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More explicit formulas for Bernoulli and Euler numbers

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By directly considering Taylor coefficients and composite generating functions, we employ a generalized Faà di Bruno formula for higher partial derivatives using vector partitions to obtain identities that include explicit formulas for the Bernoulli and Euler numbers. The formulas we obtain are generalized analogs of the formulas obtained by D. C. Vella.

1. Introduction

The purpose of this paper is to extend the results of Vella [2008] using vector partitions. Recall that the sequences of Bernoulli numbers B_n and Euler numbers E_n have exponential generating functions $x/(e^x-1)$ and $\operatorname{sech} x$ respectively. Vella obtained the identities

$$B_n = \sum_{\pi\in\mathcal{P}_n} \frac{(-1)^m}{1+m}\binom{m}{\lambda(\pi)}\binom{n}{\pi} = \sum_{\pi\in\mathcal{C}_n} \frac{(-1)^m}{1+m}\binom{n}{\pi},$$

$$B_n = \sum_{1\le m\le n} \frac{(-1)^m m!}{1+m}\, S(n,m),$$

$$E_n = \sum_{\substack{\pi\in\mathcal{P}_n\\ \text{even parts}}} (-1)^m \binom{m}{\lambda(\pi)}\binom{n}{\pi} = \sum_{\substack{\pi\in\mathcal{C}_n\\ \text{even parts}}} (-1)^m \binom{n}{\pi},$$

$$E_n = \sum_{1\le m\le n} (-1)^m m!\, S(n,m,\text{even}),$$

$$1 = \sum_{1\le r\le j} \frac{(-1)^r}{(2r)!} E_{2r} \sum_{\substack{\pi\in\mathcal{P}_{2j,2r}\\ \text{odd parts}}} \binom{2r}{\lambda(\pi)}\binom{2j}{\pi} \prod_{s=0}^{j} [E_{2s}]^{\pi_{2s+1}} \quad \text{for all } j>0,$$

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where $\mathcal{P}_n$ is the set of integer partitions of n, $\mathcal{C}_n$ is the set of all ordered partitions (i.e., compositions) of n, m is the length of π, $\lambda(\pi)$ is the multiset of multiplicities of π, $S(n, m)$ is the Stirling number of the second kind, that is, the number of ways of partitioning a set of n elements into exactly m nonempty subsets, and $S(n, m, \text{even})$ is the number of ways of partitioning a set of n elements into exactly m nonempty subsets each with even cardinality.

Let

$$h_B(x_1, \dots, x_\nu) = \frac{x_1 + \cdots + x_\nu}{e^{x_1+\cdots+x_\nu} - 1} \quad \text{and} \quad h_E(x_1, \dots, x_\nu) = \operatorname{sech}(x_1 + \cdots + x_\nu)$$

be functions from $\mathbb{R}^\nu$ into $\mathbb{R}$, where $\nu \in \mathbb{N}$. For a multiindex $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\nu) \in \mathbb{N}_0^\nu$ we consider the generalized Bernoulli number $B_{\boldsymbol{\alpha}}$ to be $\boldsymbol{\alpha}!$ times the $\boldsymbol{\alpha}$-th Taylor coefficient of h_B. We define generalized Euler numbers analogously. These generalized Bernoulli numbers and Euler numbers were recently introduced and studied in [Di Nardo and Oliva 2012] in connection with multivariable Lévy processes. Note that although it wasn't explicitly said in [Di Nardo and Oliva 2012], $B_{\boldsymbol{\alpha}} = B_{|\boldsymbol{\alpha}|}$, where $|\boldsymbol{\alpha}| = \sum_{k=1}^{\nu} \alpha_k$, and thus $B_{\boldsymbol{\alpha}}$ is simply the $|\boldsymbol{\alpha}|$-th Bernoulli number. Also notice that if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^\nu$ and $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ then $B_{\boldsymbol{\alpha}} = B_{\boldsymbol{\beta}}$. The same can also be said for the Euler numbers. In the present paper, we prove the precise analogues of the identities above for these Bernoulli and Euler numbers by applying the multivariable Faà di Bruno formula found in [Constantine and Savits 1996].

The point of view adopted in [Vella 2008] is that thinking explicitly about Taylor coefficients yields tools with a lot of combinatorial leverage. The results of the present paper rely even more heavily on this point of view. For example, it would be interesting to have a combinatorial interpretation for the analogue of $S(n, m)$ that appears in our new formulas, but we obtain these formulas without such a combinatorial interpretation.

2. Notation and review of vector partitions

In this section, we fix notation that parallels that used in [Constantine and Savits 1996] but will in the end yield formulas looking like those in [Vella 2008]. We also restate the results from [Constantine and Savits 1996] in our notation. Below let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{N}_0$ the set of nonnegative integers. We regard finite cartesian powers, such as $\mathbb{N}_0^\nu$, and $\mathbb{N}^\nu$, where $\nu \in \mathbb{N}$, as sitting in the natural way in the real vector space $\mathbb{R}^\nu$ throughout.

