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Crossings of complex line segments

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The crossing lemma holds in $\mathbb{R}^2$ because a real line separates the plane into two disjoint regions. In $\mathbb{C}^2$ removing a complex line keeps the remaining point-set connected. We investigate the crossing structure of affine line segment-like objects in $\mathbb{C}^2$ by defining two notions of line segments between two points and give computational results on combinatorics of crossings of line segments induced by a set of points. One way we define the line segments motivates a related problem in $\mathbb{R}^3$, which we introduce and solve.

1. Introduction

A graph is planar if it can be drawn on the plane such that none of its edges cross. For any graph G, we define the crossing number $\mathrm{cr}(G)$ to be the smallest possible number of edge crossings over all the planar drawings of G. In this paper, we will study and present some computational results in the two-dimensional complex plane motivated by the crossing number inequality. The crossing number inequality is a well-known tool in discrete geometry as it gives a lower bound for the crossing number of a graph [Ajtai et al. 1982]:

Theorem 1.1 (crossing number inequality). *If an undirected graph with n vertices and m edges satisfies $m > 4n$, then we have* $\mathrm{cr}(G) \geq m^3/64n^2$.

One of the applications of the inequality is a short proof [Székely 1997] of the Szemerédi–Trotter theorem [1983]:

Theorem 1.2 (Szemerédi–Trotter theorem). *Given n points and m lines in the plane, the number of point-line pairs such that the point lies on the line is*

$$O(n^{2/3}m^{2/3} + n + m).$$

Theorem 1.2 generalizes to the two-dimensional complex plane [Tóth 2003] with lines of complex variable and points in the two-dimensional complex plane, and in a slightly weaker form to spaces of higher dimension [Solymosi and Tao 2012].

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The main motivation of our work is the question of whether a suitable generalization of the crossing number inequality could yield a simple proof for the complex generalization of the Szemerédi–Trotter theorem in similar vein as in the real counterpart. The answer to this question is still out of reach and very little is known. One significant difficulty in understanding the crossing number of a graph in $\mathbb{C}^2$ is that interpreting an edge in such a graph as a line segment is not as straightforward as in $\mathbb{R}^2$. One natural way to attempt to understand crossings of graphs in $\mathbb{C}^2$ is to look for complete graphs without crossings. In $\mathbb{R}^2$ it is well known that the complete graph with five or more vertices always has at least one crossing. Analogously, given a set of five or more points in $\mathbb{R}^2$, if we connect all the points with line segments, at least two of the line segments will cross. It is not clear to what extent the same is true in $\mathbb{C}^2$, and this will be the main focus of our study. In Section 2, we will present two ways to define a complex line segment and devise an algorithm that looks for configurations of n points such that the corresponding complete graph has no crossings. We will discuss the results and based on them give two conjectures regarding arrangements of points in $\mathbb{C}^2$ and crossings of the line segments between them. In Section 3, we introduce and present a solution to a problem in $\mathbb{R}^3$ motivated by our earlier discussion.

2. Line segments in $\mathbb{C}^2$

The two-dimensional complex plane is the set of points

$$\mathbb{C}^2 = \{(z_1, z_2) : z_1, z_2 \in \mathbb{C}\},$$

and a complex line determined by the constants $a, b \in \mathbb{C}$ is the subset

$$\{(u, v) \in \mathbb{C}^2 : v = au + b\}.$$

The two-dimensional complex plane can be considered as a four-dimensional real Euclidean space with complex lines being two-dimensional affine subspaces. Since lines in $\mathbb{C}^2$ are two-dimensional, it is not obvious how to define a line segment between two points $z_1, z_2 \in \mathbb{C}^2$. In general, we want a line segment to be a region enclosed by a simply connected curve on the complex line that contains the points z_1, z_2. For simplicity, we focus on two particular types of line segments: one given by the closed disk that has z_1 and z_2 as its antipodal points and another that is the union of the two closed disks centered at z_1 and z_2, both having radius $\|z_1 - z_2\|$.

