geq 0$, and L_x a two-dimensional subspace of E^c_x.

Definition 8. We say that a compact and invariant set Λ of X is *sectional-hyperbolic* if every singularity contained in Λ is hyperbolic and it has a nontrivial partially hyperbolic set with a sectionally expanding central subbundle.

Now, we recall the notion of sectional axiom A vector field, given in [Metzger and Morales 2008]; see also [Morales et al. 1999].

Definition 9. A vector field X is sectional axiom A if there is a finite disjoint decomposition

$$\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_k,$$

where each Λ_i is a hyperbolic basic set or a sectional-hyperbolic attractor up to time reversion.

2.3. ***Diffeomorphisms.*** If p is a hyperbolic periodic point, of period k, of a diffeomorphism f, then its stable manifold is the set

$$W^s(p) = \{y \in M : d(f^{kn}(y), p) \to 0 \text{ as } n \to \infty\}.$$

This set is in fact an immersed manifold. The stable manifold of the orbit of p is the union of the stable manifolds of $f^i(p)$ for $i = 0, \ldots, k-1$, and it is denoted by $W^s(O(p))$. Analogously, we define the unstable manifold of p and the orbit of p.

Definition 10. The homoclinic class of p is the set

$$H(p, f) = \overline{W^s(O(p)) \pitchfork W^u(O(p)}.$$

A diffeomorphism is *tame* if its nonwandering set decomposes as a finite number of homoclinic classes and finitely many sinks or sources. Analogous definitions hold for vector fields.

Given two nonempty open sets U and V, we define the set of times that the orbit of U visits V as

$$N(U, V) = \{i \geq 1 : f^i(U) \cap V \neq \varnothing\}.$$

The following definition can be found in [Abdenur and Crovisier 2012]:

Definition 11. An invariant and compact subset Λ of f is topologically ergodic if for every two nonempty open sets $U, V \subset \Lambda$, we have

$$\limsup_{n\to\infty} \frac{\# N(U, V) \cap \{1, \ldots, n\}}{n} > 0.$$

3. Proof of Theorem 1

First we observe that since attractors are isolated, there are at most countably many of them. Let $\Lambda_1, \Lambda_2, \ldots$ be the attractors of a generic vector field X. Denote by $B(\Lambda_1), B(\Lambda_2), \ldots$ its basins.

We select $\Lambda_1, \ldots, \Lambda_r$ such that

$$\sum_{i=1}^{r} m(B(\Lambda_i)) \geq m(B(X)) - \varepsilon.$$

There exist compact sets $K_1, \dots, K_r$ such that $\Lambda_i \subset K_i \subset B(\Lambda_i)$ for $i = 1, \dots, r$ and such that

$$m(B(\Lambda_i) - K_i) < \frac{\varepsilon}{r}.$$

Now, we recall a result of Abdenur. Actually, he works with diffeomorphisms, but his proof holds for vector fields with the necessary adaptations. Also, he states his theorem for Ω-isolated transitive sets, but we will only state it in the case of attractors, which is the context here.

Theorem 12 [Abdenur 2003]. *There exists a residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}$ and Λ is an attractor of X with local basin U which does not reduce to a singularity, then there exists a neighborhood $\mathcal{U}$ of X such that for any $Y \in \mathcal{U} \cap \mathcal{R}$, $\Lambda(Y) = \bigcap_{t \geq 0} Y_t(U)$ is an attractor. Moreover, there exists a periodic orbit $O(p)$ such that $\Lambda(Y) = H(O(p), Y)$.*

Thus, there are local basins U_i of Λ_i such that these local basins persist in a C^1-generic neighborhood of X. Since $B(\Lambda_i) = \bigcup_{t \geq 0} X_{-t}(U_i)$ and $K_i \subset B(\Lambda_i)$ is a compact set, there is $T > 0$ such that

$$K_i \subset \bigcup_{t \in [0,T]} X_{-t}(U_i).$$

The set on the right is open. So, if Y is C^1-close to X, we obtain that

$$K_i \subset \bigcup_{t \in [0,T]} Y_{-t}(U_i).$$

Thus, if Y is generic and C^1-close to X, we have $m(B(\Lambda_i(Y))) \geq m(K_i)$.