Since the generalized Faà di Bruno formula found in [Constantine and Savits 1996] is expressed as a sum over the vector partitions of $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\nu) \in \mathbb{N}_0^\nu$, we begin with a review of the vector partition notation we have adopted in this paper. A *vector partition* $\pi = (\boldsymbol{m}_1, \dots, \boldsymbol{m}_s; \boldsymbol{p}_1, \dots, \boldsymbol{p}_s)$ of $\boldsymbol{\alpha}$ is a multiset of *vector parts* $\boldsymbol{p}_1, \dots, \boldsymbol{p}_s \in \mathbb{N}_0^\nu$ and their respective *vector multiplicities* $\boldsymbol{m}_1, \dots, \boldsymbol{m}_s \in \mathbb{N}_0^\mu$ with

$\mu, s \in \mathbb{N}$, where

$$\sum_{i=1}^{s} \boldsymbol{m}_i = \boldsymbol{m} = (r_1, \ldots, r_\mu) \in \mathbb{N}_0^\mu, \quad |\boldsymbol{m}_i| = \sum_{j=1}^{\mu} m_{ij} > 0, \quad \sum_{i=1}^{s} |\boldsymbol{m}_i| \boldsymbol{p}_i = \boldsymbol{\alpha}.$$

Additionally, we require that the parts are lexicographically ordered, that is,

$$\boldsymbol{0} \prec \boldsymbol{p}_1 \prec \cdots \prec \boldsymbol{p}_s,$$

where $\boldsymbol{p}_i \prec \boldsymbol{p}_j$ means $\boldsymbol{p}_i$ and $\boldsymbol{p}_j$ satisfy one of the following:

- $|\boldsymbol{p}_i| < |\boldsymbol{p}_j|$.
- $|\boldsymbol{p}_i| = |\boldsymbol{p}_j|$ and $p_{i1} < p_{j1}$.
- $|\boldsymbol{p}_i| = |\boldsymbol{p}_j|$ and $p_{i1} = p_{j1}, p_{i2} = p_{j2}, \ldots, p_{ik} = p_{jk}$ and $p_{i(k+1)} < p_{j(k+1)}$ for some $1 \le k < \nu$.

One readily checks that $\prec$ defines a total ordering on $\mathbb{N}_0^\nu$.

The set of vector partitions of $\boldsymbol{\alpha}$ of *size* s and *total multiplicity* $\boldsymbol{m}$ is denoted by $p_s(\boldsymbol{\alpha}, \boldsymbol{m})$. Note that the size s is the number of vector parts in the partition and this number differs from the total multiplicity of the partition. We let

$$p(\boldsymbol{\alpha}, \boldsymbol{m}) = \bigcup_{s=1}^{|\boldsymbol{\alpha}|} p_s(\boldsymbol{\alpha}, \boldsymbol{m}) \quad \text{and} \quad p(\boldsymbol{\alpha}) = \bigcup_{1 \le |\boldsymbol{m}| \le |\boldsymbol{\alpha}|} p(\boldsymbol{\alpha}, \boldsymbol{m}),$$

and we always set $0^0 = 1$. The above definitions can be clarified by working through the example below.

Example 2.1. Let $\nu = 3$ and $\mu = 2$. Take $\boldsymbol{\alpha} = (1, 2, 1)$ where $|\boldsymbol{\alpha}| = 4$. We will verify that

$$\pi = \big((1,1), (1,0); (0,1,0), (1,0,1)\big) \in p_2(\boldsymbol{\alpha}, \boldsymbol{m}),$$

where $\boldsymbol{m} = (2, 1)$. First observe that $\sum_{i=1}^{2} \boldsymbol{m}_i = (1,1) + (1,0) = (2,1) = \boldsymbol{m}$ and $|\boldsymbol{m}_1| = 2 > 0$ and $|\boldsymbol{m}_2| = 1 > 0$. Now observe that

$$\sum_{i=1}^{2} |\boldsymbol{m}_i| \boldsymbol{p}_i = 2(0,1,0) + 1(1,0,1) = (1,2,1) = \boldsymbol{\alpha}.$$

Finally, our last condition is met because $\boldsymbol{p}_1 \prec \boldsymbol{p}_2$ since $|\boldsymbol{p}_1| = 1 < 2 = |\boldsymbol{p}_2|$.

We will also make use of ordered vector partitions of $\boldsymbol{\alpha}$. The set of *ordered vector partitions* of $\boldsymbol{\alpha}$ of total multiplicity $\boldsymbol{m}$ is denoted by $s^+(\boldsymbol{\alpha}, \boldsymbol{m})$. In order to define $s^+(\boldsymbol{\alpha}, \boldsymbol{m})$, we must first define the following:

$$s(\boldsymbol{\alpha}, \boldsymbol{m}) = \left\{ (\boldsymbol{p}_1^{(1)}, \ldots, \boldsymbol{p}_{r_1}^{(1)}; \ldots; \boldsymbol{p}_1^{(\mu)}, \ldots, \boldsymbol{p}_{r_\mu}^{(\mu)}) : \boldsymbol{p}_j^{(i)} \in \mathbb{N}_0^\nu \text{ and } \sum_{i=1}^{\mu} \sum_{j=1}^{r_i} \boldsymbol{p}_j^{(i)} = \boldsymbol{\alpha} \right\}.$$

This allows us to define our set of ordered vector partitions as follows:

$$s^{+}(\boldsymbol{\alpha},\boldsymbol{m})=\big\{(\boldsymbol{p}_1^{(1)},\ldots,\boldsymbol{p}_{r_1}^{(1)};\ldots;\boldsymbol{p}_1^{(\mu)},\ldots,\boldsymbol{p}_{r_\mu}^{(\mu)})\in s(\boldsymbol{\alpha},\boldsymbol{m}):\\ \boldsymbol{p}_j^{(i)}\neq\boldsymbol{0},\ i\in\{1,\ldots,\mu\},\ j\in\{1,\ldots,r_i\}\big\}.$$

We let $s^{+}(\boldsymbol{\alpha})=\bigcup_{1\le|\boldsymbol{m}|\le|\boldsymbol{\alpha}|}s^{+}(\boldsymbol{\alpha},\boldsymbol{m})$. The definition of an ordered vector partition of $\boldsymbol{\alpha}$ of total multiplicity $\boldsymbol{m}$ can be clarified by working through the example below.

