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Nathaniel Karst, Jessica Oehrlein, Denise Sakai Troxell and Junjie Zhu



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An $L(2, 1)$ -labeling of a graph G is a function assigning a nonnegative integer to each vertex such that adjacent vertices are labeled with integers differing by at least 2 and vertices at distance two are labeled with integers differing by at least 1. The minimum span across all $L(2, 1)$ -labelings of G is denoted $\lambda(G)$. An $L'(2, 1)$ -labeling of G and the number $\lambda'(G)$ are defined analogously, with the additional restriction that the labelings must be injective. We determine $\lambda(H)$ when H is a join-page amalgamation of graphs, which is defined as follows: given $p \geq 2$, H is obtained from the pairwise disjoint union of graphs H_0, H_1, \dots, H_p by adding all the edges between a vertex in H_0 and a vertex in H_i for $i = 1, 2, \dots, p$. Motivated by these join-page amalgamations and the partial relationships between $\lambda(G)$ and $\lambda'(G)$ for general graphs G provided by Chang and Kuo, we go on to show that $\lambda'(G) = \max\{n_G - 1, \lambda(G)\}$, where n_G is the number of vertices in G .

1. Introduction

In a well-studied model of the classic channel assignment problem introduced in [Hale 1980], each vertex of a graph G represents a transmitter in a communications network, and edges connect vertices corresponding to transmitters operating in close proximity which must receive sufficiently different frequencies to avoid interference. In a simplified instance of the problem, a frequency assignment is represented by an $L(2, 1)$ -labeling of G , which is a function f from the vertex set to the nonnegative integers such that $|f(x) - f(y)| \geq 2$ if vertices x and y are adjacent and $|f(x) - f(y)| \geq 1$ if x and y are at distance two. $L(2, 1)$ -labelings and their variations have been studied extensively since their introduction in [Griggs and Yeh 1992] (see the surveys [Calamoneri 2011; Griggs and Král 2009; Yeh 2006]) and continue to generate a rich literature to this date (see a sample of the

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most recent works in [Calamoneri 2013; Franks 2015; Karst et al. 2015; Li and Zhou 2013; Lin and Dai 2015; Lu and Zhou 2013; Shao and Solis-Oba 2013]).

An $L(2, 1)$ -labeling of a graph G that uses labels in the set $\{0, 1, \dots, k\}$ will be called a k - $L(2, 1)$ -labeling. The minimum k so that G has a k - $L(2, 1)$ -labeling is called the λ -number of G , denoted by $\lambda(G)$. Griggs and Yeh [1992] conjectured that $\lambda(G) \leq \Delta^2(G)$, where $\Delta(G)$ denotes the maximum degree of G . This conjecture holds for $\Delta(G) \geq 10^{69}$ [Havet et al. 2012], but it remains open even when $\Delta(G) = 3$. The best general upper bound yet established is $\lambda(G) \leq \Delta^2(G) + \Delta(G) - 2$ [Gonçalves 2008]. Recently, it has been proven that this conjecture also holds for small enough graphs, namely, graphs with at most $(\lfloor \Delta(G)/2 \rfloor + 1)(\Delta^2(G) - \Delta(G) + 1) - 1$ vertices [Franks 2015]. As the general problem of determining $\lambda(G)$ is NP-hard [Georges et al. 1994], a significant body of literature has focused on finding bounds or exact λ -numbers for particular classes of graphs. In particular, [Adams et al. 2013] focused on the amalgamations of graphs.

Definition 1.1. Let H_1, H_2, \dots, H_p be $p \geq 2$ graphs each containing a fixed induced subgraph isomorphic to a graph H_0 . The *amalgamation* of H_1, H_2, \dots, H_p along H_0 is the simple graph $H = \text{Amalg}(H_0; H_1, H_2, \dots, H_p)$ obtained by identifying H_1, H_2, \dots, H_p at the vertices in the fixed subgraphs isomorphic to H_0 in each H_1, H_2, \dots, H_p respectively. H_0 is referred to as the *spine* and H_k as the k -th *page* of the amalgamation for $k = 1, 2, \dots, p$. (We refer the reader to [Adams et al. 2013] for some concrete examples.)

