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Bas Lemmens and Christopher Parsons



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(Communicated by Kenneth S. Berenhaut)

In this note, it is shown that the maximum number of pairwise touching translates of an  $n$ -simplex is at least  $n + 3$  for  $n = 7$ , and for all  $n \geq 5$  such that  $n \equiv 1 \pmod{4}$ . The current best known lower bound for general  $n$  is  $n + 2$ . For  $n = 2^k - 1$  and  $k \geq 2$ , we will also present an alternative construction to give  $n + 2$  touching simplices using Hadamard matrices.

## 1. Introduction

A classic problem in discrete geometry is to determine for a given convex body  $K$  in  $\mathbb{R}^n$  the maximum number of pairwise touching translates of  $K$ . This number is called the *touching number of  $K$*  and is denoted by  $t(K)$ . It is well known that for any convex body  $K$  in  $\mathbb{R}^n$ ,

$$t(K) \leq 2^n,$$

and equality holds if, and only if,  $K$  is a parallelotope; see [Danzer and Grünbaum 1962; Petty 1971; Soltan 1975]. On the other hand, it is unknown if for each convex body  $K$  in  $\mathbb{R}^n$  the inequality  $t(K) \geq n + 1$  holds when  $n \geq 4$ ; see [Bezdek 2010, Section 2.3].

This paper concerns the touching number of  $n$ -dimensional simplices,  $\Delta_n$ . This number was studied by Koolen, Laurent and Schrijver [Koolen et al. 2000]. They showed, among other things, that  $t(\Delta_n) \geq n + 2$  for all  $n \geq 3$  and  $t(\Delta_3) = 5$ , see Figure 1. Lemmens [2007] gave examples that showed that  $t(\Delta_4) \geq 7$  and  $t(\Delta_5) \geq 9$ .

The main goal of this short note is to present a construction that gives the following small improvement of the lower bound for  $t(\Delta_n)$ .

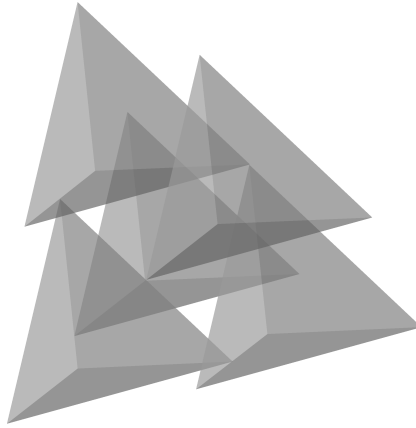
**Theorem 1.1.** *For  $n = 7$  and  $n \equiv 1 \pmod{4}$ , with  $n \geq 5$ , we have that*

$$t(\Delta_n) \geq n + 3.$$

*MSC2010:* primary 52C17; secondary 05B40, 46B20.

*Keywords:* touching number, simplices, equilateral sets,  $\ell_1$ -norm.

Parsons is grateful for the support from School of Mathematics, Statistics and Actuarial Science at the University of Kent and from the HE STEM project “Communicating Mathematical Sciences”.



**Figure 1.** Five pairwise touching tetrahedra.

The problem of determining  $t(\Delta_n)$  is known to be equivalent to finding the maximum size of  $\ell_1$ -norm equilateral sets in a hyperplane [Koolen et al. 2000; Lemmens 2007]. We will discuss the equivalence between these two problems in the next section.

## 2. Equilateral sets

A convex body  $K$  in  $\mathbb{R}^n$  which is centrally symmetric, i.e.,  $x \in K$  if and only if  $-x \in K$ , is the unit ball of a norm  $\|\cdot\|_K$  on  $\mathbb{R}^n$ . Indeed, for  $x \in \mathbb{R}^n$ , we can define the norm by

$$\|x\|_K = \inf\{\lambda > 0 : x \in \lambda K\}.$$

A set  $S$  in a normed space  $(\mathbb{R}^n, \|\cdot\|)$  is called an *equilateral set* if there exists a constant  $\delta > 0$  such that