Example 2.2. Take $\nu=3$ and $\mu=2$, as before, with $\boldsymbol{\alpha}=(1,2,1)$. We will first verify that $\pi=((0,1,0),(1,0,1);(0,1,0),(0,0,0))\in s(\boldsymbol{\alpha},\boldsymbol{m})$ where $\boldsymbol{m}=(2,1)$. Notice that the size of this ordered partition is 4, but $|\boldsymbol{m}|=3$. Now observe that $\sum_{i=1}^{2}\sum_{j=1}^{r_i}\boldsymbol{p}_j^{(i)}=(0,1,0)+(1,0,1)+(0,1,0)+(0,0,0)=(1,2,1)=\boldsymbol{\alpha}$. Now we can construct an element $\pi'\in s^{+}(\boldsymbol{\alpha},\boldsymbol{m})$ by removing all elements of π equal to $(0,0,0)$. Thus $\pi'=((0,1,0),(1,0,1);(0,1,0))\in s^{+}(\boldsymbol{\alpha},\boldsymbol{m})$. Notice that π is a different element of $s(\boldsymbol{\alpha},\boldsymbol{m})$ than $\pi''=((1,0,1),(0,1,0);(0,1,0),(0,0,0))$ and yields an element of $s^{+}(\boldsymbol{\alpha},\boldsymbol{m})$ not equal to π'.

3. The generalized Faà di Bruno formula

We begin this section by restating the multiindex notation found on page 504 in [Constantine and Savits 1996], which will be used in the generalized Faà di Bruno formula. In what follows, let $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\nu)\in\mathbb{N}_0^\nu$, $\boldsymbol{x}=(x_1,\ldots,x_\nu)\in\mathbb{R}^\nu$ and

$$\boldsymbol{\alpha}!=\prod_{i=1}^{\nu}(\alpha_i!),\qquad \boldsymbol{x}^{\boldsymbol{\alpha}}=\prod_{i=1}^{\nu}x_i^{\alpha_i},$$

$$D_{\boldsymbol{x}}^{\boldsymbol{0}}=\text{identity operator},\qquad D_{\boldsymbol{x}}^{\boldsymbol{\alpha}}=\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_\nu^{\alpha_\nu}}\quad\text{for } |\boldsymbol{\alpha}|>0.$$

Note that for $\boldsymbol{w}=(w_1,\ldots,w_\nu)\in\mathbb{N}_0^\nu$, we write $\boldsymbol{w}\le\boldsymbol{\alpha}$ if $w_k\le\alpha_k$ for $k=1,2,\ldots,\nu$. A function h is an element of $C_{\boldsymbol{\alpha}}(\boldsymbol{x}^0)$ if $D_{\boldsymbol{x}}^{\boldsymbol{w}}h$ exists and is continuous in a neighborhood of $\boldsymbol{x}^0$ for all $\boldsymbol{w}\le\boldsymbol{\alpha}$. Additionally, a function h is an element of $C^n(\boldsymbol{x}^0)$ if $h\in C_{\boldsymbol{w}}(\boldsymbol{x}^0)$ for all $|\boldsymbol{w}|\le n$.

Now let $g:\mathbb{R}^\nu\longrightarrow\mathbb{R}^\mu$ and $f:\mathbb{R}^\mu\to\mathbb{R}$ be functions and $h:\mathbb{R}^\nu\to\mathbb{R}$ their composition; that is, let

$$h(x_1,\ldots,x_\nu)=f\big[g^{(1)}(x_1,\ldots,x_\nu),\ldots,g^{(\mu)}(x_1,\ldots,x_\nu)\big].$$

Assume that $\boldsymbol{0}\neq\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\nu)\in\mathbb{N}_0^\nu$ and $\boldsymbol{x}^0=(x_1^0,\ldots,x_\nu^0)\in\mathbb{R}^\nu$ are given, $g^{(1)},\ldots,g^{(\mu)}\in C_{\boldsymbol{\alpha}}(\boldsymbol{x}^0)$ and $f\in C^{|\boldsymbol{\alpha}|}(\boldsymbol{y}^0)$, where $\boldsymbol{y}^0=(g^{(1)}(\boldsymbol{x}^0),\ldots,g^{(\mu)}(\boldsymbol{x}^0))$. Then, setting $h_{\boldsymbol{\alpha}}=D_{\boldsymbol{x}}^{\boldsymbol{\alpha}}h(\boldsymbol{x}^0)$, $f_{\boldsymbol{m}}=D_{\boldsymbol{y}}^{\boldsymbol{m}}f(\boldsymbol{y}^0)$, $g_{\boldsymbol{k}}^{(i)}=D_{\boldsymbol{x}}^{\boldsymbol{k}}g^{(i)}(\boldsymbol{x}^0)$, and $\boldsymbol{g}_{\boldsymbol{k}}=(g_{\boldsymbol{k}}^{(1)},\ldots,g_{\boldsymbol{k}}^{(\mu)})$, we can state the generalized Faà di Bruno formula that appears as the main result (Theorem 2.1) of [Constantine and Savits 1996]:

