involve
a journal of mathematics

On the ε-ascent chromatic index of complete graphs

Jean A. Breytenbach and C. M. (Kieka) Mynhardt

msp

2015 vol. 8, no. 2

INVOLVE 8:2 (2015)
dx.doi.org/10.2140/involve.2015.8.295

On the ε-ascent chromatic index of complete graphs

Jean A. Breytenbach and C. M. (Kieka) Mynhardt

(Communicated by Jerrold Griggs)

An edge ordering of a graph $G=(V, E)$ is an injection $f : E \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. A path in G for which the edge ordering f increases along its edge sequence is called an f-ascent; an f-ascent is maximal if it is not contained in a longer f-ascent. The depression $\varepsilon(G)$ of G is the smallest integer k such that any edge ordering f has a maximal f-ascent of length at most k. Applying the concept of ascents to edge colourings rather than edge orderings, we consider the problem of determining the minimum number $\chi_\varepsilon(K_n)$ of colours required to edge colour K_n, $n \geq 4$, such that the length of a shortest maximal ascent is equal to $\varepsilon(K_n)=3$. We obtain new upper and lower bounds for $\chi_\varepsilon(K_n)$, which enable us to determine $\chi_\varepsilon(K_n)$ exactly for $n=7$ and $n \equiv 2 \pmod 4$ and to bound $\chi_\varepsilon(K_{4m})$ by $4m \leq \chi_\varepsilon(K_{4m}) \leq 4m+1$.

1. Introduction

Following [Schurch 2013a; 2013b], we consider the following question:

Question 1. For $n \geq 4$, what is the smallest integer $r(n)$ for which there exists a proper edge colouring of K_n in colours $1, \dots, r(n)$ such that a shortest maximal path of increasing edge labels has length three?

Schurch showed that $r(n) \leq 2n-3$ for all $n \geq 4$. This bound enabled him to determine $r(n)$ for $n \in \{4, 5\}$ and to show that $7 \leq r(6) \leq 8$. In Section 2 we give a lower bound for $r(n)$ and in Section 3 we improve the general upper bound to

$$r(n) \leq \left\lfloor \frac{3n-3}{2} \right\rfloor.$$

We then improve this bound for even values of n. Consequently, we obtain $r(7)=9$, $r(n)=n+1$ if $n \equiv 2 \pmod 4$, and $n \leq r(n) \leq n+1$ if $n \equiv 0 \pmod 4$ and $n \geq 8$.

MSC2010: 05C15, 05C78, 05C38.
Keywords: edge ordering of a graph, increasing path, depression, edge colouring.
Breytenbach was a second year undergraduate student, enrolled for the degree BSc in Mathematical Sciences (stream Computer Science) at Stellenbosch University, while this paper was being prepared. The paper earned him extra credit for the Foundations of Abstract Mathematics I course. Mynhardt was supported by an NSERC discovery grant.

We begin with a short historical account of the background to this problem. An *edge ordering* of a finite, simple graph G is an injection $f : E(G) \to \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. Denote the set of all edge orderings of G by $\mathcal{F}(G)$. A path $v_1, \ldots, v_k$ (where $v_k \neq v_1$) in G such that $f(v_1) < \cdots < f(v_k)$ is called an *f-ascent*; an f-ascent is *maximal* if it is not contained in a longer f-ascent. The *height $H(f)$* of an edge ordering f is the length of a longest f-ascent, and the *flatness* of f, denoted by $h(f)$, is the length of a shortest maximal f-ascent of G.

Chvátal and Komlós [1971] posed the problem of determining

$$\alpha(K_n) = \min_{f \in \mathcal{F}(K_n)} \{H(f)\}$$

of the complete graph K_n. This is a difficult problem and $\alpha(K_n)$ is known only for $1 \leq n \leq 8$ (see [Burger et al. 2005; Chvátal and Komlós 1971]). The parameter $\alpha(G)$ for complete and other finite graphs was also investigated in [Bialostocki and Roditty 1987; Burger et al. 2005; Calderbank et al. 1984; Graham and Kleitman 1973; Mynhardt et al. 2005; Roditty et al. 2001; Yuster 2001].

For an arbitrary finite graph G, Cockayne et al. [2006] considered the problem of determining $\varepsilon(G) = \max_{f \in \mathcal{F}(G)}\{h(f)\}$, that is, the maximum length, taken over all edge orderings $f \in \mathcal{F}(G)$, of a shortest maximal f-ascent. The parameter $\varepsilon(G)$ is known as the *depression* of G and its computation is likewise a difficult problem. Another interpretation of the depression of G is that any edge ordering f of G has a maximal f-ascent of length at most $\varepsilon(G)$, and $\varepsilon(G)$ is the smallest integer for which this statement is true. Graphs with depression two were characterized in [Cockayne et al. 2006], while trees with depression three were characterized in [Mynhardt 2008]. Graphs with no adjacent vertices of degree three or higher that have depression three were characterized in [Mynhardt and Schurch 2013]. Further work on depression can be found in [Cockayne and Mynhardt 2006; Gaber-Rosenblum and Roditty 2009; Schurch and Mynhardt 2014; 2014; Schurch 2013a; 2013b].