$$\|s - t\| = \delta \quad \text{for all } s \neq t \text{ in } S.$$

The maximum size of an equilateral set in  $(\mathbb{R}^n, \|\cdot\|)$  is the *equilateral dimension* of  $(\mathbb{R}^n, \|\cdot\|)$  and is denoted by  $e(\mathbb{R}^n, \|\cdot\|)$ . Note that the constant  $\delta > 0$  does not play a role, as we can always scale the equilateral set. Clearly, if  $K$  is a centrally symmetric body in  $\mathbb{R}^n$ , then  $S = \{s_1, \dots, s_p\}$  is an equilateral set in  $(\mathbb{R}^n, \|\cdot\|_K)$  with pairwise distance 2 if, and only if, the set of unit balls with centers  $s_1, \dots, s_p$  is a configuration of  $p$  pairwise touching translates of  $K$ .

The equilateral dimension has been studied for many normed spaces; see, for example, [Alon and Pudlák 2003; Swanepoel 2004a; Swanepoel 2004b]. Particular attention has been given to so-called  $\ell_p$ -norms which are defined as follows. For  $1 \leq p < \infty$ , the  $\ell_p$ -norm on  $\mathbb{R}^n$  is given by  $\|x\|_p = (\sum_i |x_i|^p)^{1/p}$ . For the  $\ell_1$ -norm,

$n = 5$	$n = 6$	$n = 8$
(4, 0, 1, 1, 2)	(4, 0, 1, 1, 1, 1)	(0, 4, 2, 2, 0, 4, 2, 2)
(0, 4, 1, 1, 2)	(0, 4, 1, 1, 1, 1)	(4, 0, 2, 2, 4, 0, 2, 2)
(1, 1, 4, 0, 2)	(1, 1, 4, 0, 1, 1)	(2, 2, 0, 4, 2, 2, 0, 4)
(1, 1, 0, 4, 2)	(1, 1, 0, 4, 1, 1)	(2, 2, 4, 0, 2, 2, 4, 0)
(2, 2, 0, 0, 4)	(1, 1, 1, 1, 4, 0)	(8, 2, 1, 1, 0, 2, 1, 1)
(0, 0, 2, 2, 4)	(1, 1, 1, 1, 0, 4)	(4, 4, 4, 4, 0, 0, 0, 0)
(2, 2, 2, 2, 0)	(2, 2, 2, 2, 0, 0)	(4, 4, 0, 0, 4, 4, 0, 0)
	(2, 2, 0, 0, 2, 2)	(4, 4, 0, 0, 0, 0, 4, 4)
	(0, 0, 2, 2, 2, 2)	(4, 0, 4, 0, 0, 4, 0, 4)
		(4, 0, 0, 4, 0, 4, 4, 0)

**Table 1.** Equilateral sets.

it has been conjectured by Kusner that  $e(\mathbb{R}^n, \|\cdot\|_1) = 2n$  [Guy 1983], but at present this has only been confirmed for  $1 \leq n \leq 4$ ; see [Bandelt et al. 1998; Koolen et al. 2000]. Obviously,  $2n$  is a lower bound for  $e(\mathbb{R}^n, \|\cdot\|_1)$ , as the set of standard basis vectors and their opposites form an equilateral set. The best known upper bound is  $Cn \log n$ , where  $C > 0$  is a constant, which was obtained using probabilistic methods by Alon and Pudlák [2003].

Finding the touching number for the  $n$ -dimensional simplex turns out to be equivalent to determining the maximum size of an  $\ell_1$ -norm equilateral set contained in a hyperplane. More precisely, if  $h(n)$  is the maximum size of an  $\ell_1$ -norm equilateral set in  $H_\alpha = \{x \in \mathbb{R}^n : \sum_i x_i = \alpha\}$  for some  $\alpha \in \mathbb{R}$ , then

$$t(\Delta_n) = h(n + 1) \quad \text{for all } n \geq 1; \tag{2-1}$$

see [Koolen et al. 2000; Lemmens 2007]. For example, the  $\ell_1$ -norm equilateral set

$$S = \{(2, 0, 1, 1), (0, 2, 1, 1), (1, 1, 2, 0), (1, 1, 0, 2), (2, 2, 0, 0)\} \tag{2-2}$$

in the hyperplane  $H_4 \subseteq \mathbb{R}^4$  corresponds to the configuration of five pairwise touching translates of a tetrahedron depicted in Figure 1. The examples of equilateral sets in Table 1 were found with the aid of a computer. In particular, we see that  $t(\Delta_7) \geq 10$ , which settles the  $n = 7$  case in Theorem 1.1.