Theorem 3.1. $$h_{\boldsymbol{\alpha}} = \sum_{1\le|\boldsymbol{m}|\le|\boldsymbol{\alpha}|} f_{\boldsymbol{m}} \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{\pi\in p_s(\boldsymbol{\alpha},\boldsymbol{m})} (\boldsymbol{\alpha}!) \prod_{j=1}^{s} \frac{[\boldsymbol{g}_{\boldsymbol{p}_j}]^{\boldsymbol{m}_j}}{(\boldsymbol{m}_j!)[\boldsymbol{p}_j!]^{|\boldsymbol{m}_j|}}.$$

The proof of the above theorem found in [Constantine and Savits 1996] takes into account issues of convergence. Now we can rigorously rewrite this generalized formula to resemble the single variable formula used in [Vella 2008]. First let

$$\binom{\boldsymbol{\alpha}}{\pi} = \frac{\boldsymbol{\alpha}!}{\pi!}, \quad \pi! = \prod_{j=1}^{s}[\boldsymbol{p}_j!]^{|\boldsymbol{m}_j|} \quad \text{and} \quad \lambda(\pi)! = \prod_{j=1}^{s}(\boldsymbol{m}_j!).$$

Now observe,

$$\begin{aligned}
h_{\boldsymbol{\alpha}} &= \sum_{1\le|\boldsymbol{m}|\le|\boldsymbol{\alpha}|} f_{\boldsymbol{m}} \sum_{s=1}^{|\boldsymbol{\alpha}|} \sum_{\pi\in p_s(\boldsymbol{\alpha},\boldsymbol{m})} (\boldsymbol{\alpha}!) \prod_{j=1}^{s} \frac{[\boldsymbol{g}_{\boldsymbol{p}_j}]^{\boldsymbol{m}_j}}{(\boldsymbol{m}_j!)[\boldsymbol{p}_j!]^{|\boldsymbol{m}_j|}} \\
&= \sum_{1\le|\boldsymbol{m}|\le|\boldsymbol{\alpha}|} (\boldsymbol{\alpha}!) f_{\boldsymbol{m}} \sum_{\pi\in p(\boldsymbol{\alpha},\boldsymbol{m})} \prod_{j=1}^{s} \frac{[\boldsymbol{g}_{\boldsymbol{p}_j}]^{\boldsymbol{m}_j}}{(\boldsymbol{m}_j!)[\boldsymbol{p}_j!]^{|\boldsymbol{m}_j|}} \\
&= \sum_{\pi\in p(\boldsymbol{\alpha})} \frac{\boldsymbol{\alpha}!}{\prod_{j=1}^{s}(\boldsymbol{m}_j!)[\boldsymbol{p}_j!]^{|\boldsymbol{m}_j|}} f_{\boldsymbol{m}} \prod_{j=1}^{s}[\boldsymbol{g}_{\boldsymbol{p}_j}]^{\boldsymbol{m}_j} \\
&= \sum_{\pi\in p(\boldsymbol{\alpha})} \frac{\binom{\boldsymbol{\alpha}}{\pi}}{\lambda(\pi)!} f_{\boldsymbol{m}} \prod_{j=1}^{s}[\boldsymbol{g}_{\boldsymbol{p}_j}]^{\boldsymbol{m}_j}. \qquad (1)
\end{aligned}$$

Our formula for Taylor coefficients of $h_{\boldsymbol{\alpha}}$ follows:

Corollary 3.2.

$$T_{\boldsymbol{\alpha}}(h;\boldsymbol{x}^0) = \sum_{1\le|\boldsymbol{m}|\le|\boldsymbol{\alpha}|} T_{\boldsymbol{m}}(f;\boldsymbol{y}^0) \sum_{\pi\in p(\boldsymbol{\alpha},\boldsymbol{m})} \binom{\boldsymbol{m}}{\lambda(\pi)} \prod_{j=1}^{s}\prod_{k=1}^{\mu}[T_{\boldsymbol{p}_j}(g^{(k)};\boldsymbol{x}^0)]^{(\boldsymbol{m}_j)_k}. \qquad (2)$$

Proof. This follows directly from (1), since

$$\begin{aligned}
T_{\boldsymbol{\alpha}}(h;\boldsymbol{x}^0) = \frac{h_{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} &= \sum_{\pi\in p(\boldsymbol{\alpha})} \frac{f_{\boldsymbol{m}}}{\pi!\,\lambda(\pi)!} \prod_{j=1}^{s}[\boldsymbol{g}_{\boldsymbol{p}_j}]^{\boldsymbol{m}_j} \\
&= \sum_{\pi\in p(\boldsymbol{\alpha})} \frac{\boldsymbol{m}!\,f_{\boldsymbol{m}}}{\boldsymbol{m}!\,\lambda(\pi)!} \prod_{j=1}^{s} \frac{[\boldsymbol{g}_{\boldsymbol{p}_j}]^{\boldsymbol{m}_j}}{[\boldsymbol{p}_j!]^{|\boldsymbol{m}_j|}} \\
&= \sum_{\pi\in p(\boldsymbol{\alpha})} \binom{\boldsymbol{m}}{\lambda(\pi)} \frac{f_{\boldsymbol{m}}}{\boldsymbol{m}!} \prod_{j=1}^{s} \frac{\prod_{k=1}^{\mu}[\boldsymbol{g}_{\boldsymbol{p}_j}^{(k)}]^{(\boldsymbol{m}_j)_k}}{[\boldsymbol{p}_j!]^{\sum_{k=1}^{\mu}(\boldsymbol{m}_j)_k}}
\end{aligned}$$