An edge ordering of G is also a *proper edge colouring* — a labelling of the edges of G such that adjacent edges have different labels. The minimum number of labels, also called *colours*, is called the *edge chromatic number* or the *chromatic index* $\chi'(G)$. It is well known (see [Chartrand et al. 2011, Section 10.2], for example) that $\chi'(K_n) = n - 1$ if n is even and $\chi'(K_n) = n$ if n is odd. A *1-factor* of G is a 1-regular spanning subgraph of G, and G is *1-factorable* if $E(G)$ can be partitioned into 1-factors. If G is 1-factorable, then G is r-regular for some r and $\chi'(G) = r$. König's theorem (see [Chartrand et al. 2011, Theorem 10.15]) states that every r-regular bipartite graph is 1-factorable. In particular, the chromatic index of the complete bipartite graph $K_{n,n}$ is given by $\chi'(K_{n,n}) = n$.

Noticing that the labels of some edges in an edge ordering of G may be unimportant when determining $\varepsilon(G)$, Schurch applied the concept of ascents to edge colourings and called the minimum number of colours in a proper edge colouring c of G such that $h(c) = \varepsilon(G)$ the *ε-ascent chromatic index* of G, denoted $\chi_\varepsilon(G)$. Unlike the case for general graphs, the depression of K_n is easy to determine: $\varepsilon(K_1) = 0$, $\varepsilon(K_2) = 1$, $\varepsilon(K_3) = 2$ and $\varepsilon(K_n) = 3$ for all $n \geq 4$ (see [Cockayne et al. 2006]); that is, there does not exist an edge ordering or an edge colouring of K_n such that a shortest maximal ascent has length four or more. Note that $\chi_\varepsilon(K_1) = 0$, $\chi_\varepsilon(K_2) = 1$, $\chi_\varepsilon(K_3) = 3$, and determining $\chi_\varepsilon(K_n)$ for $n \geq 4$ is equivalent to finding the smallest integer $r(n)$ such that there exists a proper edge colouring c of K_n in colours $1, \dots, r(n)$ with $h(c) = 3$, as formulated in Question 1.

2. Lower bound for the ε-ascent chromatic index of K_n

We begin with a simple lower bound for $\chi_\varepsilon(K_n)$, which slightly improves the bound in [Schurch 2013b, Proposition 8] in the special case where $G = K_n$.

Theorem 1. *If $n \geq 4$, then*

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod 4, \\ n+1 & \text{if } n \equiv 1, 2 \pmod 4, \\ n+2 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proof. Let c be a proper edge colouring of K_n in colours $1, \dots, r$ such that $h(c) = 3$. Such a colouring exists because $\varepsilon(K_n) = 3$ if $n \geq 4$. For $i = 1, \dots, r$, define

$$E_i = \{e \in E(K_n) : c(e) = i\}.$$

Then $|E_i| \leq \lfloor n/2 \rfloor$ for each i. Also, no vertex v is incident with an edge $e \in E_1$ and an edge $e' \in E_r$, otherwise e, e' is a maximal c-ascent of length two, which contradicts $h(c) = 3$. Thus $|E_1 \cup E_r| \leq \lfloor n/2 \rfloor$ and $E_1 \cup E_r$ is an independent set of edges, that is, $E_1 \cup E_r, E_2, \dots, E_{r-1}$ is also a proper edge colouring of K_n. Hence $r \geq \chi'(K_n) + 1$. In particular,

$$\chi_\varepsilon(K_n) \geq \begin{cases} n & \text{if } n \equiv 0 \pmod 4, \\ n+1 & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Assume $n \equiv 2 \pmod 4$; say $n = 4p + 2$. Then K_n has $(2p + 1)(4p + 1)$ edges. Suppose $r = \chi'(K_n) + 1 = n$. The upper bound

$$|E_1 \cup E_r|, |E_2|, \dots, |E_{r-1}| \leq \left\lfloor \frac{n}{2} \right\rfloor$$

implies that

$$|E_1 \cup E_r| = |E_2| = \cdots = |E_{r-1}| = \left\lfloor \frac{n}{2} \right\rfloor = 2p + 1.$$

Since $|E_1| + |E_r| = 2p + 1$, an odd number, $|E_1| \neq |E_r|$. Without loss of generality say $|E_1| = k$, where $k \leq p$, and $|E_r| = 2p + 1 - k$. Suppose $e \in E_2$ is not adjacent to any edge in E_1. Since $|E_1 \cup E_r| = 2p+1 = \lfloor n/2 \rfloor$, e is adjacent to an edge $e' \in E_r$. But then e, e' is a maximal c-ascent of length two, which contradicts $h(c) = 3$. Therefore each edge in E_2 is adjacent to an edge in E_1, and since c is a proper edge colouring, $|E_2| \le 2|E_1| = 2k \le 2p < \lfloor n/2 \rfloor$, a contradiction. Thus $r \ge n+1$ as required.

