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Finite groups with some weakly s-permutably embedded and weakly s-supplemented subgroups

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Let G be a finite group. A subgroup H of G is called weakly s-permutably embedded in G if there is a subnormal subgroup T of G and an s-permutably embedded subgroup $H_{s\mathrm{e}}$ of G contained in H such that $G = HT$ and $H \cap T \leq H_{s\mathrm{e}}$. The subgroup H is called weakly s-supplemented in G if G has a subgroup K such that $HK = G$ and $H \cap K \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H. In this paper, we investigate the influence of weakly s-permutably embedded and weakly s-supplemented subgroups on the structure of finite groups. Some recent results are generalized.

1. Introduction

Throughout only finite groups are considered. We use conventional terminology and notation, as in [Robinson 1982]. Let G denote a group and $|G|$ denote the order of G. Let $B \trianglelefteq A \leq G$. Then A/B is a section of G. In the theory of groups, G is said to be A_4-free if G does not posses a section isomorphic to A_4.

Let $\mathcal{F}$ be a class of groups. Then $\mathcal{F}$ is called a formation provided that (1) if $G \in \mathcal{F}$ and $H \lhd G$, then $G/H \in \mathcal{F}$, and (2) if G/M and G/N are in $\mathcal{F}$, then $G/M \cap N$ is in $\mathcal{F}$ for all normal subgroups M, N of G. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$, where $\Phi(G)$ denotes the Frattini subgroup of G.

Two subgroups H and K of G are said to be permutable if $HK = KH$. Following [Kegel 1962], the subgroup H of G is said to be s-permutable in G if H permutes with every Sylow subgroup of G, that is, $HP = PH$ for any Sylow subgroup P of G. Schmid [1998] showed that if both H and K are s-permutable subgroups of G, then both $H \cap K$ and $\langle H, K\rangle$ are s-permutable in G. Recently, Ballester-Bolinches and Pedraza-Aguilera [1998] generalized s-permutable subgroups to s-permutably

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embedded subgroups. A subgroup H is said to be s-permutably embedded in G provided every Sylow subgroup of H is a Sylow subgroup of some s-permutable subgroup of G. By applying this concept, Ballester-Bolinches and Pedraza-Aguilera got new criteria for the supersolvability of groups. Moreover, a nice result in [Li et al. 2005] on the p-nilpotency of a group could be stated as follows: Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If G is A_4-free and all 2-maximal subgroups of P are s-permutably embedded in G, then G is p-nilpotent.

In recent years, it has been of interest to use supplementation properties of subgroups to characterize properties of a group. Wang [1996] first introduced the concept of c-normal subgroups. Furthermore, Li, Qiao, and Wang [Li et al. 2009] continued to promote this concept and introduced weakly s-permutably embedded subgroups, which are a generalization of both c-normality [Wang 1996] and s-permutably embedding. A subgroup H of G is called weakly s-permutably embedded in G if there is a subnormal subgroup T of G and an s-permutably embedded subgroup H_{se} of G contained in H such that $G = HT$ and $H \cap T \leq H_{se}$. In the meantime, Skiba [2007] introduced the definition of a weakly s-supplemented subgroup. A subgroup H is said to be weakly s-supplemented in G if G has a subgroup T such that $HT = G$ and $H \cap T \leq H_{sG}$, where H_{sG} is the largest s-permutable subgroup of G contained in H.

We note that weakly s-permutably embedded subgroups and weakly s-supplemented subgroups are two distinct concepts. There are examples that show that weakly s-permutably embedded subgroups are not weakly s-supplemented subgroups, and, in general, the converse is also false.

Example 1.1. Let $G = A_5$ be the alternating group of degree 5. Then the Sylow 2-subgroups of G are weakly s-permutably embedded in G, but not weakly s-supplemented in G.

Example 1.2. Let $H = S_4$ be the symmetric group of degree 4, let V be an irreducible and faithful module for H over $\mathbb{F}_3$, the finite field of 3 elements, and consider $G = [V]H$, the corresponding semidirect product. If X is a Sylow 3-subgroup of H, then X is weakly s-supplemented in G but not weakly s-permutably embedded in G.