Hence,

$$m(B(Y)) \geq \sum_{i=1}^{r} m(K_i) \geq m(B(X)) - \varepsilon.$$

This proves lower semicontinuity.

4. Proof of Theorems 2 and 3

Let Λ be a sectional-hyperbolic set for X. We recall that its strong stable manifold is the set

$$W^{ss}(x) = \Big\{ y \in M : \lim_{t \to \infty} d(X_t(x), X_t(y)) = 0 \Big\}.$$

Its local strong stable manifold is an ε-ball $W^{ss}_{\varepsilon}(x)$ in $W^{ss}(x)$ centered at x for some $\varepsilon > 0$.

Given $A \subset M$, we define $\alpha(A)$ as the set of points $y = \lim_{n \to \infty} X_{t_n}(z_n)$ for some sequences $t_n \to -\infty$ and $z_n \in A$. We say that a sectional-hyperbolic set is homogeneous if the splitting $E^s \oplus E^c$ given by the definition is such that $\dim E^s$ is constant.

The following result improves the main theorem in [Morales 2007] since we do not require transitivity.

Theorem 13. *Let $\Lambda \subset \Omega(X)$ be a homogeneous sectional-hyperbolic set for X. Denote by R the union of the hyperbolic repellers contained in Λ. Then $\Lambda - R$ does not contain any local strong stable manifold.*

Proof. By hypothesis, the map $x \in \Lambda \mapsto W^{ss}_{\varepsilon}(x)$ is continuous if $\varepsilon > 0$ is small, but fixed. Assume that $\Lambda - R$ contains some $W^{ss}_{\varepsilon}(x)$. Let $\delta < \varepsilon$ and take $H = \alpha(W^{ss}_{\delta}(x)) \subset \Lambda - R$, which is compact and invariant. Observe also that the set $\Lambda - R$ is compact and invariant, since $\Lambda \subset \Omega(X)$.

If H has a singularity σ then, by definition, $\sigma = \lim X_{t_n}(z_n)$ for some sequences $t_n \to -\infty$ and $z_n \in W^{ss}_{\delta}(x)$. Moreover, $W^{ss}_{\delta}(X_{t_n}(z_n)) \subset \Lambda$ for any natural number n. Taking the limit as $n \to \infty$, we obtain that $W^{ss}_{\delta}(\sigma) \subset \Lambda$.

However, by [Bautista and Morales 2011], since Λ is sectional-hyperbolic, we have that $\Lambda \cap W^{ss}(\sigma) = \{\sigma\}$, and this is a contradiction.

If H does not have a singularity, then by the hyperbolic lemma [Bautista and Morales 2011], H is a hyperbolic set. Now, let y be a cluster point of $X_{t_n}(x)$, with $t_n \to -\infty$. We will show that $W^{ss}(y) \subset H$. Indeed, let $z \in W^{ss}(y)$ and let $\varepsilon > 0$ be small enough. There exists $T > 0$ such that

$$d(X_T(z), X_T(y)) < \varepsilon.$$

Also, there exists n_0 such that for any $n \geq n_0$, we have

$$d(X_t(y), X_{t_n+T}(x)) < \varepsilon.$$

Finally, for any n large, there exists $z_n \in W^{ss}_{\delta}(x)$ such that

$$d(X_{t_n+T}(x), X_{t_n+T}(z_n)) < \varepsilon.$$

This implies that if n is large enough then

$$d(X_T(z), X_{t_n+T}(z_n)) < \varepsilon.$$

In particular, we can assume that $t_n + T \to -\infty$. Thus, $X_T(z) \in H$, and by invariance, $z \in H$. Thus H is a repeller inside $\Lambda - R$, a contradiction. □

Remark 14. We could remove the homogeneity assumption. Indeed, the sets $\{x \in \Lambda : \dim(E^s(x)) = i\}$ for $1 \leq i \leq d-1$ are compact. Hence, we could use the argument restricting ourselves to each of these sets.