In [Adams et al. 2013], upper bounds for the λ -number of the amalgamation of graphs along a given graph were established by determining the exact λ -number of amalgamations of complete graphs along a complete graph. They also provided the exact λ -numbers of amalgamations of rectangular grids along a path, or more specifically, of the Cartesian products of a path and a star with spokes of arbitrary lengths. This focus on the Cartesian products motivated us to investigate amalgamations of the join of graphs.

Definition 1.2. Let G_1 and G_2 be two disjoint graphs. The *union* $G_1 \cup G_2$ is the graph with vertex (resp., edge) set equal to the union of the vertex (resp., edge) sets of G_1 and G_2 . The *join* $G_1 + G_2$ is obtained from $G_1 \cup G_2$ by adding an edge between each vertex in G_1 and each vertex in G_2 .

Definition 1.3. Let G_0, G_1 , and G_2 be pairwise disjoint graphs. The graph $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2)$ is called a *join-page amalgamation* of G_1, G_2 along G_0 . Note that G is isomorphic to $G_0 + (G_1 \cup G_2)$.

Definitions 1.2 and 1.3 can be extended for more than two graphs G_1, G_2 . The λ -numbers of the union and join of graphs are well known as stated in the next two results.

Result 1.4 [Chang and Kuo 1996, Lemma 3.1]. *For any two graphs G and H , $\lambda(G \cup H) = \max\{\lambda(G), \lambda(H)\}$.*

Result 1.5 [Georges et al. 1994, Corollary 4.6]. *For any two graphs G and H with n_G and n_H vertices respectively,*

$$\lambda(G + H) = \max\{n_G - 1, \lambda(G)\} + \max\{n_H - 1, \lambda(H)\} + 2.$$

In Section 2, we provide the exact λ -number for all join-page amalgamations. Motivated by a connection between this λ -number and the minimum span over injective $L(2, 1)$ -labelings, Section 3 revisits these labelings for general graphs which were first introduced in [Chang and Kuo 1996]. More specifically, we establish a new exact relationship between the λ -number of a graph and the minimum span over all injective $L(2, 1)$ -labelings of this graph.

2. The λ -number of join-page amalgamations

Theorem 2.1. *Let $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \dots, G_0 + G_p)$ be a join-page amalgamation, where G_i is a graph with $n_i \geq 1$ vertices for $i = 0, 1, \dots, p \geq 2$ so that $n_1 \geq n_j$ for $j = 2, 3, \dots, p$, and let $n = n_1 + n_2 + \dots + n_p$. Then,*

$$\lambda(G) = \max\{n_0 - 1, \lambda(G_0)\} + \max\{n - 1, \lambda(G_1)\} + 2.$$

Proof. Since G is isomorphic to $G_0 + (G_1 \cup G_2 \cup \dots \cup G_p)$, using Results 1.4 and 1.5,

$$\begin{aligned} \lambda(G) &= \lambda(G_0 + (G_1 \cup G_2 \cup \dots \cup G_p)) \\ &= \max\{n_0 - 1, \lambda(G_0)\} + \max\{n - 1, \lambda(G_1 \cup G_2 \cup \dots \cup G_p)\} + 2 \\ &= \max\{n_0 - 1, \lambda(G_0)\} + \max\{n - 1, \lambda(G_1), \lambda(G_2), \dots, \lambda(G_p)\} + 2. \end{aligned}$$

For $i = 2, 3, \dots, p$, we have $\lambda(G_i) \leq \lambda(K_{n_i}) = 2n_i - 2 \leq n_1 + n_i - 2 < n - 1$, where K_{n_i} denotes the complete graph with n_i vertices, and therefore

$$\max\{n - 1, \lambda(G_1), \lambda(G_2), \dots, \lambda(G_p)\} = \max\{n - 1, \lambda(G_1)\},$$

and the desired result follows. \square

It is worth noting that Theorem 2.1 implies that $\lambda(G)$ depends on the number of vertices in G_2, G_3, \dots, G_p but not on their particular λ -numbers.