Hence it is natural to ask the following question: can these two concepts and the related results be unified and generalized? The purpose of this article is to present an answer to the above question. By using these subgroup properties, we determine the structure of G based on the assumption that all 2-maximal subgroups of a Sylow subgroup of G are either weakly s-permutably embedded or weakly s-supplemented subgroups in G. Our results unify and generalize the above mentioned result and some other results in the literature on p-nilpotency and formation theory of finite groups.

2. Preliminaries

For the sake of convenience, we include the following results.

Lemma 2.1 [Ballester-Bolinches and Pedraza-Aguilera 1998, Lemma 1]. *Let H be a subgroup of G.*

(1) *If H is s-permutably embedded in G and $H \leq M \leq G$, then H is s-permutably embedded in M.*

(2) *Let $N \lhd G$ and assume that H is s-permutably embedded in G. Then HN is s-permutably embedded in G and HN/N is s-permutably embedded in G/N.*

Lemma 2.2 [Li et al. 2009, Lemma 2.5]. *Let U be a weakly s-permutably embedded subgroup of G and N a normal subgroup of G. Then*:

(1) *If $U \leq H \leq G$, then U is weakly s-permutably embedded in H.*

(2) *If $N \leq U$, then U/N is weakly s-permutably embedded in G/N.*

(3) *Let π be a set of primes, U a π-subgroup and N a π'-subgroup. Then UN/N is weakly s-permutably embedded in G/N.*

Lemma 2.3 [Skiba 2007, Lemma 2.10]. *Let H be a subgroup of a group G.*

(1) *If H is weakly s-supplemented in G and $H \leq M \leq G$, then H is weakly s-supplemented in M.*

(2) *Let $N \lhd G$ and $N \leq H$. If H is weakly s-supplemented in G, then H/N is weakly s-supplemented in G/N.*

(3) *Let π be a set of primes, H a π-subgroup of G and N a normal π'-subgroup of G. If H is weakly s-supplemented in G, then HN/N is weakly s-supplemented in G/N.*

Lemma 2.4 [Guo and Shum 2003, Lemma 3.12]. *Let P be a Sylow p-subgroup of a group G, where p is the smallest prime dividing $|G|$. If G is A_4-free and $|P| \leq p^2$, then G is p-nilpotent.*

Lemma 2.5 [Guo et al. 2009, Lemma 2.12]. *Let p be a prime, and let G be a group with $(|G|, p-1) = 1$. Suppose that P is a Sylow p-subgroup of G such that each maximal subgroup of P has a p-nilpotent supplement in G. Then G is p-nilpotent.*

Lemma 2.6 [Li et al. 2005]. (1) *If P is an s-permutable p-subgroup of G for some prime p, then $O^p(G) \leq N_G(P)$.*

(2) *Suppose that H is s-permutable in G and P is a Sylow p-subgroup of H, where p is a prime. If $H_G = 1$, then P is s-permutable in G.*

(3) *Suppose that P is a p-subgroup of G contained in $O_p(G)$. If P is s-permutably embedded in G, then P is s-permutable in G.*

Lemma 2.7 [Li and Guo 2000, Lemma 2.6]. *Let H be a nontrivial solvable normal subgroup of G. If every minimal normal subgroup of G which is contained in H is not contained in $\Phi(G)$, then the Fitting subgroup $F(H)$ of H is the direct product of minimal normal subgroups of G which are contained in H.*

Lemma 2.8 [Doerk and Hawkes 1992, A, Lemma 1.2]. *Let U, V and W be subgroups of G. The following statements are equivalent*:

(1) $U \cap VW = (U \cap V)(U \cap W)$.

(2) $UV \cap UW = U(V \cap W)$.