Assume $n \equiv 3 \pmod 4$; say $n = 4p+3$. Then $|E(K_n)| = (4p+3)(2p+1)$. Suppose $r = \chi'(K_n) + 1 = n+1$. As in the case $n \equiv 2 \pmod 4$, we obtain that $|E_1 \cup E_r| = |E_2| = \cdots = |E_{r-1}| = \lfloor n/2 \rfloor = 2p+1$ and that each edge in E_2 is adjacent to an edge in E_1. There is one vertex v that is not incident with any edge in $E_1 \cup E_r$, but an edge in E_2 incident with v also needs to be adjacent to an edge in E_1. We obtain a contradiction as above and the result follows. □

3. Upper bounds for the ε-ascent chromatic index of K_n

In Section 3.1 we provide a new general upper bound for $\chi_\varepsilon(K_n)$. We improve this bound for even values of n in Sections 3.2 (the case $n \equiv 0 \pmod 4$) and 3.3 (the case $n \equiv 2 \pmod 4$).

3.1. ***A general bound.*** For $n \ge 6$, we now describe an edge colouring c of K_n in $\lfloor (3n-3)/2 \rfloor$ colours, as illustrated in Figure 1 for $n \in \{6, 7\}$, and prove in Theorem 3 that $h(c) = 3$. Let $V(K_n) = \{v_0, \dots, v_{n-1}\}$ and $p = \lceil n/2 \rceil$.

- For $i \in \{0, \dots, p-1\}$ and $j \in \{i+1, \dots, n-1\}$, let $c(v_i v_j) = i + j$.
- For $i \in \{p, \dots, n-2\}$ and $j \in \{i+1, \dots, n-1\}$, let $c(v_i v_j) = i + j - 2p$.

Lemma 2. *For all $n \ge 6$, the colouring c defines a proper edge colouring of K_n in $\lfloor (3n-3)/2 \rfloor$ colours.*

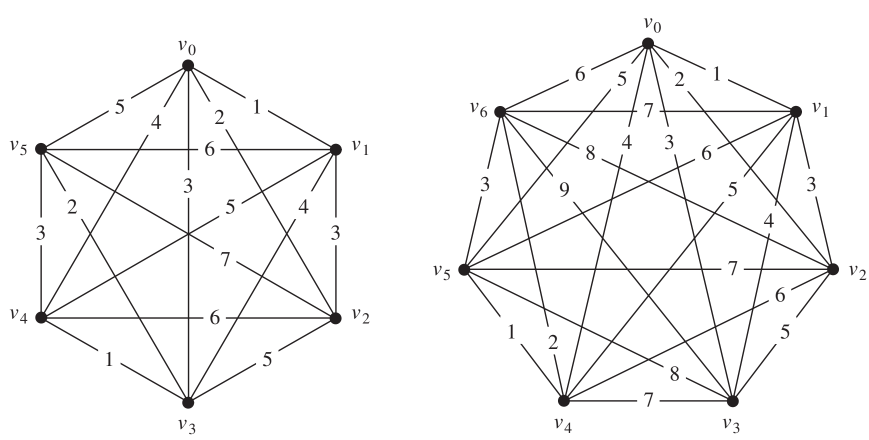

Figure 1. Edge colourings of K_6 and K_7 with flatness three.

Proof. Suppose that $c(v_i v_j) = c(v_i v_{j'})$ for some $j < j'$. After a brief reflection, we deduce that $i + j = i + j' - 2p$. But $i + j \geq i$ and

$$i + j' - 2p \leq i + n - 1 - 2\lceil n/2 \rceil \leq i - 1,$$

hence $c(v_i v_j) > c(v_i v_{j'})$, contradicting our assumption.