Now, we observe that X_1 is a partially hyperbolic diffeomorphism over Λ since the dominated splitting $T_\Lambda M = E^s \oplus E^c$ has a contracting subbundle E^s. A strong stable disk of X_1 is a disk which is tangent to the subbundle E^s over Λ. Obviously, a strong stable disk of X_1 is a local strong stable manifold for some point $x \in \Lambda$. However, the following result was proved in [Alves et al. 2007, Theorem 2.2]:

Theorem 15. *Let $f : M \to M$ be a C^2 diffeomorphism and $\Lambda \subset M$ a partially hyperbolic set with positive volume. Then Λ contains a strong stable disk.*

Together with Theorem 13, we obtain the following:

Corollary 16. *Let Λ be a proper subset of M. If Λ is a homogeneous sectional-hyperbolic set of a C^2 vector field X and $\Lambda \subset \Omega(X)$, then $m(\Lambda) = 0$.*

Proof. First we remark that there are only countably many repellers in Λ, since they are isolated. Moreover, by [Bowen 1975], the measure of any hyperbolic repeller (or attractor) is zero if X is C^2.

On the other hand, if R denotes the union of the hyperbolic repellers of Λ and $m(\Lambda - R) > 0$, then by Theorem 15, there exists a strong stable disk on $\Lambda - R$, and this contradicts Theorem 13. □

For any open set U, let $\Lambda_Y(U) = \bigcap_{t\in\mathbb{R}} Y_t(\bar{U})$. These sets have an upper semicontinuity property: $\limsup \Lambda_{X_n}(U) \subset \Lambda_X(U)$. Indeed, let $x \in \limsup \Lambda_{X_n}(U)$. So, there exists $x_n \in \Lambda_{X_n}(U)$ such that $x_n \to x$. Fix $t \in \mathbb{R}$. We have $(X_n)_t(x_n) \in \bar{U}$. Thus, $X_t(x) \in \bar{U}$. Since this holds for every $t \in \mathbb{R}$, this implies that $x \in \Lambda_X(U)$.

Now, let $\{U_k\}$ be a countable basis of the topology and $\{O_k\}$ the set of finite unions of the U_k. For every $n, k \in \mathbb{N}$, we define $\mathcal{U}_{n,k}$ as the set of vector fields Y such that $m(\Lambda_Y(O_k)) < 1/n$.

Lemma 17. *$\mathcal{U}_{n,k}$ is an open set.*

Proof. Let $Y \in \mathcal{U}_{n,k}$, and suppose that $m(\Lambda_Y(O_k)) = 1/n - \varepsilon$. There exists T large enough that

$$m\biggl(\bigcap_{t=-T}^{T} Y_t(\bar{O}_k)\biggr) < m(\Lambda_Y(O_k)) + \frac{\varepsilon}{2}.$$

Let W be a neighborhood of $\bigcap_{t=-T}^{T} Y_t(\bar{O}_k)$ such that

$$m(W) < m\biggl(\bigcap_{t=-T}^{T} Y_t(\bar{O}_k)\biggr) + \frac{\varepsilon}{2}.$$

If Z is close enough to Y, we have that $\bigcap_{t=-T}^{T} Z_t(\bar{O}_k) \subset W$. Thus

$$\begin{aligned} m(\Lambda_Z(O_k)) &\le m\biggl(\bigcap_{t=-T}^{T} Z_t(\bar{O}_k)\biggr) < m\biggl(\bigcap_{t=-T}^{T} Y_t(\bar{O}_k)\biggr) + \frac{\varepsilon}{2} \\ &\le m(\Lambda_Y(O_k)) + \varepsilon = \frac{1}{n}. \end{aligned}$$ □

Now, we prove Theorem 2.