Lemma 2.9 [Guo and Shum 2003, Lemma 3.16]. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type. Also let P be a normal p-subgroup of G such that $G/P \in \mathcal{F}$. If G is A_4-free and $|P| \leq p^2$, then $G \in \mathcal{F}$.*

Lemma 2.10 [Zhang and Li 2012, Lemma 2.11]. *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If G is A_4-free and every 2-maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent.*

Lemma 2.11 [Yang et al. 2012, Lemma 2.12]. *If a p-subgroup H is s-permutable in G, then $H \leq O_p(G)$.*

3. Main results

Our first result unifies and improves the results [Ballester-Bolinches and Guo 1999, Theorem 3; Guo and Shum 2001, Theorem 3.2; Wang 2000, Theorem 4.2] on the p-nilpotency of a group.

Theorem 3.1. *Let p be the smallest prime dividing $|G|$ and P a Sylow p-subgroup of G. If G is A_4-free and every 2-maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the statement is false and let G be a counterexample of minimal order. We proceed with the following steps.

<u>Step 1</u>: By Lemma 2.4, $|P| \geq p^3$ and thus every 2-maximal subgroup of P is nontrivial.

<u>Step 2</u>: G is not a nonabelian simple group.

Assume that G is nonabelian simple. By Lemma 2.5, P has a maximal subgroup P_1 which has no p-nilpotent supplement in G. It follows that any 2-maximal subgroup P_2 of P contained in P_1 has no p-nilpotent supplement in G. From the hypothesis, P_2 is either weakly s-permutably embedded or weakly s-supplemented in G. If P_2 is weakly s-permutably embedded in G, then there is a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_2)_{se}$ of G contained in P_2 such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. Clearly, $T = G$ and thus $P_2 = (P_2)_{se}$

is s-permutably embedded in G. Thus there is an s-permutable subgroup K of G such that P_2 is a Sylow p-subgroup of K. Since G is simple, we get $K_G = 1$. By Lemma 2.6, P_2 is s-permutable in G. Consequently, $1 \neq P_2 \leq O_p(G)$ by Lemma 2.11, which is a contradiction. If P_2 is weakly s-supplemented in G, then there is a non-p-nilpotent subgroup T of G such that

$$G = P_2T \quad \text{and} \quad P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = 1$$

by Lemma 2.11. By Lemma 2.4, T is p-nilpotent, a contradiction.

Step 3: G has a unique minimal normal subgroup N, and G/N is p-nilpotent. Furthermore, $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. Consider the factor group G/N; we will prove that G/N meets the hypotheses of the theorem. Since P is a Sylow p-subgroup of G, PN/N is a Sylow p-subgroup of G/N. If $|PN/N| \leq p^2$, then G/N is p-nilpotent by Lemma 2.4. Hence we assume $|PN/N| \geq p^3$. Let M_2/N be a 2-maximal subgroup of PN/N. Then $M_2 = N(M_2 \cap P)$. Let $P_2 = M_2 \cap P$. It follows that $P_2 \cap N = M_2 \cap P \cap N = P \cap N$ is a Sylow p-subgroup of N. Since

$$p^2 = |PN/N : M_2/N| = |PN : (M_2 \cap P)N| = |P : M_2 \cap P| = |P : P_2|,$$

P_2 is a 2-maximal subgroup of P. If P_2 is weakly s-supplemented in G, then there is a subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{sG}$. So

$$G/N = M_2/N \cdot TN/N = P_2N/N \cdot TN/N.$$

Since $(|N : P_2 \cap N|, |N : T \cap N|) = 1$,

$$(P_2 \cap N)(T \cap N) = N = N \cap G = N \cap P_2T.$$

By Lemma 2.8, $(P_2N) \cap (TN) = (P_2 \cap T)N$. It follows that

$$(P_2N/N) \cap (TN/N) = (P_2N \cap TN)/N = (P_2 \cap T)N/N \leq (P_2)_{sG}N/N.$$

By Lemma 2.6(2) of [Skiba 2007], we know that $(P_2)_{sG}N/N$ is s-permutable in G and thus $(P_2)_{sG}N/N \leq (P_2N/N)_{sG}$. Hence M_2/N is weakly s-supplemented in G/N. If P_2 is weakly s-permutably embedded in G, by Lemma 2.1, it follows analogously that M_2/N is weakly s-permutably embedded in G/N, too. Consequently, G/N meets the hypotheses of the theorem. The minimal choice of G implies that G/N is p-nilpotent. The uniqueness of N and $\Phi(G) = 1$ are clear.