Since the smallest colour is $0 + 1 = 1$ and the largest colour is

$$p - 1 + n - 1 = \left\lceil \frac{n}{2} \right\rceil + n - 2 = \left\lfloor \frac{n-1}{2} \right\rfloor + n - 1 = \left\lfloor \frac{3n-3}{2} \right\rfloor,$$

the colouring c uses exactly $\lfloor (3n-3)/2 \rfloor$ colours. □

Theorem 3. *For all $n \geq 6$, the colouring c of K_n has flatness equal to three.*

Proof. To prove that $h(c) = 3$, it is sufficient to prove this:

Statement. *For any $v_i \in V(K_n)$ and edges $e = v_j v_i$ and $f = v_i v_k$ such that $c(e) < c(f)$, there exists*

(Sa) *an edge $g = v_{j'} v_j$, $j' \notin \{i, j, k\}$, such that $c(g) < c(e)$, or*

(Sb) *an edge $g = v_k v_{k'}$, $k' \notin \{i, j, k\}$, such that $c(f) < c(g)$.*

Hence suppose there exist indices $i, j, k \in I = \{0, \ldots, n-1\}$ such that for edges $e = v_j v_i$ and $f = v_i v_k$, we have $c(e) < c(f)$, but neither (Sa) nor (Sb) holds. Then

$$c(v_{j'} v_j) > c(e) \quad \text{for all } j' \in I - \{i, j, k\}, \tag{1}$$

and

$$c(v_k v_{k'}) < c(f) \quad \text{for all } k' \in I - \{i, j, k\}. \tag{2}$$

We consider three cases, depending on the values of i and j.

Case 1: $j \leq p - 1$. Then, regardless of the values of i and j', $c(v_{j'} v_j) = j + j'$ and $c(e) = i + j$. By (1), $j' > i$ for all $j' \in I - \{i, j, k\}$. Hence $i \leq 2$. But $p \geq 3$ since $n \geq 6$, and therefore $i \leq p - 1$. Now $i + j = c(e) < c(f) = i + k$ implies that $j < k$. Therefore one of the following three subcases holds:

(i) $j = 0$, $k = 1$ and $i = 2$,

(ii) $j = 0$ and $k > i = 1$,

(iii) $i = 0$ and $k > j > 0$.

If (i) holds, then $c(v_j v_k) = 1$. Since $n \geq 6$, there exists $k' \in I - \{0, 1, 2\}$ such that $c(v_k v_{k'}) = k + k' \geq k' + 1 \geq 4 > c(f) = i + k = 3$, contradicting (2). If (ii) holds, then $c(f) = 1 + k$. If $k \leq p - 1$, then v_k is adjacent to v_p, where $p \notin \{0, 1, k\}$, and $c(v_k v_p) = k + p > c(f)$, contradicting (2); while if $k \geq p$, then v_k is adjacent to v_2 and $c(v_2 v_k) = k + 2 > c(f)$, again a contradiction. If (iii) holds, then $c(e) = j < k = c(f)$. If $k \leq p - 1$, then $j < p - 1$ and v_k is adjacent to v_p, where $p \notin \{0, j, k\}$, giving a contradiction as in (ii). If $k \geq p$, then there exists $\ell \in \{1, 2\} - \{j\}$ such that $c(v_k v_\ell) = k + \ell > k$, once again a contradiction.

Case 2: $j \geq p$ and $i \leq p-1$. Then $c(e) = i + j$. Since $i \leq p-1$ and $n \geq 6$, there exists $j' \in I - \{i, j, k\}$ such that $j' \geq p$. Then $c(v_{j'}v_j) = j + j' - 2p > i + j$ by (1); that is, $i < j' - 2p \leq 0$, which is impossible.

Case 3: $\min\{i, j\} \geq p$. Then $c(e) = i + j - 2p$. Suppose there exists $j' \in I - \{i, j, k\}$ such that $j' \geq p$. Then $c(v_{j'}v_j) = j + j' - 2p$ and thus $j' > i$ by (1). Since $i, j' \geq p$,

$$c(f) = c(v_i v_k) = \begin{cases} i + k & \text{if } k \leq p-1, \\ i + k - 2p & \text{if } k \geq p, \end{cases}$$

and

$$c(v_k v_{j'}) = \begin{cases} j' + k & \text{if } k \leq p-1, \\ j' + k - 2p & \text{if } k \geq p. \end{cases}$$

Thus, regardless of the value of k, $c(v_k v_{j'}) > c(f)$. Since $j' \in I - \{i, j, k\}$, this contradicts (2). Hence there does not exist $j' \in I - \{i, j, k\}$ such that $j' \geq p$. Since $n \geq 6$, we have $|\{p, \dots, n-1\}| \geq 3$. We deduce that $n \in \{6, 7\}$ and $\{p, \dots, n-1\} = \{i, j, k\}$ so that $c(e) = i + j - 2p$ and $c(f) = i + k - 2p$, where $j < k$ since $c(e) < c(f)$. For either value of n, $c(f) \leq 3$ and $k \geq 4$. Let $j' = 0 < p$. Then $j' \in I - \{i, j, k\}$ and $c(v_{j'}v_k) = j' + k = k \geq 4 > 3 \geq c(f)$, again contradicting (2). □

The following corollary to Lemma 2 and Theorem 3 improves Theorem 17 of [Schurch 2013b].

