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On distance labelings of amalgamations and injective labelings of general graphs

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An $L(2,1)$-*labeling* of a graph G is a function assigning a nonnegative integer to each vertex such that adjacent vertices are labeled with integers differing by at least 2 and vertices at distance two are labeled with integers differing by at least 1. The minimum span across all $L(2,1)$-labelings of G is denoted $\lambda(G)$. An $L'(2,1)$-*labeling* of G and the number $\lambda'(G)$ are defined analogously, with the additional restriction that the labelings must be injective. We determine $\lambda(H)$ when H is a join-page amalgamation of graphs, which is defined as follows: given $p \geq 2$, H is obtained from the pairwise disjoint union of graphs $H_0, H_1, \ldots, H_p$ by adding all the edges between a vertex in H_0 and a vertex in H_i for $i = 1, 2, \ldots, p$. Motivated by these join-page amalgamations and the partial relationships between $\lambda(G)$ and $\lambda'(G)$ for general graphs G provided by Chang and Kuo, we go on to show that $\lambda'(G) = \max\{n_G - 1, \lambda(G)\}$, where n_G is the number of vertices in G.

1. Introduction

In a well-studied model of the classic channel assignment problem introduced in [Hale 1980], each vertex of a graph G represents a transmitter in a communications network, and edges connect vertices corresponding to transmitters operating in close proximity which must receive sufficiently different frequencies to avoid interference. In a simplified instance of the problem, a frequency assignment is represented by an $L(2,1)$-*labeling* of G, which is a function f from the vertex set to the nonnegative integers such that $|f(x) - f(y)| \geq 2$ if vertices x and y are adjacent and $|f(x) - f(y)| \geq 1$ if x and y are at distance two. $L(2,1)$-labelings and their variations have been studied extensively since their introduction in [Griggs and Yeh 1992] (see the surveys [Calamoneri 2011; Griggs and Král 2009; Yeh 2006]) and continue to generate a rich literature to this date (see a sample of the

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most recent works in [Calamoneri 2013; Franks 2015; Karst et al. 2015; Li and Zhou 2013; Lin and Dai 2015; Lu and Zhou 2013; Shao and Solis-Oba 2013]).

An $L(2, 1)$-labeling of a graph G that uses labels in the set $\{0, 1, \ldots, k\}$ will be called a *k-$L(2, 1)$-labeling*. The minimum k so that G has a k-$L(2, 1)$-labeling is called the *λ-number* of G, denoted by $\lambda(G)$. Griggs and Yeh [1992] conjectured that $\lambda(G) \leq \Delta^2(G)$, where $\Delta(G)$ denotes the maximum degree of G. This conjecture holds for $\Delta(G) \geq 10^{69}$ [Havet et al. 2012], but it remains open even when $\Delta(G) = 3$. The best general upper bound yet established is $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 2$ [Gonçalves 2008]. Recently, it has been proven that this conjecture also holds for small enough graphs, namely, graphs with at most $(\lfloor \Delta(G)/2 \rfloor + 1)(\Delta^2(G) - \Delta(G) + 1) - 1$ vertices [Franks 2015]. As the general problem of determining $\lambda(G)$ is NP-hard [Georges et al. 1994], a significant body of literature has focused on finding bounds or exact λ-numbers for particular classes of graphs. In particular, [Adams et al. 2013] focused on the amalgamations of graphs.

Definition 1.1. Let $H_1, H_2, \ldots, H_p$ be $p \geq 2$ graphs each containing a fixed induced subgraph isomorphic to a graph H_0. The *amalgamation* of $H_1, H_2, \ldots, H_p$ along H_0 is the simple graph $H = \text{Amalg}(H_0; H_1, H_2, \ldots, H_p)$ obtained by identifying $H_1, H_2, \ldots, H_p$ at the vertices in the fixed subgraphs isomorphic to H_0 in each $H_1, H_2, \ldots, H_p$ respectively. H_0 is referred to as the *spine* and H_k as the *k-th page* of the amalgamation for $k = 1, 2, \ldots, p$. (We refer the reader to [Adams et al. 2013] for some concrete examples.)

In [Adams et al. 2013], upper bounds for the λ-number of the amalgamation of graphs along a given graph were established by determining the exact λ-number of amalgamations of complete graphs along a complete graph. They also provided the exact λ-numbers of amalgamations of rectangular grids along a path, or more specifically, of the Cartesian products of a path and a star with spokes of arbitrary lengths. This focus on the Cartesian products motivated us to investigate amalgamations of the join of graphs.