Proof of Theorem 2. By the previous lemma, $\mathcal{U}_{n,k}$ is an open set. Now, we define $\mathcal{N}_{n,k} = \mathfrak{X}^1(M) - \overline{\mathcal{U}}_{n,k}$. Consider the residual subset

$$\mathcal{R} = \bigcap_n \bigcap_k (\mathcal{U}_{n,k} \cup \mathcal{N}_{n,k}).$$

Let $X \in \mathcal{R}$ and let $\Gamma = \Lambda_1 \cup \cdots \cup \Lambda_k$, as in the statement of Theorem 2. Suppose that Λ_i is a homogeneous sectional-hyperbolic set for X. Since Λ_i is invariant, there exists $k(i)$ such that $\Lambda_i \subset \Lambda_X(O_{k(i)})$ and $\Lambda_X(O_{k(i)})$ is a homogeneous sectional-hyperbolic set. A similar argument holds when Λ_i is a homogeneous sectional-hyperbolic set for $-X$.

Now, suppose that $m(\Lambda_X(O_{k(i)})) > 0$ for some i. Thus, there exists n such that $m(\Lambda_X(O_{k(i)})) \geq 1/n$. So, $X \in \mathcal{N}_{n,k(i)}$. Since $\mathcal{N}_{n,k(i)}$ is an open set, there exists a neighborhood $\mathcal{V}$ of X such that $m(\Lambda_Y(O_{k(i)})) \geq 1/n$ for every $Y \in \mathcal{V}$.

Using the semicontinuity property, mentioned above, and the sectional hyperbolicity of $\Lambda_X(O_{k(i)})$, we can assume, shrinking $\mathcal{V}$ if necessary, that $\Lambda_Y(O_{k(i)})$ is a homogeneous sectional-hyperbolic set for every $Y \in \mathcal{V}$.

Now, we can choose a C^2 vector field $Y \in \mathcal{V}$ and by Corollary 16, we have that $m(\Lambda_Y(O_k)) = 0$, a contradiction. □

Proof of Theorem 3. The arguments given above show that there exists a residual subset $\mathcal{S}$ such that if $X \in \mathcal{S}$ and Λ is a proper saddle-type isolated transitive sectional-hyperbolic set, then $m(\Lambda) = 0$.

Indeed, let U be an open set, and define $\mathcal{U}(U)$ as the (open) set formed by vector fields Y such that $\Lambda_Y(U)$ is hyperbolic of saddle type. Let $\mathcal{U}_n(U) = \{Y \in \mathcal{U}(U) : m(B(\Lambda_Y(U))) < 1/n\}$. Using the same argument as in the proof of Lemma 17, we obtain that $\mathcal{U}_n(U)$ is an open set.

Moreover, if $Y \in \mathcal{U}(U)$ is C^2, we have that $m(B(\Lambda_Y(U))) = 0$ [Bowen 1975, p. 68]. So, $\mathcal{U}_n(U)$ is dense in $\mathcal{U}(U)$.

Defining O_k as above, we set $S = \bigcap_{k,n} \mathcal{U}_n(O_k)$.

Let $X \in \mathcal{R} \cap \mathcal{S}$ be a sectional axiom A vector field. By definition, we have a spectral decomposition $\Omega(X) = \Lambda_1 \cup \cdots \cup \Lambda_k$, formed by sectional-hyperbolic attractors, repellers and basic saddle-type hyperbolic sets. Moreover, since these sets have a dense orbit, they are homogeneous.

If $m(\Omega(X)) > 0$, then there exists $1 \leq i \leq k$ such that $m(\Lambda_i) > 0$. By the previous argument and Theorem 2, we have that $\Lambda_i = M$. If Λ_i is a saddle-type hyperbolic set, then X is Anosov. If Λ_i is a sectional-hyperbolic attractor then it cannot have any singularity. Indeed, if σ is a singularity, we must have $W^{ss}(\sigma) \cap \Lambda_i = \{\sigma\}$, but if $\Lambda_i = M$ this cannot be true. Hence, by the hyperbolic lemma, M would be hyperbolic again and X would be Anosov. If Λ_i is a sectional-hyperbolic attractor for $-X$, the same holds.