It is interesting to note that in these examples all the nonzero coordinates are powers of 2. We have looked into those type of examples in more detail, which let to the construction in Proposition 4.1. At present, however, we have no clear understanding of why these coordinate values generate large examples.

Before we prove Theorem 1.1, we mention that the inequalities

$$h(n) \leq e(\mathbb{R}^n, \|\cdot\|_1) \leq h(2n - 1)$$

are known to hold for all  $n \geq 1$  [Koolen et al. 2000; Lemmens 2007]. Thus,  $e(\mathbb{R}^n, \|\cdot\|_1)$  grows linearly in  $n$  if, and only if,  $h(n)$  does.

### 3. Proof of Theorem 1.1

For each  $n \equiv 2 \pmod{4}$  with  $n \geq 6$ , we shall construct an  $\ell_1$ -norm equilateral set in  $H_\alpha = \{x \in \mathbb{R}^n : \sum_i x_i = \alpha\}$  of size  $n+2$ , where  $\alpha = (n-2)^2/2$ . The result then follows from Equation (2-1). So let  $n \equiv 2 \pmod{4}$  with  $n \geq 6$ . Define

$$\begin{aligned} v^1 &= (b, 0, a, a, \dots, a, a), \\ v^2 &= (0, b, a, a, \dots, a, a), \\ v^3 &= (a, a, b, 0, \dots, a, a), \\ v^4 &= (a, a, 0, b, \dots, a, a), \\ &\vdots \\ v^{n-1} &= (a, a, a, a, \dots, b, 0), \\ v^n &= (a, a, a, a, \dots, 0, b), \end{aligned}$$

in  $\mathbb{R}^n$ , where  $a = (n-4)/2$  and  $b = n-2$ . Furthermore let

$$v^{n+1} = (\overbrace{y, y, \dots, y}^k, \overbrace{z, z, \dots, z}^{n-k}) \quad \text{and} \quad v^{n+2} = (\overbrace{z, z, \dots, z}^{n-k}, \overbrace{y, y, \dots, y}^k)$$

in  $\mathbb{R}^n$ . We now show that if we take

$$k = \frac{n-2}{2}, \quad y = \frac{n-6}{2}, \quad \text{and} \quad z = \frac{n-2}{2},$$

then  $V = \{v^1, \dots, v^{n+2}\}$  is an  $\ell_1$ -norm equilateral set in  $H_\alpha$ , where  $\alpha = (n-2)^2/2$  and the distance is  $2(n-2)$ .

To verify this we note first that  $b \geq z \geq a \geq y \geq 0$ . For  $i = 1, \dots, n$ , the coefficient sum of  $v^i$  is given by

$$b + (n-2)a = n-2 + \frac{(n-2)(n-4)}{2} = \frac{(n-2)^2}{2}.$$

Similarly the coefficient sum for the vectors  $v^{n+1}$  and  $v^{n+2}$  is equal to

$$(n-k)z + ky = \frac{(n+2)(n-2)}{4} + \frac{(n-2)(n-6)}{4} = \frac{(n-2)^2}{2}.$$

Let  $1 \leq i \neq j \leq n$ . For  $i = 2k-1$  and  $j = 2k$ , the distance between  $v^i$  and  $v^j$  is given by

$$\|v^i - v^j\|_1 = |b-0| + |0-b| = 2(n-2),$$

and for all other  $i \neq j$ ,

$$\|v^i - v^j\|_1 = |b - a| + |0 - a| + |a - b| + |a - 0| = 2(b - a) + 2a = 2(n - 2).$$

Also

$$\|v^{n+1} - v^{n+2}\|_1 = k|z - y| + k|y - z| = (n - 2)\left(\frac{n-2}{2} - \frac{n-6}{2}\right) = 2(n - 2).$$