Definition 1.2. Let G_1 and G_2 be two disjoint graphs. The *union* $G_1 \cup G_2$ is the graph with vertex (resp., edge) set equal to the union of the vertex (resp., edge) sets of G_1 and G_2. The *join* $G_1 + G_2$ is obtained from $G_1 \cup G_2$ by adding an edge between each vertex in G_1 and each vertex in G_2.

Definition 1.3. Let G_0, G_1, and G_2 be pairwise disjoint graphs. The graph $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2)$ is called a *join-page amalgamation* of G_1, G_2 along G_0. Note that G is isomorphic to $G_0 + (G_1 \cup G_2)$.

Definitions 1.2 and 1.3 can be extended for more than two graphs G_1, G_2. The λ-numbers of the union and join of graphs are well known as stated in the next two results.

Result 1.4 [Chang and Kuo 1996, Lemma 3.1]. *For any two graphs G and H*, $\lambda(G \cup H) = \max\{\lambda(G), \lambda(H)\}$.

Result 1.5 [Georges et al. 1994, Corollary 4.6]. *For any two graphs G and H with n_G and n_H vertices respectively*,

$$\lambda(G + H) = \max\{n_G - 1, \lambda(G)\} + \max\{n_H - 1, \lambda(H)\} + 2.$$

In Section 2, we provide the exact λ-number for all join-page amalgamations. Motivated by a connection between this λ-number and the minimum span over injective $L(2, 1)$-labelings, Section 3 revisits these labelings for general graphs which were first introduced in [Chang and Kuo 1996]. More specifically, we establish a new exact relationship between the λ-number of a graph and the minimum span over all injective $L(2, 1)$-labelings of this graph.

2. The λ-number of join-page amalgamations

Theorem 2.1. *Let $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \dots, G_0 + G_p)$ be a join-page amalgamation, where G_i is a graph with $n_i \geq 1$ vertices for $i = 0, 1, \dots, p \geq 2$ so that $n_1 \geq n_j$ for $j = 2, 3, \dots, p$, and let $n = n_1 + n_2 + \cdots + n_p$. Then,*

$$\lambda(G) = \max\{n_0 - 1, \lambda(G_0)\} + \max\{n - 1, \lambda(G_1)\} + 2.$$

Proof. Since G is isomorphic to $G_0 + (G_1 \cup G_2 \cup \cdots \cup G_p)$, using Results 1.4 and 1.5,

$$\begin{aligned} \lambda(G) &= \lambda(G_0 + (G_1 \cup G_2 \cup \cdots \cup G_p)) \\ &= \max\{n_0 - 1, \lambda(G_0)\} + \max\{n - 1, \lambda(G_1 \cup G_2 \cup \cdots \cup G_p)\} + 2 \\ &= \max\{n_0 - 1, \lambda(G_0)\} + \max\{n - 1, \lambda(G_1), \lambda(G_2), \dots, \lambda(G_p)\} + 2. \end{aligned}$$

For $i = 2, 3, \dots, p$, we have $\lambda(G_i) \leq \lambda(K_{n_i}) = 2n_i - 2 \leq n_1 + n_i - 2 < n - 1$, where K_{n_i} denotes the complete graph with n_i vertices, and therefore

$$\max\{n - 1, \lambda(G_1), \lambda(G_2), \dots, \lambda(G_p)\} = \max\{n - 1, \lambda(G_1)\},$$

and the desired result follows. □

It is worth noting that Theorem 2.1 implies that $\lambda(G)$ depends on the number of vertices in $G_2, G_3, \dots, G_p$ but not on their particular λ-numbers.

The following corollary is equivalent to Theorem 2.3 in [Adams et al. 2013] but with an alternative and more compact proof.

Corollary 2.2. *Let $G = \text{Amalg}(K_0; K_0 + K_1, K_0 + K_2, \dots, K_0 + K_p)$ be a join-page amalgamation, where K_i is the complete graph with $n_i \geq 1$ vertices for $i = 0, 1, \dots, p \geq 2$ so that $n_1 \geq n_j$ for $j = 2, 3, \dots, p$, and let $n = n_1 + n_2 + \cdots + n_p$. Then $\lambda(G) = 2n_0 + \max\{n - 1, 2n_1 - 2\}$.*

Proof. By Theorem 2.1,

$$\begin{aligned}\lambda(G) &= \max\{n_0 - 1, \lambda(K_0)\} + \max\{n - 1, \lambda(K_1)\} + 2\\ &= \max\{n_0 - 1, 2n_0 - 2\} + \max\{n - 1, 2n_1 - 2\} + 2\\ &= 2n_0 - 2 + \max\{n - 1, 2n_1 - 2\} + 2\\ &= 2n_0 + \max\{n - 1, 2n_1 - 2\}.\end{aligned}$$ □

3. A connection between join-page amalgamation and injective $L(2, 1)$-labelings

When examining the $L(2, 1)$-labelings of a join-page amalgamation of the form $G = \mathrm{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \dots, G_0 + G_p)$, as described in Theorem 2.1 in Section 2, we noticed that we could extend an injective $L(2, 1)$-labeling of G_0 of minimum span over all its injective labelings to a $\lambda(G)$-$L(2, 1)$-labeling of the entire G. We suspected that this was not a coincidence, which led us to revisit the following variation of $L(2, 1)$-labelings introduced in [Chang and Kuo 1996].