So, assuming that X is not Anosov, $m(\Omega(X)) = 0$. Using Lemma 2.2 of [Shub 1978], we have that

$$M = W^s(\Lambda_1) \cup \cdots \cup W^s(\Lambda_k).$$

Since $X \in \mathcal{R}$, if Λ_i is a repeller then $W^s(\Lambda_i) = \Lambda_i$ and $m(\Lambda_i) = 0$. Since $X \in \mathcal{S}$, if Λ_i is a hyperbolic basic set then $m(W^s(\Lambda_i)) = 0$. Thus, the measure of the union of the basins of the attractors is total. □

5. Proof of Proposition 5

In the following, we will work with the topology relative to the homoclinic class. First, we will show that any homoclinic class is topologically ergodic.[2]

Let $H(p, f)$ be a homoclinic class. Denote by k the period of p. We recall that the local stable manifold of p is the set $W^s_\varepsilon(p) = \{y \in M : d(f^n(y), f^n(p)) \leq \varepsilon\}$.

Fix two nonempty open subsets U and V of $H(p, f)$. Since the stable manifold of its orbit is dense, there exist $\varepsilon > 0$ and $N > 0$ such that

$$f^{-N}(W^s_\varepsilon(p)) \cap U \neq \varnothing.$$

In particular, there exists a disk $D \subset f^N(U)$ transversal to $W^s_\varepsilon(p)$. Moreover, since $W^u(O(p))$ is dense, there exists $K > 0$ such that $f^K(W^u_\varepsilon(p)) \cap V \neq \varnothing$. Using the λ-lemma [Palis and de Melo 1982], there exist m_0 and $0 \leq i < k$ such that for every $m \geq m_0$, we have that $f^{km+i}(D) \cap V \neq \varnothing$. Let $l \in \mathbb{N}$ such that $A = lk + (N + i)$ is the largest integer less than or equal to n; in particular, $n < A + k$. By the previous remark, $\#N(U, V) \cap \{1, \ldots, n\} \geq l - m_0$. So,

$$\limsup_{n\to\infty} \frac{\#N(U, V) \cap \{1, \ldots, n\}}{n} \geq \limsup_{l\to\infty} \frac{l - m_0}{(l+1)k + N + i} = \frac{1}{k}.$$

This shows that the homoclinic class is topologically ergodic.

6. Proof of Proposition 6

We recall that an invariant and compact subset $A \subset M$ is called Lyapunov stable if given U, an open neighborhood of A, there exists another neighborhood V of A such that $f^n(V) \subset U$ for every $n \in \mathbb{N}$.

Lemma 18 [Carballo et al. 2003, Lemma 3.4]. *If f is a C^1-generic diffeomorphism, then $\overline{W^u(O(p))}$ is Lyapunov stable for f.*

Another source of Lyapunov stable sets is the following, which is [Morales and Pacifico 2002, Theorem A]:

Theorem 19. *There exists a residual subset $R^* \subset \mathrm{Diff}^1(M)$ such that if $g \in R^*$, then the set $S = \{x \in M : \omega(x)$ is Lyapunov stable$\}$ is a residual subset of M.*

[2] We want to thank Professor Abdenur for pointing out this short argument to us.

We also recall Hayashi's connecting lemma [1997], one of the most useful techniques in the C^1-generic theory of dynamical systems. The formulation that we give here is taken from [Wen and Xia 2000].