Corollary 4. *For $n \geq 6$, we have $\chi_\varepsilon(K_n) \leq \lfloor (3n-3)/2 \rfloor$.*

Combining Theorem 1 and Corollary 4 we improve Proposition 20 of [Schurch 2013b] and also obtain the new value $\chi_\varepsilon(K_7)$.

Corollary 5. $$\chi_\varepsilon(K_6) = 7 \quad \textit{and} \quad \chi_\varepsilon(K_7) = 9.$$

3.2. ***The case $n \equiv 0$ (mod 4).*** Our next result is an improved upper bound for $\chi_\varepsilon(K_n)$ in the case where $n \equiv 0 \pmod 4$ and $n \geq 8$. Say $n = 4m$ and $V(K_n) = \{u_0, \dots, u_{2m-1}, v_0, \dots, v_{2m-1}\}$. Let G and H be the subgraphs of K_n induced by $\{u_0, \dots, u_{2m-1}\}$ and $\{v_0, \dots, v_{2m-1}\}$, respectively. Then $G \cong H \cong K_{2m}$ and each of them is $(2m-1)$-edge colourable. We describe a colouring c_1 of K_n in the colours $1, \dots, 4m+1$ as follows.

- In G, let c_1 be any proper edge colouring of K_{2m} in the $2m-1$ colours $\{1, 2\} \cup \{m+3, \dots, 3m-1\}$.
- In H, let c_1 be any proper edge colouring of K_{2m} in the $2m-1$ colours $\{4m, 4m+1\} \cup \{m+3, \dots, 3m-1\}$.
- We still need to colour the edges of the complete bipartite graph $F \cong K_{2m,2m}$ induced by the edges $u_i v_j$, with $i, j \in \{0, \dots, 2m-1\}$. But $\chi'(K_{2m,2m}) = 2m$ and there are $2m$ unused colours $3, \dots, m+2$ and $3m, \dots, 4m-1$. Colour the edges of F with these colours.

It is clear that c_1 is a proper edge colouring of K_{4m} in $4m+1$ colours.

Theorem 6. *For all $m \geq 2$, the colouring c_1 of K_{4m} has flatness equal to three.*

Proof. Let F, G and H be the subgraphs of K_{4m} defined above and let $e, f \in E(K_{4m})$ be adjacent edges such that $c_1(e) < c_1(f)$. We show that (Sa) or (Sb) holds, as stated in the proof of Theorem 3. We consider three cases, depending on the choice of e and f.

Case 1: $\{e, f\} \cap E(F) = \varnothing$. Assume first $e, f \in E(G)$; say $e = u_j u_i$ and $f = u_i u_k$. Then $c_1(e) < c_1(f) \leq 3m-1$, and u_k is adjacent to some vertex $v_\ell \in V(H)$ such that $c_1(u_k v_\ell) = 4m-1 > c_1(f)$. Hence (Sb) holds. Similarly, if $e, f \in E(H)$, say $e = v_j v_i$ and $f = v_i v_k$, then $c_1(f) > c_1(e) \geq m+3$, and v_j is adjacent to some vertex $u_\ell \in V(G)$ such that $c_1(v_j u_\ell) = 3 < c_1(e)$. Hence (Sa) holds.

Case 2: $|\{e, f\} \cap E(F)| = 1$. By symmetry we may assume that $e \in E(F)$; say $e = u_i v_j$. If $f \in E(G)$, say $f = u_i u_k$, then $c_1(e) \in \{3, \ldots, m+2\}$ and $c_1(f) \in \{m+3, m+4, \ldots, 3m-1\}$. Since $m \geq 2$, u_k is adjacent to at least two vertices v_{t_1}, v_{t_2} of H such that $c_1(u_k v_{t_\ell}) \in \{3m, \ldots, 4m-1\}$ for $\ell = 1, 2$, and we may choose a subscript t_ℓ, say t_1, such that $t_1 \neq j$. Then v_j, u_i, u_k, v_{t_1} is a c_1-ascent of length three and (Sb) holds. On the other hand, if $f \in E(H)$, say $f = v_j v_k$, then $c_1(e) \geq 3$. In this case u_i is adjacent to a vertex u_ℓ such that $c_1(u_\ell u_i) \in \{1, 2\}$ and (Sa) holds.

Case 3: $\{e, f\} \subseteq E(F)$. First, if $e = u_i v_j$ and $f = v_j u_k$, then there exists at least one index $\ell \in \{0, \ldots, 2m-1\} - \{i, k\}$ such that $c_1(u_\ell u_i) \in \{1, 2\}$. Then u_ℓ, u_i, v_j, u_k is a c_1-ascent of length three and (Sa) holds. Finally, if $e = v_i u_j$ and $f = u_j v_k$, then there exists at least one index $\ell \in \{0, \ldots, 2m-1\} - \{i, k\}$ such that $c_1(v_k v_\ell) \in \{4m, 4m+1\}$. Then v_i, u_j, v_k, v_ℓ is a c_1-ascent of length three and (Sb) holds. □

Combining Theorems 1 and 6 we narrow down $\chi_\varepsilon(K_n)$ to two possible values in infinitely many cases.