Definition 3.1. An $L'(2, 1)$-*labeling* of a graph G is an injective $L(2, 1)$-labeling of G. The definitions of k-$L'(2, 1)$-*labeling*, λ'-*number* and $\lambda'(G)$ are analogous to those of k-$L(2, 1)$-labeling, λ-number, and $\lambda(G)$ when restricted to injective labelings.

The following basic properties were previously known.

Result 3.2 [Chang and Kuo 1996, Lemmas 2.1, 2.2, 2.3]. *For any graph G with n_G vertices,*

(i) $\lambda'(H) \le \lambda'(G)$ *for any subgraph H of G*;

(ii) $\lambda(G) \le \lambda'(G)$ *with equality if G has diameter at most two*; *and*

(iii) $c(G) = \lambda'(G^c) - n_G + 2$, *where $c(G)$ is the path covering number of G, i.e., the smallest number of vertex-disjoint paths needed to cover all the vertices of the graph G, and G^c is the complement of G.*

In Theorem 3.4, we will strengthen Result 3.2(ii) by providing a surprisingly simple exact relationship between $\lambda(G)$ and $\lambda'(G)$ for any graph G. We will be using the following auxiliary result in the proof of Theorem 3.4.

Result 3.3 [Georges et al. 1994, Theorem 1.1]. *For any graph G on n_G vertices,*

(i) $\lambda(G) \le n_G - 1$ *if and only if* $c(G^c) = 1$; *and*

(ii) $\lambda(G) = n_G + c(G^c) - 2$ *if and only if* $c(G^c) \ge 2$.

Theorem 3.4. *For any graph G with n_G vertices,*

$$\lambda'(G) = \max\{n_G - 1, \lambda(G)\}.$$

Proof. Suppose $\lambda(G) \le n_G - 1$. By Result 3.3(i), $c(G^c) = 1$, and Result 3.2(iii) implies $1 = c(G^c) = \lambda'(G) - n_G + 2$. Therefore,

$$\lambda'(G) = n_G - 1 = \max\{n_G - 1, \lambda(G)\}.$$

Assume, on the other hand, that $\lambda(G) > n_G - 1$. Item (i) in Result 3.3 implies $c(G^c) \ge 2$, and item (ii) implies $\lambda(G) = n_G + c(G^c) - 2$, or equivalently, $c(G^c) = \lambda(G) - n_G + 2$. Finally, Result 3.2(iii) implies

$$\begin{aligned}\lambda'(G) &= c(G^c) + n_G - 2 \\ &= (\lambda(G) - n_G + 2) + n_G - 2 = \lambda(G) = \max\{n_G - 1, \lambda(G)\}. \qquad \square\end{aligned}$$

In view of Theorem 3.4, the general problem of determining the λ'-number of graphs is as complex as determining their λ-numbers, which, as mentioned previously, is known to be an NP-hard problem. Furthermore, the exact λ'-numbers of families of graphs, such as the ones derived in [Chang and Kuo 1996] using more involved techniques (e.g., paths, cycles, union and join of two graphs), can be readily obtained using Theorem 3.4 and the vast list of known exact λ-numbers in the $L(2, 1)$-labeling literature.

If $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \dots, G_0 + G_p)$ and we apply Theorem 3.4 to G_0 in Theorem 2.1, we obtain a relationship between $\lambda(G)$ and $\lambda'(G_0)$, confirming the connection between injective $L(2, 1)$-labelings of G_0 and $L(2, 1)$-labelings of G we mentioned in the first paragraph of this section. The following corollary provides this relationship.

Corollary 3.5. *Let* $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \dots, G_0 + G_p)$ *be a join-page amalgamation, where* G_i *is a graph with* n_i *vertices for* $i = 0, 1, \dots, p \ge 2$ *so that* $n_1 \ge n_j$ *for* $j = 2, 3, \dots, p$, *and let* $n = n_1 + n_2 + \cdots + n_p$. *Then* $\lambda(G) = \lambda'(G_0) + \max\{n - 1, \lambda(G_1)\} + 2$.

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