Before making these notions precise, let us briefly discuss the problem we will study: any arrangement of five points in $\mathbb{R}^2$ is such that if we draw the line segments between all the points, then at least two of the line segments cross[1]. The same is

[1] By crossing of line segments we mean an intersection of two line segments that is not an endpoint of either line segment.

not true for every configuration of four points. This is equivalent to saying that the smallest complete graph with nonzero crossing number is the one with five vertices, K_5. We are interested in studying to what extent this is true for complex line segments in $\mathbb{C}^2$, or in particular, what is the number of points such that the induced line segments necessarily contain at least one crossing? We will present a computational algorithm that looks for configurations of points with no crossings for a given number of points. Using the algorithm, we can look for a lower bound for the number of points such that the induced graph does not have a crossing.

Let us denote the set of points in $\mathbb{C}^2$ by

$$\begin{aligned} P &= \{z_1, z_2, \ldots, z_n\} \\ &= \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}, \quad u_i, v_i \in \mathbb{C}, \end{aligned}$$

and a line containing the points z_i, z_j by

$$L_{ij} = \{(u, au + b) \subset \mathbb{C}^2 : a, b \in \mathbb{C} \text{ s.t. } au_k + b = v_k, k = i, j\}.$$

We can now introduce the two notions of line segments.

Definition. Call the set

$$S_{\mathrm{I}}(z_1, z_2) = \left\{ z \in L_{12} : \left\| z - \frac{z_1 + z_2}{2} \right\| \le \left\| \frac{z_1 - z_2}{2} \right\| \right\}$$

a textitline segment of type I.

Definition. Call the set

$$S_{\mathrm{II}}(z_1, z_2) = \{z \in L_{12} : \|z - z_1\| \le \|z_1 - z_2\| \text{ or } \|z - z_2\| \le \|z_1 - z_2\|\}$$

a *line segment of type II.*

If the type of the line segment is irrelevant, we will just write $S(z_1, z_2)$. We say that the line segments $S(z_i, z_j)$ and $S(z_k, z_l)$ (where no two points are equal) have a crossing if and only if

$$S(z_i, z_j) \cap S(z_k, z_l) = L_{ij} \cap L_{kl} \neq \varnothing.$$

Computational setup. We observe that if two line segments do not cross, then the intersection point of the lines defined by the points lies outside of at least one of the line segments. This motivates us to look for configurations of points where the intersection point of any two lines is in some sense close to the boundary of the curve defining the line segment.

Let z_i, z_j, z_k, z_l be distinct points of the set P. Denote by $z = L_{ij} \cap L_{kl}$ the intersection of one of the pairs of lines induced by the points. For an intersection of line segments of type I, set

$$r^{\mathrm{I}}_{ij} = \frac{\|z - \frac{z_i+z_j}{2}\|}{\frac{1}{2}\|z_i - z_j\|}$$

to measure the relative distance of the intersection point from the center of the circle defining the line segment. For the lines L_{ij} and L_{kl}, define

$$\rho^{\mathrm{I}}_{ij,kl} = \max\{r^{\mathrm{I}}_{ij}, r^{\mathrm{I}}_{kl}\}.$$

For each pair of lines, $\rho^{\mathrm{I}}_{ij,kl}$ picks the one for which the intersection point of the lines is relatively further from the center of the circle defining the line segment. Finally, set

$$\rho^{\mathrm{I}} = \min_{z_i,z_j,z_k,z_l \in P} \{\rho^{\mathrm{I}}_{ij,kl}, \rho^{\mathrm{I}}_{ik,jl}, \rho^{\mathrm{I}}_{il,jk}\},$$

where all the points z_i, z_j, z_k, z_l are distinct. Similarly, for an intersection of line segments of type II, set

$$r^{\mathrm{II}}_{ij} = \min\left\{ \frac{\|z - z_i\|}{\|z_i - z_j\|}, \frac{\|z - z_j\|}{\|z_i - z_j\|} \right\},$$

and define the quantities $\rho^{\mathrm{II}}_{ij,kl}$ and ρ^{II} in the same way we did for the line segment of type I. In what follows, we will just write ρ instead of ρ^{I} or ρ^{II} when it does not matter which type of line segment is in question. Furthermore, notice that ρ is a function of the set of points P, but to simplify notation we will leave it unwritten.