$$= \sum_{\pi \in p(\boldsymbol{\alpha})} \binom{\boldsymbol{m}}{\lambda(\pi)} \frac{f_{\boldsymbol{m}}}{\boldsymbol{m}!} \prod_{j=1}^{s} \prod_{k=1}^{\mu} \left(\frac{g^{(k)}_{\boldsymbol{p}_j}}{\boldsymbol{p}_j!} \right)^{(\boldsymbol{m}_j)_k}$$

$$= \sum_{1 \le |\boldsymbol{m}| \le |\boldsymbol{\alpha}|} \frac{f_{\boldsymbol{m}}}{\boldsymbol{m}!} \sum_{\pi \in p(\boldsymbol{\alpha}, \boldsymbol{m})} \binom{\boldsymbol{m}}{\lambda(\pi)} \prod_{j=1}^{s} \prod_{k=1}^{\mu} \left(\frac{g^{(k)}_{\boldsymbol{p}_j}}{\boldsymbol{p}_j!} \right)^{(\boldsymbol{m}_j)_k}$$

$$= \sum_{1 \le |\boldsymbol{m}| \le |\boldsymbol{\alpha}|} T_{\boldsymbol{m}}(f; \boldsymbol{y}^0) \sum_{\pi \in p(\boldsymbol{\alpha}, \boldsymbol{m})} \binom{\boldsymbol{m}}{\lambda(\pi)} \prod_{j=1}^{s} \prod_{k=1}^{\mu} [T_{\boldsymbol{p}_j}(g^{(k)}; \boldsymbol{x}^0)]^{(\boldsymbol{m}_j)_k}. \qquad \square$$

We will also want to make use of the generalized Faà di Bruno formula that considered ordered vector partitions. This is given by Theorem 3.4 of [Constantine and Savits 1996]:

Theorem 3.3.

$$h_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}! \sum_{1 \le |\boldsymbol{m}|} \frac{f_{\boldsymbol{m}}}{\boldsymbol{m}!} \sum_{\pi \in s(\boldsymbol{\alpha}, \boldsymbol{m})} \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} \frac{[g^{(i)}_{\boldsymbol{p}^{(i)}_j}]}{[\boldsymbol{p}^{(i)}_j!]}. \tag{3}$$

Proposition 3.4.

$$T_{\boldsymbol{\alpha}}(h; \boldsymbol{x}^0) = \sum_{s^+(\boldsymbol{\alpha})} T_{\boldsymbol{m}}(f; \boldsymbol{y}^0) \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} T_{\boldsymbol{p}^{(i)}_j}(g^{(i)}; \boldsymbol{x}^0).$$

Proof. This follows directly from (3) by substituting formulas 3.3 and 3.8 of [Constantine and Savits 1996] as follows:

$$T_{\boldsymbol{\alpha}}(h; \boldsymbol{x}^0) = \frac{h_{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} = \sum_{1 \le |\boldsymbol{m}|} \frac{f_{\boldsymbol{m}}}{\boldsymbol{m}!} \sum_{\pi \in s(\boldsymbol{\alpha}, \boldsymbol{m})} \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} \frac{[g^{(i)}_{\boldsymbol{p}^{(i)}_j}]}{[\boldsymbol{p}^{(i)}_j!]}$$

$$= \sum_{1 \le |\boldsymbol{m}| \le |\boldsymbol{\alpha}|} f_{\boldsymbol{m}} \sum_{\pi \in p(\boldsymbol{\alpha}, \boldsymbol{m})} \prod_{j=1}^{|\boldsymbol{\alpha}|} \frac{[\boldsymbol{g}_{\boldsymbol{p}_j}]^{\boldsymbol{m}_j}}{(\boldsymbol{m}_j!)[\boldsymbol{p}_j!]^{|\boldsymbol{m}_j|}}$$

$$= \sum_{1 \le |\boldsymbol{m}| \le |\boldsymbol{\alpha}|} \frac{\boldsymbol{m}! f_{\boldsymbol{m}}}{\boldsymbol{m}!} \sum_{\pi \in p(\boldsymbol{\alpha}, \boldsymbol{m})} \prod_{j=1}^{|\boldsymbol{\alpha}|} \frac{[\boldsymbol{g}_{\boldsymbol{p}_j}]^{\boldsymbol{m}_j}}{(\boldsymbol{m}_j!)[\boldsymbol{p}_j!]^{|\boldsymbol{m}_j|}}$$

$$= \sum_{1 \le |\boldsymbol{m}| \le |\boldsymbol{\alpha}|} \frac{f_{\boldsymbol{m}}}{\boldsymbol{m}!} \sum_{\pi \in s^+(\boldsymbol{\alpha}, \boldsymbol{m})} \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} \frac{[g^{(i)}_{\boldsymbol{p}^{(i)}_j}]}{[\boldsymbol{p}^{(i)}_j!]}$$

$$= \sum_{\pi \in s^+(\boldsymbol{\alpha})} \frac{f_{\boldsymbol{m}}}{\boldsymbol{m}!} \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} \frac{[g^{(i)}_{\boldsymbol{p}^{(i)}_j}]}{[\boldsymbol{p}^{(i)}_j!]}.$$

We used formula 3.3 in going from the first line to the second, and formula 3.8 in going from the third to the fourth. $\square$