Corollary 7. *For all $n \geq 8$ and $n \equiv 0 \pmod 4$, we have $n \leq \chi_\varepsilon(K_n) \leq n+1$.*

3.3. ***The case $n \equiv 2 \pmod 4$.*** We now assume that $n \equiv 2 \pmod 4$ and $n \geq 10$. Say $n = 4m+2$ and $V(K_n) = \{u_0, \ldots, u_{2m}, v_0, \ldots, v_{2m}\}$. Let G and H be the subgraphs of K_n induced by $\{u_0, \ldots, u_{2m}\}$ and $\{v_0, \ldots, v_{2m}\}$, respectively. Then $G \cong H \cong K_{2m+1}$ and each of them is $(2m+1)$-edge colourable. We describe an edge colouring c_2 of K_n in the colours $1, \ldots, 4m+3$. This colouring is similar to the colouring c_1 above, but not quite as straightforward. See Figure 2 for a partial colouring of K_{10}.

- In G, let c_2 be any proper edge colouring of K_{2m+1} in the $2m+1$ colours $\{1, 2\} \cup \{m+3, \ldots, 3m+1\}$.
- In H, let c_2 be any proper edge colouring of K_{2m+1} in the $2m+1$ colours $\{4m+2, 4m+3\} \cup \{m+3, \ldots, 3m+1\}$.

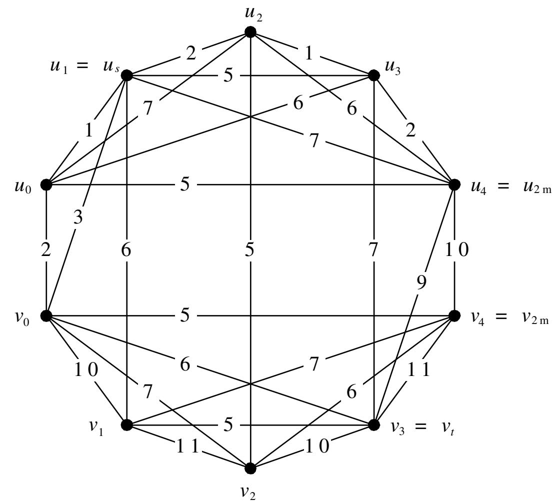

Figure 2. Part of the edge colouring c_2 of K_{10}.

We still need to colour the edges of the complete bipartite graph $F \cong K_{2m+1,2m+1}$ induced by the edges $u_i v_j$, with $i, j \in \{0, \ldots, 2m\}$. By König's theorem, F is 1-factorable. Note that for each colour k in the edge colouring of G there is exactly one vertex that is not incident with an edge coloured k, and conversely, for each vertex u_i there is exactly one colour that does not occur as colour of an edge incident with u_i. A similar remark holds for H. Without loss of generality, say colour 2 does not occur at u_0, colour 1 does not appear at u_{2m}, colour $4m+3$ does not appear at v_0 and colour $4m+2$ does not appear at v_{2m}. Since colour 2 does not occur at u_0, all other colours of the colouring do and thus there exists a vertex $u_s \in V(G)$ such that $c_2(u_0 u_s) = 1$. Since colour $4m+2$ does not appear at v_{2m}, there exists a vertex $v_t \in V(H)$ such that $c_2(v_{2m} v_t) = 4m+3$.

- Colour the edges $u_0 v_0$ and $u_{2m} v_{2m}$ of F with colours 2 and $4m+2$, respectively. For $i, j \in \{1, \ldots, 2m-1\}$ and $k \in \{m+3, \ldots, 3m+1\}$, colour $u_i v_j$ with colour k if and only if no edge incident with u_i in G or with v_j in H is coloured k.

We have now coloured a 1-factor F_0 of F, and $F - F_0$ is a $2m$-regular bipartite graph, which is 1-factorable by König's theorem. Let F_1' be a 1-factor of $F - F_0$ that contains the edge $v_0 u_s$. If $u_{2m} v_t \notin F_1'$, let $F_1 = F_1'$, and if $u_{2m} v_t \in F_1'$, let $u_i v_j \in F_1' - \{v_0 u_s, u_{2m} v_t\}$ and define $F_1 = (F_1' - \{u_i v_j, u_{2m} v_t\}) \cup \{u_i v_t, u_{2m} v_j\}$. Now $F - F_0 - F_1$ is 1-factorable. Let F_2 be a 1-factor of $F - F_0 - F_1$ that contains $u_{2m} v_t$.