Theorem 20 (connecting lemma). *Let $f \in \mathrm{Diff}^1(M)$, and let z be a nonperiodic point of f. Given a neighborhood $\mathcal{U}$ of f, there exist $\rho > 1$, $L \in N$ and $\delta_0 > 0$ with the following property. Let $0 < \delta < \delta_0$ and*

$$p, q \notin \Delta(\delta) := \bigcup_{n=1}^{L} \big(f^{-n}(B(z, \delta))\big).$$

If there exist $a > L$ such that $f^a(p) \in B(z, \delta/\rho)$ and $b \geq 0$ such that $f^{-b}(q) \in B(z, \delta/\rho)$, then there exists $g \in \mathcal{U}$ such that q is a future g-iterate of p and $g \equiv f$ outside $\Delta(\delta)$.

We remark that the method used in the proof of Theorem 1 could be used to prove the topological semicontinuity of the basins of generic attractors. However, in the C^1-topology a stronger property can be obtained, which, together with the continuity given by the stable manifold theorem, quickly implies this semicontinuity in this topology.

Proposition 21. *C^1-generically, if a diffeomorphism has an attractor, then there exists a periodic point inside the attractor such that its stable manifold is dense in the basin of the attractor.*

Proof. Let U be an open set. We define the set

$$\mathcal{U}(U, m) = \left\{ f \in \mathrm{Diff}^1(M) : \exists p \in \bigcap_{n \geq 0} f^n(U) \cap \mathrm{Per}_h(f) \text{ with } W^s(p, f)\ 1/m\text{-dense in } U \right\}.$$

If $f \in \mathcal{U}(U, m)$, then it has a hyperbolic periodic point in U such that its stable manifold is $1/m$-dense in U. Since this point is hyperbolic, there exists V, a C^1-neighborhood of f such that if $g \in V$ then $p(g) \in U$. Take $y \in U$ and $B = B(y, 1/m)$, so for f, we have that $W^s(p, f) \cap B \neq \varnothing$. By the stable manifold theorem, we have that $W^s(p(g)) \cap B \neq \varnothing$, so $W^s(p(g))$ is $1/m$-dense in U and $g \in \mathcal{U}(U, m)$. This proves that the set $\mathcal{U}(U, m)$ is open.

Let $\{U_k\}$ be a countable basis of open sets of M, and let $\{O_n\}$ be the set of all possible unions of the elements U_k. Define

$$A(O_n, m) = \mathcal{U}(O_n, m) \cup \overline{\mathcal{U}(O_n, m)^c}.$$

Now, by the previous remark, and by construction, this set is open and dense in $\mathrm{Diff}^1(M)$. So $R_1 = \bigcap_{n,m} A(O_n, m)$ is a residual subset. Let R_2 be the residual subset given in [Abdenur 2003].

Let $R = R_1 \cap R_2$. If $f \in R$ and Λ is an attractor of f, then there exists $p \in \mathrm{Per}_h(f)$ such that $\Lambda = H(p, f)$. Fix n, m such that O_n is a local basin of Λ. Now, we must prove that $f \in \mathcal{U}(O_n, m)$. Suppose that $f \in \overline{\mathcal{U}(O_n, m)^c}$. Since this set is open, there is $W \subset \overline{\mathcal{U}(O_n, m)^c}$, a small open C^1-neighborhood of f. The next step is to prove that we can find $g \in W$ such that $g \in \mathcal{U}(O_n, m)$, which will be a contradiction. To prove this we will use the C^1-connecting lemma, and we will also need the following lemmas. From now on we will fix f and W as above.

Lemma 22. *The function $\Phi(g) = \overline{W^s(p(g), g)}$ for $g \in W$ is continuous in a residual subset of W.*

Proof. The map Φ is lower semicontinuous in W by the stable manifold theorem. Then, it is continuous in a residual subset $W^* \subset W$. □

Thus we have that the map Φ is continuous in $W^* \cap R$. Now, since $f \in \overline{\mathcal{U}(O_n, m)^c}$, there exists an $x \in O_n$ such that $B(x, 1/m) \cap \Phi(f) = \varnothing$.