Finally the distance between any of the first  $n$  vectors and the last two is calculated as in either the case of  $v^1$  and  $v^{n+1}$ ,

$$\begin{aligned} \|v^1 - v^{n+1}\|_1 &= |b - y| + |0 - y| + (k - 2)|a - y| + (n - k)|a - z| \\ &= n - 2 + \frac{n-6}{2} + \frac{n+2}{2} \\ &= 2(n - 2), \end{aligned}$$

or, as in the case of  $v^1$  and  $v^{n+2}$ ,

$$\begin{aligned} \|v^1 - v^{n+2}\|_1 &= |b - z| + |0 - z| + (n - k - 2)|a - z| + k|a - y| \\ &= n - 2 + \frac{n-2}{2} + \frac{n-2}{2} \\ &= 2(n - 2). \end{aligned}$$

Thus,  $V$  is an  $\ell_1$ -norm equilateral set in  $H_\alpha$  of size  $n + 2$ . Table 2 shows examples in dimensions  $n = 6, 10$  and  $14$ .

#### 4. Hadamard matrices

In this section, we will give an alternative construction that shows that  $t(\Delta_n) \geq n + 2$  for all  $n = 2^k - 1$  with  $k \geq 2$  using  $\ell_1$ -norm equilateral sets and Hadamard matrices. Recall that an  $n \times n$  matrix  $H = [h_{ij}]$  with entries  $h_{ij} \in \{-1, 1\}$  for all  $i$  and  $j$  is called a *Hadamard matrix* if  $HH^T = nI$ . There exists a simple well-known construction of Hadamard matrices of size  $2^k$ . Define  $H_1 = [1]$  and

$$H_{2^{k+1}} = \begin{bmatrix} H_{2^k} & H_{2^k} \\ H_{2^k} & -H_{2^k} \end{bmatrix}$$

for all  $k \geq 1$ . So,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad \dots$$

$n = 6$	$n = 10$	$n = 14$
(4, 0, 1, 1, 1, 1)	(8, 0, 3, 3, 3, 3, 3, 3, 3, 3)	(12, 0, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(0, 4, 1, 1, 1, 1)	(0, 8, 3, 3, 3, 3, 3, 3, 3, 3)	(0, 12, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(1, 1, 4, 0, 1, 1)	(3, 3, 8, 0, 3, 3, 3, 3, 3, 3)	(5, 5, 12, 0, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(1, 1, 0, 4, 1, 1)	(3, 3, 0, 8, 3, 3, 3, 3, 3, 3)	(5, 5, 0, 12, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(1, 1, 1, 1, 4, 0)	(3, 3, 3, 3, 8, 0, 3, 3, 3, 3)	(5, 5, 5, 5, 12, 0, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(1, 1, 1, 1, 0, 4)	(3, 3, 3, 3, 0, 8, 3, 3, 3, 3)	(5, 5, 5, 5, 0, 12, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)
(2, 2, 2, 2, 0, 0)	(3, 3, 3, 3, 3, 3, 8, 0, 3, 3)	(5, 5, 5, 5, 5, 5, 12, 0, 5, 5, 5, 5, 5, 5, 5, 5)
(0, 0, 2, 2, 2, 2)	(3, 3, 3, 3, 3, 3, 0, 8, 3, 3)	(5, 5, 5, 5, 5, 5, 0, 12, 5, 5, 5, 5, 5, 5, 5, 5)
	(3, 3, 3, 3, 3, 3, 3, 3, 8, 0)	(5, 5, 5, 5, 5, 5, 5, 5, 12, 0, 5, 5, 5, 5, 5, 5)
	(3, 3, 3, 3, 3, 3, 3, 3, 0, 8)	(5, 5, 5, 5, 5, 5, 5, 5, 0, 12, 5, 5, 5, 5, 5, 5)
	(4, 4, 4, 4, 4, 4, 2, 2, 2, 2)	(5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 12, 0, 5, 5)
	(2, 2, 2, 2, 4, 4, 4, 4, 4, 4)	(5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 0, 12, 5, 5)
		(5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 12, 0)
		(5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 0, 12)
		(6, 6, 6, 6, 6, 6, 6, 6, 6, 4, 4, 4, 4, 4, 4, 4)
		(4, 4, 4, 4, 4, 4, 4, 6, 6, 6, 6, 6, 6, 6, 6, 6)