- Colour the edges in F_1 with colour 3 and the edges in F_2 with colour $4m+1$. Colouring $F - F_0 - F_1 - F_2$ with the $2m-2$ unused colours $4, \ldots, m+2$ and $3m+2, \ldots, 4m$ yields a proper edge colouring of K_{4m+2}.

Theorem 8. *For all $m \geq 2$, the colouring c_2 of K_{4m+2} has flatness equal to three.*

Proof. Let F, J, G and H be the subgraphs of K_{4m+2} defined above and let $e, f \in E(K_{4m+2})$ be adjacent edges such that $c_2(e) < c_2(f)$. We show that (Sa) or (Sb) holds, as stated in the proof of Theorem 3. If $\{e, f\} \cap E(F) = \varnothing$, the proof follows similar to Case 1 in the proof of Theorem 6. We consider two further cases.

Case 1: $|\{e, f\} \cap E(F)| = 1$. By symmetry we may assume that $e \in E(F)$; say $e = u_i v_j$. First suppose that $f \in E(G)$, say $f = u_i u_k$. Since $c_2(f) > c_2(e) \geq 2$, $c_2(f) \in \{m+3, \ldots, 3m+1\}$. As in Case 2 of the proof of Theorem 3, (Sb) holds. Now suppose $f = v_j v_k \in E(H)$. If $c_2(e) = 2$, then $i = j = 0$ and $c_2(u_0 u_s) = 1$. If $c_2(e) \neq 2$ then $c_2(e) > 2$ and there exists an index ℓ such that $c_2(u_i u_\ell) \in \{1, 2\}$. Thus u_s, u_i, v_j, v_k or u_ℓ, u_i, v_j, v_k is a c_2-ascent of length three and (Sa) holds.

Case 2: $\{e, f\} \subseteq E(F)$. Suppose $e = u_i v_j$ and $f = v_j u_k$. If $e = u_0 v_0$ and $f = v_0 u_s$, then $c_2(e) = 2$ and $c_2(f) = 3$. Therefore there exists a vertex u_ℓ such that $c_2(u_s u_\ell) \in \{m+3, \ldots, 3m+1\}$ and (Sb) holds. If $e = u_0 v_0$ and $k \neq s$, then u_s, u_0, v_0, v_k is a c_2-ascent of length three and (Sa) holds. For all other choices of $e = u_i v_j$ and $f = v_j u_k$ it follows as in Case 3 of the proof of Theorem 3 that (Sa) or (Sb) holds. Suppose $e = v_i u_j$ and $f = u_j v_k$. If $e = v_t u_{2m}$ and $f = u_{2m} v_{2m}$, then $c_2(e) = 4m+1$ and $c_2(f) = 4m+2$. There exists a vertex v_ℓ such that $c_2(v_\ell v_t) \in \{m+3, \ldots, 3m+1\}$ and thus (Sa) holds. If $f = u_{2m} v_{2m}$ and $i \neq t$, then v_i, v_{2m}, u_{2m}, v_t is a c_2-ascent of length three and (Sb) holds. All other cases are dealt with as in Case 3 of the proof of Theorem 3. □

Combining Theorems 1 and 8 and Corollary 4 determines $\chi_\varepsilon(K_n)$ for all $n \equiv 2 \pmod{10}$, $n \geq 6$.

Corollary 9. *For all $n \geq 6$ and $n \equiv 2 \pmod{10}$, we have $\chi_\varepsilon(K_n) = n + 1$.*

4. Conclusion

In Theorem 1 we proved a lower bound for $\chi_\varepsilon(K_n)$, and in Corollary 4 we improved the previously known general upper bound for $\chi_\varepsilon(K_n)$ from $2n-3$ to $\lfloor (3n-3)/2 \rfloor$. Corollary 7 improves this bound for $n \equiv 0 \pmod 4$ and allows us to bound $\chi_\varepsilon(K_{4m})$ by $4m \leq \chi_\varepsilon(K_n) \leq 4m+1$. Finally, Corollary 9 determines $\chi_\varepsilon(K_n)$ for all $n \equiv 2 \pmod 4$, $n \geq 6$. Based on the results for even n and the values $\chi_\varepsilon(K_5) = 7$ and $\chi_\varepsilon(K_7) = 9$, we formulate the following conjecture.

Conjecture 10. *For all $n \geq 4$, we have $\chi_\varepsilon(K_n) = \chi'(K_n) + 2$.*

Acknowledgements

Jean Breytenbach wishes to thank Professor Jan van Vuuren of the Department of Industrial Engineering, Stellenbosch University, for fuelling his interest in graph theory. Both authors hereby also express their gratitude towards Professor van Vuuren for introducing them and providing a wonderful research environment to work in.