The following corollary is equivalent to Theorem 2.3 in [Adams et al. 2013] but with an alternative and more compact proof.

Corollary 2.2. *Let $G = \text{Amalg}(K_0; K_0 + K_1, K_0 + K_2, \dots, K_0 + K_p)$ be a join-page amalgamation, where K_i is the complete graph with $n_i \geq 1$ vertices for $i = 0, 1, \dots, p \geq 2$ so that $n_1 \geq n_j$ for $j = 2, 3, \dots, p$, and let $n = n_1 + n_2 + \dots + n_p$. Then $\lambda(G) = 2n_0 + \max\{n - 1, 2n_1 - 2\}$.*

Proof. By [Theorem 2.1](#),

$$\begin{aligned}
 \lambda(G) &= \max\{n_0 - 1, \lambda(K_0)\} + \max\{n - 1, \lambda(K_1)\} + 2 \\
 &= \max\{n_0 - 1, 2n_0 - 2\} + \max\{n - 1, 2n_1 - 2\} + 2 \\
 &= 2n_0 - 2 + \max\{n - 1, 2n_1 - 2\} + 2 \\
 &= 2n_0 + \max\{n - 1, 2n_1 - 2\}.
 \end{aligned}
 \quad \square$$

3. A connection between join-page amalgamation and injective $L(2, 1)$ -labelings

When examining the $L(2, 1)$ -labelings of a join-page amalgamation of the form $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \dots, G_0 + G_p)$, as described in [Theorem 2.1](#) in [Section 2](#), we noticed that we could extend an injective $L(2, 1)$ -labeling of G_0 of minimum span over all its injective labelings to a $\lambda(G)$ - $L(2, 1)$ -labeling of the entire G . We suspected that this was not a coincidence, which led us to revisit the following variation of $L(2, 1)$ -labelings introduced in [\[Chang and Kuo 1996\]](#).

Definition 3.1. An $L'(2, 1)$ -labeling of a graph G is an injective $L(2, 1)$ -labeling of G . The definitions of k - $L'(2, 1)$ -labeling, λ' -number and $\lambda'(G)$ are analogous to those of k - $L(2, 1)$ -labeling, λ -number, and $\lambda(G)$ when restricted to injective labelings.

The following basic properties were previously known.

Result 3.2 [\[Chang and Kuo 1996, Lemmas 2.1, 2.2, 2.3\]](#). *For any graph G with n_G vertices,*

- (i) $\lambda'(H) \leq \lambda'(G)$ for any subgraph H of G ;
- (ii) $\lambda(G) \leq \lambda'(G)$ with equality if G has diameter at most two; and
- (iii) $c(G) = \lambda'(G^c) - n_G + 2$, where $c(G)$ is the path covering number of G , i.e., the smallest number of vertex-disjoint paths needed to cover all the vertices of the graph G , and G^c is the complement of G .

In [Theorem 3.4](#), we will strengthen [Result 3.2\(ii\)](#) by providing a surprisingly simple exact relationship between $\lambda(G)$ and $\lambda'(G)$ for any graph G . We will be using the following auxiliary result in the proof of [Theorem 3.4](#).

Result 3.3 [\[Georges et al. 1994, Theorem 1.1\]](#). *For any graph G on n_G vertices,*

- (i) $\lambda(G) \leq n_G - 1$ if and only if $c(G^c) = 1$; and
- (ii) $\lambda(G) = n_G + c(G^c) - 2$ if and only if $c(G^c) \geq 2$.