**Table 2.** Equilateral sets of size  $n + 2$ .

Now suppose  $k \geq 2$ . Let  $v^1, \dots, v^{2^k} \in \mathbb{R}^{2^k}$  denote the rows of the Hadamard matrix  $H_{2^k}$ , and define the set

$$V_k = \{v^3\} \cup \{v^i : i = 5, \dots, 2^k\}.$$

Furthermore, let  $W_k = \{w^1, w^2, w^3, w^4\} \in \mathbb{R}^{2^k}$  be given by

$$\begin{aligned} w^1 &= (1, -1, 0, 0, 1, -1, 0, 0, \dots, 1, -1, 0, 0), \\ w^2 &= (-1, 1, 0, 0, -1, 1, 0, 0, \dots, -1, 1, 0, 0), \\ w^3 &= (0, 0, 1, -1, 0, 0, 1, -1, \dots, 0, 0, 1, -1), \\ w^4 &= (0, 0, -1, 1, 0, 0, -1, 1, \dots, 0, 0, -1, 1). \end{aligned}$$

**Proposition 4.1.** *For each  $k \geq 2$ , the set  $V_k \cup W_k$  is an  $\ell_1$ -norm equilateral set of size  $2^k + 1$  in  $H_0 = \{x \in \mathbb{R}^{2^k} : \sum_i x_i = 0\}$ .*

*Proof.* Let  $k \geq 2$ . It is easy to show that each  $u \in V_k \cup W_k$  lies in  $H_0$ . Also note that any two distinct points  $v^i$  and  $v^j$  in  $V_k$  satisfy

$$\|v^i - v^j\|_1 = 2^k,$$

as the rows in  $H_{2^k}$  differ in exactly  $2^{k-1}$  places. The reader can check that

$$\|w^i - w^j\|_1 = 2^k \quad \text{for all } 1 \leq i \neq j \leq 4.$$

So, it remains to show that

$$\|v^i - w^j\|_1 = 2^k \quad \text{for all } v^i \in V_k \text{ and } w^j \in W_k. \quad (4-1)$$

We use induction on  $k$ . Note that if  $k = 2$ , we have that

$$V_2 \cup W_2 = \{(1, 1, -1, -1), (1, -1, 0, 0), (-1, 1, 0, 0), (0, 0, 1, -1), (0, 0, -1, 1)\},$$

which is an  $\ell_1$ -norm equilateral set with distance 4. Now suppose that (4-1) holds for  $k$ . Denote the points in  $V_{k+1}$  by  $\bar{v}^i$  and the points in  $W_{k+1}$  by  $\bar{w}^j$ . Note that for  $j = 1, \dots, 4$ , we have  $\bar{w}^j = (w^j, w^j)$ , where  $w^j \in W_k$ . Also observe that for  $i = 3, 5, \dots, 2^k$ , we have  $\bar{v}^i = (v^i, v^i)$ , and for  $i = 2^k + 1, \dots, 2^{k+1}$ , we have  $\bar{v}^i = (v^{i-2^k}, -v^{i-2^k})$ , where  $v^i \in V_k$ .

So, for  $i = 3, 5, \dots, 2^k$  and  $j = 1, \dots, 4$ , we have that

$$\|\bar{v}^i - \bar{w}^j\|_1 = \sum_{l=1}^{2^{k+1}} |\bar{v}_l^i - \bar{w}_l^j| = 2 \sum_{l=1}^{2^k} |v_l^i - w_l^j| = 2 \cdot 2^k = 2^{k+1}$$

by the induction hypothesis. Also for  $i = 2^k + 1, \dots, 2^{k+1}$  and  $j = 1, \dots, 4$ , we have that

$$\|\bar{v}^i - \bar{w}^j\|_1 = \sum_{l=1}^{2^k} (|v_l^{i-2^k} - w_l^j| + |v_l^{i-2^k} + w_l^j|) = \sum_{l=1}^{2^k} (1 - w_l^j + 1 + w_l^j) = 2^{k+1},$$

as  $v_l^i \in \{-1, 1\}$  and  $-1 \leq w_l^j \leq 1$  for all  $l$ . □

The reader should note that the equilateral set  $V_k \cup W_k$  can be seen as a generalization of the equilateral set  $S$  in (2-2), as  $V_2 \cup W_2 = S - (1, 1, 1, 1)$ . Furthermore, the example in Table 1 with  $n = 8$  is also of this type, if one ignores the point  $(8, 2, 1, 1, 0, 2, 1, 1)$ .