4. More Bernoulli and Euler number identities

Recall from Section 1 that if

$$h_B(x_1,\dots,x_\nu)=\frac{x_1+\cdots+x_\nu}{e^{x_1+\cdots+x_\nu}-1}\quad\text{and}\quad h_E(x_1,\dots,x_\nu)=\operatorname{sech}(x_1+\cdots+x_\nu),$$

then the $\boldsymbol{\alpha}$-th generalized Bernoulli and Euler numbers are $B_{\boldsymbol{\alpha}}=\boldsymbol{\alpha}!\,T_{\boldsymbol{\alpha}}(h_B;\mathbf{0})$ and $E_{\boldsymbol{\alpha}}=\boldsymbol{\alpha}!\,T_{\boldsymbol{\alpha}}(h_E;\mathbf{0})$ respectively. In [Vella 2008], the Bernoulli and Euler number identities are expressed in terms of Stirling numbers of the second kind. In this section, we will derive more Bernoulli and Euler number identities using the multivariable analog of these Stirling numbers.

Recall the *multivariable Stirling number of the second kind*,

$$S(\boldsymbol{\alpha},\boldsymbol{m})=\sum_{p(\boldsymbol{\alpha},\boldsymbol{m})}\boldsymbol{\alpha}!\prod_{j=1}^{|\boldsymbol{\alpha}|}\frac{1}{\boldsymbol{m}_j!\,(\boldsymbol{p}_j!)^{|\boldsymbol{m}_j|}}=\sum_{p(\boldsymbol{\alpha},\boldsymbol{m})}\frac{\boldsymbol{\alpha}!}{\lambda(\pi)!\,\pi!},\tag{4}$$

introduced on page 516 of [Constantine and Savits 1996]. Additionally, we define

$$p(\boldsymbol{\alpha},\boldsymbol{m},\text{even})=\{(\boldsymbol{m}_1,\dots,\boldsymbol{m}_s;\boldsymbol{p}_1,\dots,\boldsymbol{p}_s)\in p(\boldsymbol{\alpha},\boldsymbol{m}):|\boldsymbol{p}_j|\text{ even},\ \text{for all } j\in\{1,\dots,s\}\},$$

$$s^+(\boldsymbol{\alpha},\boldsymbol{m},\text{even})=\{(\boldsymbol{p}_1^{(1)},\dots,\boldsymbol{p}_{r_1}^{(1)};\dots;\boldsymbol{p}_1^{(\mu)},\dots,\boldsymbol{p}_{r_\mu}^{(\mu)})\in s^+(\boldsymbol{\alpha},\boldsymbol{m}):|\boldsymbol{p}_j^{(i)}|\text{ even},\ \text{for all } i\in\{1,\dots\mu\},\ \text{for all } j\in\{1,\dots r_\mu\}\}.$$

We analogously define $p(\boldsymbol{\alpha},\boldsymbol{m},\text{odd})$ and $s^+(\boldsymbol{\alpha},\boldsymbol{m},\text{odd})$. Let

$$p(\boldsymbol{\alpha},\text{even})=\bigcup_{1\le|\boldsymbol{m}|\le|\boldsymbol{\alpha}|}p(\boldsymbol{\alpha},\boldsymbol{m},\text{even}),$$

$$s^+(\boldsymbol{\alpha},\text{even})=\bigcup_{1\le|\boldsymbol{m}|\le|\boldsymbol{\alpha}|}s^+(\boldsymbol{\alpha},\boldsymbol{m},\text{even}),$$

and similarly define $p(\boldsymbol{\alpha},\text{odd})$ and $s^+(\boldsymbol{\alpha},\text{odd})$. We call the $\boldsymbol{p}_i$ appearing in elements of $p(\boldsymbol{\alpha},\text{even})$ and $p(\boldsymbol{\alpha},\boldsymbol{m},\text{even})$ *even parts* of $\boldsymbol{\alpha}$, and we define *odd parts* of $\boldsymbol{\alpha}$ in the same manner. Furthermore, let

$$S(\boldsymbol{\alpha},\boldsymbol{m},\text{even})=\sum_{p(\boldsymbol{\alpha},\boldsymbol{m},\text{even})}\boldsymbol{\alpha}!\prod_{j=1}^{|\boldsymbol{\alpha}|}\frac{1}{\boldsymbol{m}_j!\,(\boldsymbol{p}_j!)^{|\boldsymbol{m}_j|}}=\sum_{p(\boldsymbol{\alpha},\boldsymbol{m},\text{even})}\frac{\boldsymbol{\alpha}!}{\lambda(\pi)!\,\pi!},\tag{5}$$

and similarly define $S(\boldsymbol{\alpha},\boldsymbol{m},\text{odd})$.

Our next theorem gives more explicit identities for calculating Bernoulli numbers.

Theorem 4.1. *If $B_{\boldsymbol{\alpha}}$ is the $|\boldsymbol{\alpha}|$-th Bernoulli number, then*

(a) $\displaystyle B_{\boldsymbol{\alpha}}=\sum_{\pi\in p(\boldsymbol{\alpha})}\frac{(-1)^m}{1+m}\binom{m}{\lambda(\pi)}\binom{\boldsymbol{\alpha}}{\pi}=\sum_{\pi\in s^+(\boldsymbol{\alpha})}\frac{(-1)^m}{1+m}\binom{\boldsymbol{\alpha}}{\pi},$

(b) $B_{\boldsymbol{\alpha}} = \sum_{1 \le m \le |\boldsymbol{\alpha}|} \frac{(-1)^m m!}{1+m} S(\boldsymbol{\alpha}, m).$