Evidently if $\rho > 1$, none of the line segments defined by the points in the configuration have a crossing. We will use a randomized algorithm to search for configurations with ρ close to 1 in hope of either finding a configuration that contains no crossing of the induced line segments or a configuration that is extremal in the sense that $\rho \approx 1$.

The way our algorithm works is as follows: Initially start with a random configuration $P_0 = \{z_1, \ldots, z_n\}$. On iteration k, choose an index $j \in \{1, \ldots, n\}$ randomly using a uniform distribution and set $\hat{z}_j = z_j + \epsilon$, where $\epsilon \in \mathbb{C}^2$ is some uniformly distributed random variable with 0 mean and small variance. If the ρ computed for the new configuration is larger than the ρ of the configuration from the previous iteration, replace z_j by $\hat{z}_j$ in the configuration, otherwise do nothing.

In order to justify the algorithm, let us make the following remarks: The results of the described algorithm provide us with lower bounds for the number of points whose induced complete graph does not necessarily have a crossing. The algorithm makes small local perturbations to maximize the quantity ρ, but it is not clear whether or not there are several local optima that differ from a global optimum. Therefore, the cases where the algorithm fails to find a noncrossing configuration are inconclusive. However, when applied to $\mathbb{R}^2$, the algorithm found noncrossing configurations for four points but not for five, agreeing with known results.

Results. Our computational experiments motivate the following remark and two conjectures:

Remark. There is a configuration of seven points in $\mathbb{C}^2$ such that none of the line segments of type I between any pairs of points have a crossing.

One such configuration, with $\rho^{\mathrm{I}} \approx 1.1047$, is

$$\begin{aligned}
z_1 &= (0.4358 - 0.3796i, 0.5726 + 0.3896i),\\
z_2 &= (-0.3382 + 0.0719i, -0.1316 + 0.3220i),\\
z_3 &= (0.6391 + 0.0141i, 0.8889 - 0.3292i),\\
z_4 &= (0.6302 - 0.5513i, 0.2813 - 0.8285i),\\
z_5 &= (0.9731 - 1.3291i, 2.3615 + 0.4571i),\\
z_6 &= (1.7105 - 0.7780i, -1.4009 - 0.8982i),\\
z_7 &= (0.0099 - 0.9417i, 1.3350 - 0.9040i).
\end{aligned}$$

We were not able to produce a configuration of eight points such that $\rho^{\mathrm{I}} \geq 1$. We observed that when executing the search algorithm with 20000 iterations ten times, ρ^{I} was found to lie between 0.978347 and 0.999998. Hence we state the following conjecture:

Conjecture. *Every configuration of eight points in $\mathbb{C}^2$ has four points such that the line segments of type I induced by the points have an intersection. In particular, there exists a configuration of eight points such that $\rho^{\mathrm{I}} = 1$.*

For line segments of type II, we were not able to produce a configuration of four points such that $\rho^{\mathrm{II}} > 1$ after executing the search algorithm with 20000 iterations ten times. We noticed that there exists a configuration such that $\rho^{\mathrm{II}} = 1$; for example, consider the points

$$\begin{aligned}
z_1 &= (0, 0),\\
z_2 &= (1, 0),\\
z_3 &= \left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i, 0\right),\\
z_4 &= (u, v), \quad \text{where } u, v \in \mathbb{C}, v \neq 0.
\end{aligned}$$

It is not difficult see that this configuration has the claimed property, as z_1, z_2 and z_3 all lie on the same complex line and have equal distance from each other. Thus the following conjecture is motivated:

Conjecture. *Every configuration of four points in $\mathbb{C}^2$ is such that at least two of the line segments of type II induced by the points have an intersection.*

3. A related problem in $\mathbb{R}^3$

Line segments of type I define a disk with two given points as antipodal points. In the above treatment, we were interested in configurations of points in $\mathbb{C}^2$ such that the line segments between the points do not intersect. This motivates a similar question in $\mathbb{R}^3$, which we will introduce and produce a solution for.