Step 4: $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by Step 3. Since

$$G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$$

is p-nilpotent, we get that G is p-nilpotent, a contradiction.

Step 5: $O_p(G)=1$.

If $O_p(G)\neq 1$, Step 3 yields $N\leq O_p(G)$ and $\Phi(O_p(G))\leq\Phi(G)=1$. Hence, G has a maximal subgroup M such that $G=MN$ and $G/N\cong M$ is p-nilpotent. Since $O_p(G)\cap M$ is normalized by N and M, and also by G, the uniqueness of N yields $N=O_p(G)$. Obviously, $P=N(P\cap M)$. Since $P\cap M<P$, there exists a maximal subgroup P_1 of P such that $P\cap M\leq P_1$. Then $P=NP_1$. Pick a 2-maximal subgroup P_2 of P such that $P_2\leq P_1$. Under the hypothesis, P_2 is either weakly s-permutably embedded or weakly s-supplemented in G. If P_2 is weakly s-permutably embedded in G, then there is a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_2)_{se}$ of G contained in P_2 such that $G=P_2T$ and $P_2\cap T\leq(P_2)_{se}$. Thus there is an s-permutable subgroup K of G such that $(P_2)_{se}$ is a Sylow p-subgroup of K. If $K_G\neq 1$, then $N\leq K_G\leq K$. It follows that $N\leq(P_2)_{se}\leq P_1$, and thus $P=N(P\cap M)=NP_1=P_1$, a contradiction. If $K_G=1$, by Lemma 2.6, $(P_2)_{se}$ is s-permutable in G. It follows from Lemma 2.11 that

$$P_2\cap T\leq(P_2)_{se}\leq O_p(G)=N.$$

Hence, $(P_2)_{se}\leq P_1\cap N$. It follows that

$$((P_2)_{se})^G=1 \quad\text{or}\quad ((P_2)_{se})^G=P_1\cap N=N.$$

If $((P_2)_{se})^G=1$, then $P_2\cap T=1$ and thus $|T|_p=p^2$. Hence T is p-nilpotent by Lemma 2.4. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a normal Hall p'-subgroup of G since T is subnormal in G, which is a contradiction. If $((P_2)_{se})^G=P_1\cap N=N$, then $N\leq P_1$ and thus $P=P_1$, a contradiction. Now we may assume that P_2 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G=P_2T$ and $P_2\cap T\leq(P_2)_{sG}\leq O_p(G)=N$ by Lemma 2.11. Similarly, we get that

$$((P_2)_{sG})^G=1 \quad\text{or}\quad ((P_2)_{sG})^G=P_1\cap N=N.$$

Arguing as before we may assume that $((P_2)_{sG})^G=1$ and deduce that T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Since M is p-nilpotent, we have that M has a normal Hall p'-subgroup $M_{p'}$ and $M\leq N_G(M_{p'})\leq G$. The maximality of M and the fact that $O_{p'}(G)=1$ imply that $M=N_G(M_{p'})$. By using a deep result of Gross [1987, main theorem], there exists $g\in G$ such that $T_{p'}^g=M_{p'}$. Hence $T^g\leq N_G(T_{p'}^g)=N_G(M_{p'})=M$. But $T_{p'}$ is normalized by T, thus g can be considered to be an element of P_2. It follows that $G=P_2T^g=P_2M$ and $P=P_2(P\cap M)=P_1$, a contradiction.

Step 6: G has Hall p'-subgroups and any two Hall p'-subgroups of G are conjugate in G.