Proof. Let $g(x_1, \ldots, x_\nu) = e^{x_1+\cdots+x_\nu} - 1$ and $f(y) = \ln(1+y)/y$. Let $\boldsymbol{x}^0 = \mathbf{0} \in \mathbb{R}^\nu$. Then $T_{\boldsymbol{p}_j}(g; \mathbf{0}) = 1/\boldsymbol{p}_j!$ if $\boldsymbol{p}_j > \mathbf{0}$, while $T_m(f; \boldsymbol{y}^0) = T_m(f; \mathbf{0}) = (-1)^m/(1+m)$. By Corollary 3.2,

$$T_{\boldsymbol{\alpha}}(h; \mathbf{0}) = \sum_{1 \le m \le |\boldsymbol{\alpha}|} \frac{(-1)^m}{1+m} \sum_{\pi \in p(\boldsymbol{\alpha}, m)} \binom{m}{\lambda(\pi)} \prod_{j=1}^{s} \left[\frac{1}{\boldsymbol{p}_j!}\right]^{m_j}.$$

Since $B_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}!\, T_{\boldsymbol{\alpha}}(f \circ g; \mathbf{0})$, this yields part (a) because

$$\begin{aligned} B_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}!\, T_{\boldsymbol{\alpha}}(h; \mathbf{0}) &= \sum_{1 \le m \le |\boldsymbol{\alpha}|} \frac{(-1)^m}{1+m} \sum_{\pi \in p(\boldsymbol{\alpha}, m)} \binom{m}{\lambda(\pi)} \frac{\boldsymbol{\alpha}!}{\pi!} \\ &= \sum_{\pi \in p(\boldsymbol{\alpha})} \frac{(-1)^m}{1+m} \binom{m}{\lambda(\pi)} \binom{\boldsymbol{\alpha}}{\pi} = \sum_{\pi \in s^+(\boldsymbol{\alpha})} \frac{(-1)^m}{1+m} \binom{\boldsymbol{\alpha}}{\pi} \end{aligned}$$

by Proposition 3.4. Part (b) follows from part (a) because

$$m!\, S(\boldsymbol{\alpha}, m) = \sum_{\pi \in p(\boldsymbol{\alpha}, m)} \binom{m}{\lambda(\pi)} \binom{\boldsymbol{\alpha}}{\pi}$$

by (4). Collecting together partitions of a fixed total multiplicity yields:

$$B_{\boldsymbol{\alpha}} = \sum_{1 \le m \le |\boldsymbol{\alpha}|} \frac{(-1)^m m!}{1+m} S(\boldsymbol{\alpha}, m). \qquad \square$$

Our next theorem gives more explicit identities for calculating Euler numbers.

Theorem 4.2. *If $E_{\boldsymbol{\alpha}}$ is the $|\boldsymbol{\alpha}|$-th Euler number, then*

(a) $E_{\boldsymbol{\alpha}} = \sum_{\pi \in p(\boldsymbol{\alpha}, \text{even})} (-1)^m \binom{m}{\lambda(\pi)} \binom{\boldsymbol{\alpha}}{\pi} = \sum_{\pi \in s^+(\boldsymbol{\alpha}, \text{even})} (-1)^m \binom{\boldsymbol{\alpha}}{\pi},$

(b) $E_{\boldsymbol{\alpha}} = \sum_{1 \le m \le |\boldsymbol{\alpha}|} (-1)^m m!\, S(\boldsymbol{\alpha}, m, \text{even}).$

Proof. Let $g(x_1, \ldots, x_\nu) = \cosh(x_1, \ldots, x_\nu)$ and $f(y) = 1/y$. Let $\boldsymbol{x}^0 = \mathbf{0} \in \mathbb{R}^\nu$. Then $T_{\boldsymbol{p}_j}(g; \mathbf{0}) = 1/\boldsymbol{p}_j!$ for even parts and $T_{\boldsymbol{p}_j}(g; \mathbf{0}) = 0$ for odd parts, while $T_m(f; \boldsymbol{y}^0) = T_m(f; 1) = (-1)^m$. From Corollary 3.2, we have

$$T_{\boldsymbol{\alpha}}(h; \mathbf{0}) = \sum_{1 \le m \le |\boldsymbol{\alpha}|} (-1)^m \sum_{\pi \in p(\boldsymbol{\alpha}, m)} \binom{m}{\lambda(\pi)} \prod_{j=1}^{s} [T_{\boldsymbol{p}_j}(g; \mathbf{0})]^{m_j},$$

but if any of the parts of π are odd, the product vanishes. Thus, the sum becomes over partitions of only even parts, and

$$T_{\boldsymbol{\alpha}}(h;\mathbf{0}) = \sum_{1\le m\le|\boldsymbol{\alpha}|} (-1)^m \sum_{\pi\in p(\boldsymbol{\alpha},m,\text{even})} \binom{m}{\lambda(\pi)} \prod_{j=1}^{s} \left[\frac{1}{\boldsymbol{p}_j!}\right]^{m_j}.$$