Consider a set of n points $P = \{p_1, p_2, \dots, p_n\} \subset \mathbb{R}^3$. For each pair of points p_i, p_j, denote by T_{ij} some plane containing both points and by D_{ij} the closed disk lying on T_{ij} with antipodal points p_i, p_j. In other words,

$$D_{ij} = \left\{ x \in \mathbb{R}^3 : x \in T_{ij}, \left\| x - \frac{p_i - p_j}{2} \right\| \leq \left\| \frac{p_i - p_j}{2} \right\| \right\}.$$

We will call $\mathcal{D} = \{D_{ij} : i < j,\ i, j = 1, \dots, n\}$ a *disk system induced by P*. For a pair of such disks, $D_{ij}, D_{kl} \in \mathcal{D}$, we say that the disks *intersect properly* if $D_{ij} \cap D_{kl} \nsubseteq P$. Fixing the set P does not trivially determine if there is a pair of disks that intersect properly in $\mathcal{D}$ since there is some freedom in choosing each of the planes T_{ij} (i.e., the rotation of the disk D_{ij} around the line passing through p_i and p_j). We are now interested in determining the conditions for the set P such that none of the pairs of disks intersect properly. In what follows, we prove the following result:

Theorem 3.1. *The maximal size of the set P such that the induced disks do not intersect properly is four. In such a configuration all the points lie on a plane T, and three of the points form a triangle with one point in its interior. All the disks intersect T perpendicularly.*

Remark. Notice the differences between line segments of type I we defined in Section 2 and the disks considered here: the line segments of type I reside in four-dimensional space and their rotation along the axis given by the two points is fixed. In addition, when considering the proper intersections of the disks D_{ij} and D_{kl} here, we do not require that i, j, k, l are all different.

Proofs. We will first characterize proper intersections of two disks sharing a common point. Then using this characterization, we show that for three points, there is only one way of choosing the rotations of the disks such that no two intersect properly, which quickly implies Theorem 3.1.

Two disks. To keep notation simple, let $v, w \in \mathbb{R}^3$ be two nonparallel vectors. Let T_v, T_w be two planes such that T_v is spanned by v and some (still unspecified) vector, and T_w is similarly spanned by w and some other vector. Denote by D_v the disk lying in T_v such that the antipodal points of D_v are the origin and v, and by D_w the disk lying in T_w with the origin and w as antipodal points.

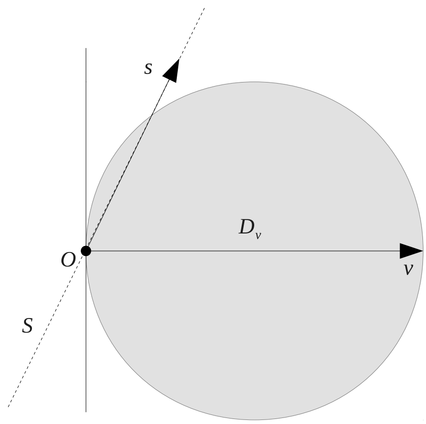

Figure 1. The disk D_v, line S and its spanning vector s on the T_v-plane.

Since T_v and T_w both contain the origin, their intersection is always nonempty. Let $S = T_v \cap T_w$ be the line given by the intersection of the two planes and s a vector such that $S = \operatorname{span} s$. Ignoring the trivial case of $\operatorname{span} v = \operatorname{span} s$ or $\operatorname{span} w = \operatorname{span} s$, we have that $T_v = \operatorname{span}(v, s)$ and $T_w = \operatorname{span}(w, s)$. Therefore, the disks D_v, D_w and thus their intersection is determined by the three vectors v, w and s.