References

[Bialostocki and Roditty 1987] A. Bialostocki and Y. Roditty, "A monotone path in an edge-ordered graph", *Internat. J. Math. Math. Sci.* **10**:2 (1987), 315–320. MR 88b:05087 Zbl 0633.05043

[Burger et al. 2005] A. P. Burger, E. J. Cockayne, and C. M. Mynhardt, "Altitude of small complete and complete bipartite graphs", *Australas. J. Combin.* **31** (2005), 167–177. MR 2005i:05160 Zbl 1080.05046

[Calderbank et al. 1984] A. R. Calderbank, F. R. K. Chung, and D. G. Sturtevant, "Increasing sequences with nonzero block sums and increasing paths in edge-ordered graphs", *Discrete Math.* **50**:1 (1984), 15–28. MR 85k:05062 Zbl 0542.05058

[Chartrand et al. 2011] G. Chartrand, L. Lesniak, and P. Zhang, *Graphs & digraphs*, 5th ed., CRC Press, Boca Raton, FL, 2011. MR 2012c:05001 Zbl 1211.05001

[Chvátal and Komlós 1971] V. Chvátal and J. Komlós, "Some combinatorial theorems on monotonicity", *Canad. Math. Bull.* **14** (1971), 151–157. MR 49 #2445 Zbl 0214.23503

[Cockayne and Mynhardt 2006] E. J. Cockayne and C. M. Mynhardt, "A lower bound for the depression of trees", *Australas. J. Combin.* **35** (2006), 319–328. MR 2007j:05118 Zbl 1094.05018

[Cockayne et al. 2006] E. J. Cockayne, G. Geldenhuys, P. J. P. Grobler, C. M. Mynhardt, and J. H. van Vuuren, "The depression of a graph", *Util. Math.* **69** (2006), 143–160. Preprint (2004) available at http://tinyurl.com/GraphDepression2004.

[Gaber-Rosenblum and Roditty 2009] I. Gaber-Rosenblum and Y. Roditty, "The depression of a graph and the diameter of its line graph", *Discrete Math.* **309**:6 (2009), 1774–1778. MR 2010d:05045 Zbl 1205.05066

[Graham and Kleitman 1973] R. L. Graham and D. J. Kleitman, "Increasing paths in edge ordered graphs", *Period. Math. Hungar.* **3** (1973), 141–148. MR 48 #5910 Zbl 0243.05116

[Mynhardt 2008] C. M. Mynhardt, "Trees with depression three", *Discrete Math.* **308**:5-6 (2008), 855–864. MR 2008j:05186 Zbl 1149.05042

[Mynhardt and Schurch 2013] C. M. Mynhardt and M. Schurch, "A class of graphs with depression three", *Discrete Math.* **313**:11 (2013), 1224–1232. MR 3034754 Zbl 1277.05083

[Mynhardt and Schurch 2014] C. M. Mynhardt and M. Schurch, "A construction of a class of graphs with depression three", *Australas. J. Combin.* **58**:2 (2014), 249–263. Zbl 1296.05087

[Mynhardt et al. 2005] C. M. Mynhardt, A. P. Burger, T. C. Clark, B. Falvai, and N. D. R. Henderson, "Altitude of regular graphs with girth at least five", *Discrete Math.* **294**:3 (2005), 241–257. MR 2006a:05079 Zbl 1062.05131

[Roditty et al. 2001] Y. Roditty, B. Shoham, and R. Yuster, "Monotone paths in edge-ordered sparse graphs", *Discrete Math.* **226**:1-3 (2001), 411–417. MR 2001i:05096 Zbl 0961.05040

[Schurch 2013a] M. Schurch, *On the depression of graphs*, Ph.D. thesis, University of Victoria, 2013, available at https://dspace.library.uvic.ca:8443/handle/1828/4527.

[Schurch 2013b] M. Schurch, "Edge colourings and the depression of a graph", *J. Combin. Math. Combin. Comput.* **85** (2013), 195–212. MR 3088160 Zbl 1274.05176

[Schurch and Mynhardt 2014] M. Schurch and C. Mynhardt, "The depression of a graph and k-kernels", *Discuss. Math. Graph Theory* **34**:2 (2014), 233–247. MR 3194034 Zbl 1290.05128

[Yuster 2001] R. Yuster, "Large monotone paths in graphs with bounded degree", *Graphs Combin.* **17**:3 (2001), 579–587. MR 2002k:05135 Zbl 1010.05044

Received: 2013-07-22 Accepted: 2013-10-26

jabreytenbach@cs.sun.ac.za *Computer Science, Department of Mathematical Sciences, Stellenbosch University, Private Bag X1, Matieland, 7602, South Africa*

kieka@uvic.ca *Department of Mathematics and Statistics, University of Victoria, P.O. Box 1700 STN CSC, Victoria, BC V8W 2Y2, Canada*

involve

msp.org/involve

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2015 is US $140/year for the electronic version, and $190/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

involve

2015 vol. 8 no. 2

1944-4176(2015)8:2;1-5