Since $E_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}!\, T_{\boldsymbol{\alpha}}(h;\mathbf{0})$, this yields part (a) because

$$\begin{aligned} E_{\boldsymbol{\alpha}} = \boldsymbol{\alpha}!\, T_{\boldsymbol{\alpha}}(h;\mathbf{0}) &= \sum_{1\le m\le|\boldsymbol{\alpha}|} (-1)^m \sum_{\pi\in p(\boldsymbol{\alpha},m,\text{even})} \binom{m}{\lambda(\pi)} \frac{\boldsymbol{\alpha}!}{\pi!} \\ &= \sum_{\pi\in p(\boldsymbol{\alpha},\text{even})} (-1)^m \binom{m}{\lambda(\pi)} \binom{\boldsymbol{\alpha}}{\pi} = \sum_{\pi\in s^+(\boldsymbol{\alpha},\text{even})} (-1)^m \binom{\boldsymbol{\alpha}}{\pi} \end{aligned}$$

by Proposition 3.4. Part (b) follows from part (a) because

$$m!\, S(\boldsymbol{\alpha},m) = \sum_{\pi\in p(\boldsymbol{\alpha},m)} \binom{m}{\lambda(\pi)} \binom{\boldsymbol{\alpha}}{\pi}$$

by (5). Collecting together partitions of a fixed total multiplicity yields

$$E_{\boldsymbol{\alpha}} = \sum_{1\le m\le|\boldsymbol{\alpha}|} (-1)^m m!\, S(\boldsymbol{\alpha},m,\text{even}). \qquad \square$$

Theorem 4.3. *If $E_{\boldsymbol{\alpha}}$ is the $|\boldsymbol{\alpha}|$-th Euler number, then*

$$1 = \sum_{1\le m\le|\boldsymbol{\alpha}|} \frac{(-1)^r}{(2r)!} E_{2r} \sum_{\pi\in p(\boldsymbol{\alpha},2r,\text{odd})} \binom{2r}{\lambda(\pi)} \binom{\boldsymbol{\alpha}}{\pi} \prod_{j=1}^{s} [E_{\boldsymbol{p}_j}]^{m_j}.$$

Proof. Let $g(x_1,\dots,x_\nu) = 2\tan^{-1}(e^{x_1+\cdots+x_\nu}) - \pi/2$ be the multivariable analogue of the gudermannian function and set $f(y) = \sec y$. Let $\boldsymbol{x}^0 = \mathbf{0}$. Notice that $h(x_1,\dots,x_\nu) = \sec(g(x_1,\dots,x_\nu)) = \cosh(x_1+\cdots+x_\nu)$. Then $T_{\boldsymbol{\alpha}}(h;\boldsymbol{x}^0) = T_{\boldsymbol{\alpha}}(h;\mathbf{0}) = 1/\boldsymbol{\alpha}!$ when $|\boldsymbol{\alpha}|$ is even and $T_{\boldsymbol{\alpha}}(h;\mathbf{0}) = 0$ otherwise, while

$$T_m(f;\boldsymbol{y}^0) = T_m(f;\mathbf{0}) = \frac{(-1)^{m/2}}{m!} E_m$$

when m is even and $T_m(f;\mathbf{0}) = 0$ when m is odd. Letting $m = 2r$, we substitute

$$T_{2r}(f;\mathbf{0}) = \frac{(-1)^r}{(2r)!} E_{2r}$$

into (2) of Corollary 3.2 to yield

$$\frac{1}{\boldsymbol{\alpha}!} = \sum_{1\le 2r\le|\boldsymbol{\alpha}|} \frac{(-1)^r}{(2r)!} E_{2r} \sum_{\pi\in p(\boldsymbol{\alpha},2r)} \binom{2r}{\lambda(\pi)} \prod_{j=1}^{s} [T_{\boldsymbol{p}_j}(g;\boldsymbol{x}^0)]^{\boldsymbol{m}_j}.$$

From the basic properties of the gudermannian function,

$$\begin{aligned} g(x_1,\ldots,x_\nu) &= \int_0^{x_i} \operatorname{sech}(x_1+\cdots+x_\nu)\,dx_i \\ &= \sum_{j_1,\ldots,j_\nu=0}^{\infty} \frac{E_{(j_1,\ldots,j_\nu)}}{j_1!\cdots j_\nu!}\int_0^{x_i} x_1^{j_1}\cdots x_\nu^{j_\nu}\,dx_i \\ &= \sum_{j_1,\ldots,j_\nu=0}^{\infty} \frac{E_{(j_1,\ldots,j_\nu)}}{j_1!\cdots(j_i+1)!\cdots j_\nu!} x_1^{j_1}\cdots x_i^{j_i+1}\cdots x_\nu^{j_\nu}. \end{aligned}$$

Thus,

$$T_{(j_1,\ldots,j_i+1,\ldots,j_\nu)}(g;\boldsymbol{x}^0) = T_{(j_1,\ldots,j_i+1,\ldots,j_\nu)}(g;\mathbf{0}) = \frac{E_{(j_1,\ldots,j_\nu)}}{j_1!\cdots(j_i+1)!\cdots j_\nu!}.$$

It follows that $T_{(j_1,\ldots,j_i+1,\ldots,j_\nu)}(g;\boldsymbol{x}^0)=0$ unless $|(j_1,\ldots,j_i+1,\ldots,j_\nu)|$ is odd because formula (a) of Theorem 4.2 implies that either $E_{(j_1,\ldots,j_\nu)}=0$ or it is possible to write $(j_1,\ldots,j_\nu)$ as the sum of only even parts. It follows that

$$1 = \sum_{1\le m\le|\boldsymbol{\alpha}|} \frac{(-1)^r}{(2r)!}E_{2r} \sum_{\pi\in p(\boldsymbol{\alpha},2r,\text{odd})} \binom{2r}{\lambda(\pi)}\binom{\boldsymbol{\alpha}}{\pi}\prod_{j=1}^{s}[E_{\boldsymbol{p}_j}]^{m_j}. \qquad \square$$

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