If every 2-maximal subgroup of P is weakly s-permutably embedded in G, then G is p-nilpotent by Lemma 2.10, a contradiction. Thus there is a 2-maximal subgroup P_2 of P such that P_2 is weakly s-supplemented in G. Then there exists a subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = 1$ by Lemma 2.11. By Lemma 2.4, T is p-nilpotent and thus T has normal p-complement $T_{p'}$. Obviously, $T_{p'}$ is also a Hall p'-subgroup of G. By [Gross 1987, main theorem], we have that any two Hall p'-subgroups of G are conjugate in G.

Step 7: The final contradiction.

If $NP < G$, then NP meets the hypotheses of the theorem. The minimal choice of G yields that NP is p-nilpotent. Let $N_{p'}$ be the normal p-complement of N. It is easy to see that $N_{p'} \lhd G$, so that $N_{p'} = 1$ by Step 4 and N is a nontrivial p-group, contrary to Step 5. Consequently, we must have $G = NP$. From Step 6, G has Hall p'-subgroups. Then we may assume that N has a Hall p'-subgroup $N_{p'}$. By the Frattini argument,

$$G = NN_G(N_{p'}) = (P \cap N)N_{p'}N_G(N_{p'}) = (P \cap N)N_G(N_{p'}),$$

and thus

$$P = P \cap G = P \cap (P \cap N)N_G(N_{p'}) = (P \cap N)(P \cap N_G(N_{p'})).$$

Since $N_G(N_{p'}) < G$, we have $P \cap N_G(N_{p'}) < P$. We pick a maximal subgroup P_1 of P such that $P \cap N_G(N_{p'}) \leq P_1$. Then $P = (P \cap N)P_1$. Let P_2 be a 2-maximal subgroup of P such that $P_2 \leq P_1$. Under the hypothesis, P_2 is either weakly s-permutably embedded or weakly s-supplemented in G. If P_2 is weakly s-permutably embedded in G, then there is a subnormal subgroup T of G and an s-permutably embedded subgroup $(P_2)_{se}$ of G contained in P_2 such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{se}$. Hence there is an s-permutable subgroup K of G such that $(P_2)_{se}$ is a Sylow p-subgroup of K. If $K_G \neq 1$, then $N \leq K_G \leq K$ and so $(P_2)_{se} \cap N$ is a Sylow p-subgroup of N. We have that $(P_2)_{se} \cap N \leq P_2 \cap N \leq P \cap N$ and $P \cap N$ is a Sylow p-subgroup of N, thus $(P_2)_{se} \cap N = P_2 \cap N = P \cap N$. Consequently,

$$P = (N \cap P)P_1 = (P_2 \cap N)P_1 = P_1,$$

which is a contradiction. Thus $K_G = 1$. By Lemma 2.6, $(P_2)_{se}$ is s-permutable in G. It follows from Lemma 2.11 that $P_2 \cap T \leq (P_2)_{se} \leq O_p(G) = 1$. Since $|T|_p = p^2$, T is p-nilpotent by Lemma 2.4. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a normal Hall p'-subgroup of G, a contradiction. Consequently, we may assume P_2 is weakly s-supplemented in G. Then there is a subgroup T of G such that $G = P_2T$ and $P_2 \cap T \leq (P_2)_{sG} \leq O_p(G) = 1$ (where $O_p(G)$ denotes the p-core of G) by Lemma 2.11. Since $|T|_p = p^2$, T is p-nilpotent by Lemma 2.4. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is a Hall p'-subgroup of G. By Step 6,

$T_{p'}$ and $N_{p'}$ are conjugate in G. Since $T_{p'}$ is normalized by T, there exists $g \in P_2$ such that $T_{p'}^g = N_{p'}$. Hence

$$G = (P_2T)^g = P_2T^g = P_2N_G(T_{p'}^g) = P_2N_G(N_{p'})$$

and

$$P = P \cap G = P \cap P_2N_G(N_{p'}) = P_2(P \cap N_G(N_{p'})) \leq P_1,$$

a final contradiction. □

The following corollaries are immediate from Theorem 3.1.