This, together with Theorem 19, implies the following corollary:

Corollary 23. *There exists a residual subset $R_W \subset W$ such that if $g \in R_W$, then there exists a residual subset $P \subset O_m$ such that if $x \in P$ then $\omega(x) = \Lambda(g)$.*

Proof. Let R^* and S be given by Theorem 19. Define $R_W := R \cap R^* \cap W$ and $P = O_n \cap S$. Hence, if $x \in P$, then $x \in O_n$ and $\omega(x) \subset \Lambda$. However, since $x \in S$ as well, we know that $\omega(x)$ is Lyapunov stable. By the previous remark, since Λ is transitive, we have that $\Lambda \subset \omega(x)$. Thus $\omega(x) = \Lambda$. □

Now, we study the consequences of the continuity of Φ.

Lemma 24. *If Φ is continuous in $g \in R_W$ and S is the set given by Theorem 19, then $O_n \cap S \subset \Phi(g)$.*

Proof. If the lemma does not hold, then there exists $x \in (O_n \cap S) - \Phi(g)$. Let U be a neighborhood of $\Phi(g)$ such that $x \notin U$. By continuity there exists a neighborhood $\mathcal{V}$ of g such that if $h \in \mathcal{V}$ then $\Phi(h) \subset U$.

Since $x \in O_n \cap S$, we have $\omega(x) = \Lambda(g)$. Thus, there exists a sequence $(l_n) \subset \mathbb{N}$ such that $g^{l_n}(x) \to p(g)$. By the Hartman–Grobman theorem [Palis and de Melo 1982], there exists another sequence $(t_n) \subset \mathbb{N}$ such that

$$g^{t_n}(x) \to q \in W^s_\varepsilon(p(g), g) - \{p(g)\}.$$

Let $\rho > 1$, $L \in \mathbb{N}$ and $\delta_0 > 0$, as given by the C^1-connecting lemma applied to q and $\mathcal{U}$. Choose δ with $0 < \delta < \delta_0$, and let V be a neighborhood of the orbit of $p(g)$ such that

$$p(g), x \notin \Delta(\delta) = \bigcup_{n=1}^{L} \big(g^{-n}(B(q, \delta))\big)$$

and

$$\bigcup_{n=1}^{L}\big(g^{-n}(B(q,\delta))\big)\cap V=\varnothing.$$

Pick $y \in B(q,\delta/\rho)\cap W^s(p(g),g)$ such that, defining $z=g^k(y)$, we have $z\in \big(W^s(p(g),g)-\{p(g)\}\big)\cap V$. By definition, we have that $g^{-k}(z)=y\in B(q,\delta/\rho)$. Using that $g^{t_n}(x)\to q$, we obtain some $n_0>L$ such that

$$g^{t_{n_0}}(x)\in B(q,\delta/\rho).$$

Applying the C^1-connecting lemma, we obtain $h\in\mathcal{V}$ such that $h=g$ outside of $\Delta(\delta)$ and x belongs to the h-negative orbit of z. However, since $z\in \big(W^s_g(p(g))-\{p(g)\}\big)\cap V$, we obtain that the h-positive orbit of z belongs to V.

Thus

$$z\in W^s_h(p(h))\quad\text{and thus}\quad x\in W^s_h(p(h)).$$

This leads to a contradiction, since $h\in\mathcal{V}$ and $x\notin U$. □

By the previous lemma, since $f\in W$, there is $g\in R_W$ such that Φ is continuous in g. So $O_n\cap S\subset\Phi(g)$, and there exists $y\in B(x,1/m)\cap S$. Then $y\in\Phi(g)$, which is a contradiction since $f\in W\subset\overline{\mathcal{U}(O_n,m)^c}$. Then $f\in\mathcal{U}(O_n,m)$, which proves the proposition. □

Now, to prove Proposition 6, it is enough to combine Proposition 21 with:

Theorem 25 [Carballo and Morales 2003]. *If f is a C^1-generic tame diffeomorphism then the union of the basins of its attractors is an open and dense subset of M.*

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