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Received: 2013-12-17

Accepted: 2014-02-23

[b.lemmens@kent.ac.uk](mailto:b.lemmens@kent.ac.uk)

*School of Mathematics, Statistics & Actuarial Science,  
University of Kent, Cornwallis Building, Canterbury,  
CT2 7NF, United Kingdom*

[cmp37@kent.ac.uk](mailto:cmp37@kent.ac.uk)

*School of Mathematics, Statistics & Actuarial Science,  
University of Kent, Cornwallis Building, Canterbury,  
CT2 7NF, United Kingdom*

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Pietro Cerone	La Trobe University, Australia <a href="mailto:P.Cerone@latrobe.edu.au">P.Cerone@latrobe.edu.au</a>	Emil Minchev	Ruse, Bulgaria <a href="mailto:eminchev@hotmail.com">eminchev@hotmail.com</a>
Scott Chapman	Sam Houston State University, USA <a href="mailto:scott.chapman@shsu.edu">scott.chapman@shsu.edu</a>	Frank Morgan	Williams College, USA <a href="mailto:frank.morgan@williams.edu">frank.morgan@williams.edu</a>
Joshua N. Cooper	University of South Carolina, USA <a href="mailto:cooper@math.sc.edu">cooper@math.sc.edu</a>	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran <a href="mailto:moslehian@ferdowsi.um.ac.ir">moslehian@ferdowsi.um.ac.ir</a>
Jem N. Corcoran	University of Colorado, USA <a href="mailto:corcoran@colorado.edu">corcoran@colorado.edu</a>	Zuhair Nashed	University of Central Florida, USA <a href="mailto:znashed@mail.ucf.edu">znashed@mail.ucf.edu</a>
Toka Diagana	Howard University, USA <a href="mailto:tdiagana@howard.edu">tdiagana@howard.edu</a>	Ken Ono	Emory University, USA <a href="mailto:ono@mathcs.emory.edu">ono@mathcs.emory.edu</a>
Michael Dorff	Brigham Young University, USA <a href="mailto:mdorff@math.byu.edu">mdorff@math.byu.edu</a>	Timothy E. O'Brien	Loyola University Chicago, USA <a href="mailto:tbriell@luc.edu">tbriell@luc.edu</a>
Sever S. Dragomir	Victoria University, Australia <a href="mailto:sever@matilda.vu.edu.au">sever@matilda.vu.edu.au</a>	Joseph O'Rourke	Smith College, USA <a href="mailto:orourke@cs.smith.edu">orourke@cs.smith.edu</a>
Behrouz Emamizadeh	The Petroleum Institute, UAE <a href="mailto:bemamizadeh@pi.ac.ae">bemamizadeh@pi.ac.ae</a>	Yuval Peres	Microsoft Research, USA <a href="mailto:peres@microsoft.com">peres@microsoft.com</a>
Joel Foisy	SUNY Potsdam <a href="mailto:foisyjs@potsdam.edu">foisyjs@potsdam.edu</a>	Y.-F. S. Pétermann	Université de Genève, Switzerland <a href="mailto:petermann@math.unige.ch">petermann@math.unige.ch</a>
Errin W. Fulp	Wake Forest University, USA <a href="mailto:fulp@wfu.edu">fulp@wfu.edu</a>	Robert J. Plemmons	Wake Forest University, USA <a href="mailto:rplemmons@wfu.edu">rplemmons@wfu.edu</a>
Joseph Gallian	University of Minnesota Duluth, USA <a href="mailto:kgallian@d.umn.edu">kgallian@d.umn.edu</a>	Carl B. Pomerance	Dartmouth College, USA <a href="mailto:carl.pomerance@dartmouth.edu">carl.pomerance@dartmouth.edu</a>
Stephan R. Garcia	Pomona College, USA <a href="mailto:stephan.garcia@pomona.edu">stephan.garcia@pomona.edu</a>	Vadim Ponomarenko	San Diego State University, USA <a href="mailto:vadim@sciences.sdsu.edu">vadim@sciences.sdsu.edu</a>