Corollary 3.2. *Let p be the smallest prime dividing $|G|$ and suppose G is A_4-free. Assume that H is a normal subgroup of G such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H such that every 2-maximal subgroup of P is either weakly s-permutably embedded or weakly s-supplemented in G, then G is p-nilpotent.*

Corollary 3.3. *Suppose that every 2-maximal subgroup of any Sylow subgroup of a group G is either weakly s-permutably embedded or weakly s-supplemented in G. If G is A_4-free, then G is a Sylow tower group of supersolvable type.*

In terms of the theory of formations, we have the following result:

Corollary 3.4. *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and suppose G is A_4-free. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every 2-maximal subgroup of any Sylow subgroup of H is either weakly s-permutably embedded or weakly s-supplemented in G.*

Proof. The necessity part is clear. We only need show the sufficiency part. Suppose that this is not true and let G be a counterexample of minimal order. By Lemmas 2.2 and 2.3, every 2-maximal subgroup of any Sylow subgroup of H is either weakly s-permutably embedded or weakly s-supplemented in H. By Corollary 3.3, H is a Sylow tower group of supersolvable type. Let p be the maximal prime divisor of $|H|$ and let P be a Sylow p-subgroup of H. Then P is normal in G. Consider the factor group G/P. It is easy to prove G/P meets the hypotheses of the theorem. By the minimal choice of G, we get $G/P \in \mathcal{F}$. Let N be a minimal normal subgroup of G contained in P. The proof is divided into two steps.

Step 1: $P = N$.

If $N < P$, then $(G/N)/(P/N) \cong G/P \in \mathcal{F}$. We will prove that $G/N \in \mathcal{F}$. If $|P/N| \leq p^2$, then $G/N \in \mathcal{F}$ by Lemma 2.4. If $|P/N| > p^2$, then every 2-maximal subgroup of P/N is either weakly s-permutably embedded or weakly s-supplemented in G/N by Lemmas 2.2 and 2.3. By the minimal choice of G, we get $G/N \in \mathcal{F}$. Since $\mathcal{F}$ is a saturated formation, N is the unique minimal normal subgroup of G contained in P and $N \nleq \Phi(G)$. It follows from Lemma 2.7 that $P = F(P) = N$, which is a contradiction.

Step 2: The final contradiction.

If $|N| \leq p^2$, then $G \in \mathcal{F}$ by Lemma 2.9, a contradiction. Then $|N| \geq p^3$. Since $N \lhd G$, we may pick a 2-maximal subgroup N_2 of N such that $N_2 \lhd G_p$, where G_p is a Sylow p-subgroup of G. Then N_2 is either weakly s-permutably embedded or weakly s-supplemented in G. Let T be a supplement of N_2 in G. Then $G = N_2T = NT$ and $N = N \cap N_2T = N_2(N \cap T)$. This means that $N \cap T \neq 1$. However, since $N \cap T$ is normal in G and N is minimal normal in G, we get $N \cap T = N$ and thus $T = G$. If N_2 is weakly s-permutably embedded in G, then $(N_2)_{se} \geq N_2 \cap G = N_2$ is s-permutably embedded in G. From Lemma 2.6, N_2 is s-permutable in G and $O^p(G) \leq N_G(N_2)$, where $O^p(G)$ denotes the p-residual subgroup.[1] Thus $N_2 \lhd G_p O^p(G) = G$. It follows that $|N| = p^2$, a contradiction. If N_2 is weakly s-supplemented in G, then $N_2 = N_2 \cap G \leq (N_2)_{sG}$. Similarly, we also get that $N_2 \lhd G$. We obtain the same contradiction, completing the proof. □

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References

[Ballester-Bolinches and Guo 1999] A. Ballester-Bolinches and X. Guo, "On complemented subgroups of finite groups", *Arch. Math.* (*Basel*) **72** (1999), 161–166. MR 2000a:20037 Zbl 0929.20015

[Ballester-Bolinches and Pedraza-Aguilera 1998] A. Ballester-Bolinches and M. C. Pedraza-Aguilera, "Sufficient conditions for supersolubility of finite groups", *J. Pure Appl. Algebra* **127**:2 (1998), 113–118. MR 99d:20048 Zbl 0928.20020