The line S is given by the intersection of the planes T_v and T_w, but what does it tell us about the intersection of the disks? First, let us see how things look on the T_v-plane (see Figure 1). If s is perpendicular to v, then clearly the disk D_v does not intersect the plane T_w outside of the origin and hence cannot intersect D_w properly. Otherwise it is clear that there exists some real $\alpha \neq 0$ such that $\alpha s \in D_v$, i.e., S intersects D_v outside the origin.

The same conclusion naturally holds for the disk D_w. Let us use this observation to prove the following lemma:

Lemma 3.2. *The disks D_v and D_w intersect properly if and only if*

$$\langle v, s\rangle\langle w, s\rangle > 0.$$

Proof. If D_v and D_w intersect properly, there is some nonzero $\alpha \in \mathbb{R}$ such that $\alpha s \in D_v \cap D_w$ since the intersection $S \cap D_v \cap D_w$ is not just the origin. Then, from the way we have defined the disks D_v, D_w to lie on the planes T_v, T_w (see Figure 1), it follows that the projection of αs to the vector v has the same direction as v, and the projection of αs to w has the same direction as w. In other words, $\langle v, \alpha s\rangle > 0$ and $\langle w, \alpha s\rangle > 0$. Multiplying these two inequalities together yields $\alpha^2\langle v, s\rangle\langle w, s\rangle > 0$.

On the other hand, if $\langle v, s\rangle\langle w, s\rangle > 0$, then either $\langle v, s\rangle$ and $\langle w, s\rangle$ are both strictly positive or negative. Assume they are both positive. This means that for an arbitrarily small $\alpha > 0$, we must have $\alpha s \in D_v$ and $\alpha s \in D_w$, i.e., $\alpha s \in D_v \cap D_w$,

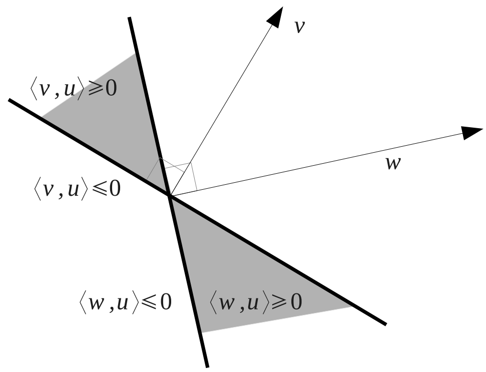

Figure 2. The region of all the vectors u satisfying $\langle v, u\rangle\langle w, u\rangle \leq 0$ on the plane spanned by v, w (shaded).

so the intersection of the disks contains points other than the origin. If both of the inner products are negative, the same conclusion holds for $-\alpha$. □

To see one useful interpretation of the above lemma, let us consider the orthogonal projection s' of s to the plane $T = \mathrm{span}(v, w)$. First, note that $\langle v, s\rangle = \langle v, s'\rangle$ and $\langle w, s\rangle = \langle w, s'\rangle$, so

$$\langle v, s\rangle\langle w, s\rangle = \langle v, s'\rangle\langle w, s'\rangle.$$

Therefore, the set of vectors s such that the disks D_v, D_w do not intersect properly, i.e., $\langle v, s\rangle\langle w, s\rangle \leq 0$, is characterized by the cone C in T (see Figure 2), where

$$C = \{u \in T : \langle u, v\rangle \leq 0 \text{ and } \langle u, w\rangle \geq 0 \quad \text{or} \quad \langle u, v\rangle \geq 0 \text{ and } \langle u, w\rangle \leq 0\}.$$

Three disks. We will now look at the implications of Lemma 3.2 to configurations of three disks. First we will show a fact from plane geometry concerning triangles and cones. Let a, b, c be noncollinear points on the plane and abc the corresponding triangle. To each vertex of the triangle we can associate a cone, as in Figure 2. Let the cones associated with the points a, b and c be called C_a, C_b and C_c (see Figure 3).

Lemma 3.3. *The intersection of the cones is empty, that is,* $C_a \cap C_b \cap C_c = \varnothing$.