Anant Godbole	East Tennessee State University, USA <a href="mailto:godbole@etsu.edu">godbole@etsu.edu</a>	Bjorn Poonen	UC Berkeley, USA <a href="mailto:poonen@math.berkeley.edu">poonen@math.berkeley.edu</a>
Ron Gould	Emory University, USA <a href="mailto:rg@mathcs.emory.edu">rg@mathcs.emory.edu</a>	James Propp	U Mass Lowell, USA <a href="mailto:jpropp@cs.uml.edu">jpropp@cs.uml.edu</a>
Andrew Granville	Université Montréal, Canada <a href="mailto:andrew.andrew@dms.umontreal.ca">andrew.andrew@dms.umontreal.ca</a>	József H. Przytycki	George Washington University, USA <a href="mailto:przytyck@gwu.edu">przytyck@gwu.edu</a>
Jerrold Griggs	University of South Carolina, USA <a href="mailto:griggs@math.sc.edu">griggs@math.sc.edu</a>	Richard Rebarber	University of Nebraska, USA <a href="mailto:rrebarbe@math.unl.edu">rrebarbe@math.unl.edu</a>
Sat Gupta	U of North Carolina, Greensboro, USA <a href="mailto:sgupta@uncg.edu">sgupta@uncg.edu</a>	Robert W. Robinson	University of Georgia, USA <a href="mailto:rwr@cs.uga.edu">rwr@cs.uga.edu</a>
Jim Haglund	University of Pennsylvania, USA <a href="mailto:jhaglund@math.upenn.edu">jhaglund@math.upenn.edu</a>	Filip Saidak	U of North Carolina, Greensboro, USA <a href="mailto:f_saidak@uncg.edu">f_saidak@uncg.edu</a>
Johnny Henderson	Baylor University, USA <a href="mailto:johnny_henderson@baylor.edu">johnny_henderson@baylor.edu</a>	James A. Sellers	Penn State University, USA <a href="mailto:sellersj@math.psu.edu">sellersj@math.psu.edu</a>
Jim Hoste	Pitzer College <a href="mailto:jhoste@pitzer.edu">jhoste@pitzer.edu</a>	Andrew J. Sterge	Honorary Editor <a href="mailto:andy@ajsterge.com">andy@ajsterge.com</a>
Natalia Hritonenko	Prairie View A&M University, USA <a href="mailto:nahritonenko@pvamu.edu">nahritonenko@pvamu.edu</a>	Ann Trenk	Wellesley College, USA <a href="mailto:atrenk@wellesley.edu">atrenk@wellesley.edu</a>
Glenn H. Hurlbert	Arizona State University, USA <a href="mailto:hurlbert@asu.edu">hurlbert@asu.edu</a>	Ravi Vakil	Stanford University, USA <a href="mailto:vakil@math.stanford.edu">vakil@math.stanford.edu</a>
Charles R. Johnson	College of William and Mary, USA <a href="mailto:crjohnso@math.wm.edu">crjohnso@math.wm.edu</a>	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy <a href="mailto:antonia.vecchio@cnr.it">antonia.vecchio@cnr.it</a>
K. B. Kulasekera	Clemson University, USA <a href="mailto:kk@ces.clemson.edu">kk@ces.clemson.edu</a>	Ram U. Verma	University of Toledo, USA <a href="mailto:verma99@msn.com">verma99@msn.com</a>
Gerry Ladas	University of Rhode Island, USA <a href="mailto:gladas@math.uri.edu">gladas@math.uri.edu</a>	John C. Wierman	Johns Hopkins University, USA <a href="mailto:wierman@jhu.edu">wierman@jhu.edu</a>
		Michael E. Zieve	University of Michigan, USA <a href="mailto:zieve@umich.edu">zieve@umich.edu</a>

## PRODUCTION


Silvio Levy, Scientific Editor

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Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW<sup>®</sup> from Mathematical Sciences Publishers.

PUBLISHED BY

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2015

vol. 8

no. 3

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