[Doerk and Hawkes 1992] K. Doerk and T. Hawkes, *Finite soluble groups*, de Gruyter Expositions in Mathematics **4**, Walter de Gruyter, Berlin, 1992. MR 93k:20033 Zbl 0753.20001

[Gross 1987] F. Gross, "Conjugacy of odd order Hall subgroups", *Bull. London Math. Soc.* **19**:4 (1987), 311–319. MR 89c:20038 Zbl 0616.20007

[Guo and Shum 2001] X. Guo and K. P. Shum, "On c-normal subgroups of finite groups", *Publ. Math. Debrecen* **58**:1-2 (2001), 85–92. MR 2001k:20034 Zbl 1062.20503

[Guo and Shum 2003] X. Guo and K. P. Shum, "Cover-avoidance properties and the structure of finite groups", *J. Pure Appl. Algebra* **181**:2-3 (2003), 297–308. MR 2004g:20027 Zbl 1028.20014

[Guo et al. 2009] W. Guo, F. Xie, and B. Li, "Some open questions in the theory of generalized permutable subgroups", *Sci. China Ser. A* **52**:10 (2009), 2132–2144. MR 2010k:20027 Zbl 1193.20021

[Kegel 1962] O. H. Kegel, "Sylow-Gruppen und Subnormalteiler endlicher Gruppen", *Math. Z.* **78** (1962), 205–221. MR 26 #5042 Zbl 0102.26802

[Li and Guo 2000] D. Li and X. Guo, "The influence of c-normality of subgroups on the structure of finite groups", *J. Pure Appl. Algebra* **150**:1 (2000), 53–60. MR 2001c:20032 Zbl 0967.20011

[1] This mean $O^p(G)$ is the intersection of all normal subgroups of G whose index in G is a power of k. The quotient $G/O^p(G)$ is the largest (not necessarily abelian) p-group onto which G surjects.

[Li et al. 2005] Y. Li, Y. Wang, and H.-Q. Wei, "On p-nilpotency of finite groups with some subgroups π-quasinormally embedded", *Acta Math. Hungar.* **108**:4 (2005), 283–298. MR 2006f:20022 Zbl 1094.20007

[Li et al. 2009] Y. Li, S. Qiao, and Y. Wang, "On weakly s-permutably embedded subgroups of finite groups", *Comm. Algebra* **37**:3 (2009), 1086–1097. MR 2010a:20041 Zbl 1177.20036

[Robinson 1982] D. J. S. Robinson, *A course in the theory of groups*, Graduate Texts in Mathematics **80**, Springer, New York, 1982. 2nd ed. published in 1996. MR 84k:20001 Zbl 0483.20001

[Schmid 1998] P. Schmid, "Subgroups permutable with all Sylow subgroups", *J. Algebra* **207**:1 (1998), 285–293. MR 99g:20037 Zbl 0910.20015

[Skiba 2007] A. N. Skiba, "On weakly s-permutable subgroups of finite groups", *J. Algebra* **315**:1 (2007), 192–209. MR 2008k:20043 Zbl 1130.20019

[Wang 1996] Y. Wang, "c-normality of groups and its properties", *J. Algebra* **180**:3 (1996), 954–965. MR 97b:20020 Zbl 0847.20010

[Wang 2000] Y. Wang, "Finite groups with some subgroups of Sylow subgroups c-supplemented", *J. Algebra* **224**:2 (2000), 467–478. MR 2001c:20036 Zbl 0953.20010

[Yang et al. 2012] N. Yang, W. Guo, J. Huang, and M. Xu, "Finite groups with weakly S-quasinormally embedded subgroups", *J. Algebra Appl.* **11**:3 (2012), 1250050, 14. MR 2928118 Zbl 1244.20021

[Zhang and Li 2012] X. Zhang and C. Li, "On weakly s-quasinormally embedded and c-supplemented subgroups of finite groups", *Southeast Asian Bull. Math.* **36**:2 (2012), 293–300. MR 2992484 Zbl 06128975

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