Proof. Assume that the angle $\alpha = \angle bac$ corresponding to the point a is the largest angle of the triangle. Since the opening angle of the cone is the same as the corresponding angle in the triangle, the opening angle of C_a is greater than the opening angles of C_b and C_c. Denote by l the line passing through the point a such that l halves the angle α. Then l divides the plane into two parts, one containing the point b and once containing the point c; denote these half-planes by H_b and H_c.

Since the opening angle of C_a is greater than the opening angle of C_b and C_c, and the opening angles of the cones are equal to the corresponding angles in the triangle, we have that the opening angles of C_b and C_c are strictly less than $\pi/2$. Thus C_b

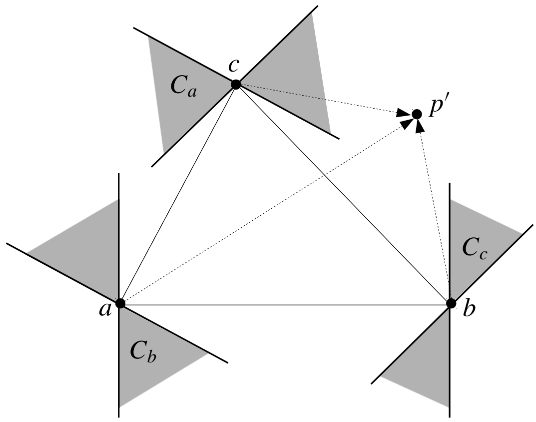

Figure 3. The triangle abc, the projection p' of p and the cones C_a, C_b and C_c.

or C_c cannot contain any points in the triangle and so $a \notin C_b \cup C_c$. Therefore, the intersection $C_a \cap C_b$ is entirely contained in $H_b \setminus l$ and the intersection $C_a \cap C_c$ in $H_c \setminus l$. □

Now we can show that there is only one way three points can induce a disk system without proper intersections. The points a, b, c lie on a plane T and determine a triangle abc. Each vertex of the triangle is a touching point of two disks, and each side of the triangle is the rotation axis of one disk. The rotation of a disk determines a plane containing the corresponding side of the triangle. If none of the three planes are equal, there are exactly two different cases for their intersection: either all three planes intersect in one point, or they are all perpendicular to the plane T and thus do not have a common intersection point.

We will show now that if the three planes have a mutual intersection point, then at least two of the disks will intersect properly. So assume there is a point p where the three planes intersect, and consider the orthogonal projection p' of p onto the plane T containing the points a, b, c (see Figure 3). As we saw earlier, if two disks touching in one vertex of the triangle do not intersect properly, then the line segment from the vertex to p' lies in the cone associated with the vertex. So to require that none of the pairs of disks intersect is the same as requiring that $p' \in C_a \cap C_b \cap C_c$, which by Lemma 3.3 is not possible.

We have justified the following:

Lemma 3.4. *The only disk system induced by three points such that no two disks intersect properly is the one where all the disks perpendicularly intersect the plane containing the points.*

Theorem 3.1 follows now without much effort. First, assume there is a configuration of four points $p_1, \ldots, p_4$ such that no two disks intersect properly and all the points do not lie on the same plane. Then by Lemma 3.4, the disks induced by p_1, p_2 and p_3 must all perpendicularly intersect the plane T containing p_1, p_2 and p_3. But the points p_1, p_2 and p_4 lie on a plane $T' \neq T$, and the induced disks have to intersect T' perpendicularly. Therefore D_{12} intersects T and T' perpendicularly, which leaves no option other than $T = T'$, which contradicts our assumption.

Hence, for any number of points, we have to have that the points lie on a plane in order to not have properly intersecting disks in the induced disk system. The points and the disks give rise to a complete graph on the plane, as we can think of the points as vertices and the rotation axes as edges of the graph. Clearly the disks intersect properly if the graph has crossing edges. Any complete graph with five or more vertices has an edge crossing, which concludes the proof of Theorem 3.1.

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