Theorem 3.4. *For any graph G with n_G vertices,*

$$\lambda'(G) = \max\{n_G - 1, \lambda(G)\}.$$

Proof. Suppose $\lambda(G) \leq n_G - 1$. By [Result 3.3\(i\)](#), $c(G^c) = 1$, and [Result 3.2\(iii\)](#) implies $1 = c(G^c) = \lambda'(G) - n_G + 2$. Therefore,

$$\lambda'(G) = n_G - 1 = \max\{n_G - 1, \lambda(G)\}.$$

Assume, on the other hand, that $\lambda(G) > n_G - 1$. Item [\(i\)](#) in [Result 3.3](#) implies $c(G^c) \geq 2$, and item [\(ii\)](#) implies $\lambda(G) = n_G + c(G^c) - 2$, or equivalently, $c(G^c) = \lambda(G) - n_G + 2$. Finally, [Result 3.2\(iii\)](#) implies

$$\begin{aligned} \lambda'(G) &= c(G^c) + n_G - 2 \\ &= (\lambda(G) - n_G + 2) + n_G - 2 = \lambda(G) = \max\{n_G - 1, \lambda(G)\}. \quad \square \end{aligned}$$

In view of [Theorem 3.4](#), the general problem of determining the λ' -number of graphs is as complex as determining their λ -numbers, which, as mentioned previously, is known to be an NP-hard problem. Furthermore, the exact λ' -numbers of families of graphs, such as the ones derived in [[Chang and Kuo 1996](#)] using more involved techniques (e.g., paths, cycles, union and join of two graphs), can be readily obtained using [Theorem 3.4](#) and the vast list of known exact λ -numbers in the $L(2, 1)$ -labeling literature.

If $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \dots, G_0 + G_p)$ and we apply [Theorem 3.4](#) to G_0 in [Theorem 2.1](#), we obtain a relationship between $\lambda(G)$ and $\lambda'(G_0)$, confirming the connection between injective $L(2, 1)$ -labelings of G_0 and $L(2, 1)$ -labelings of G we mentioned in the first paragraph of this section. The following corollary provides this relationship.

Corollary 3.5. *Let $G = \text{Amalg}(G_0; G_0 + G_1, G_0 + G_2, \dots, G_0 + G_p)$ be a join-page amalgamation, where G_i is a graph with n_i vertices for $i = 0, 1, \dots, p \geq 2$ so that $n_1 \geq n_j$ for $j = 2, 3, \dots, p$, and let $n = n_1 + n_2 + \dots + n_p$. Then $\lambda(G) = \lambda'(G_0) + \max\{n - 1, \lambda(G_1)\} + 2$.*

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References

- [Adams et al. 2013] S. S. Adams, N. Howell, N. Karst, D. S. Troxell, and J. Zhu, “On the $L(2, 1)$ -labelings of amalgamations of graphs”, *Discrete Appl. Math.* **161**:7-8 (2013), 881–888. [MR 3030574](#) [Zbl 1263.05086](#)
- [Calamoneri 2011] T. Calamoneri, “The $L(h, k)$ -labelling problem: An updated survey and annotated bibliography”, *Comput. J.* **54**:8 (2011), 1344–1371.

- [Calamoneri 2013] T. Calamoneri, “Optimal $L(\delta_1, \delta_2, 1)$ -labeling of eight-regular grids”, *Inform. Process. Lett.* **113**:10-11 (2013), 361–364. MR 3037462 Zbl 06329871
- [Chang and Kuo 1996] G. J. Chang and D. Kuo, “The $L(2, 1)$ -labeling problem on graphs”, *SIAM J. Discrete Math.* **9**:2 (1996), 309–316. MR 97b:05132 Zbl 0860.05064
- [Franks 2015] C. Franks, “The delta square conjecture holds for graphs of small order”, *Involve: J. Math.* **9**:2 (2015), to be supplied by the publisher.
- [Georges et al. 1994] J. P. Georges, D. W. Mauro, and M. A. Whittlesey, “Relating path coverings to vertex labellings with a condition at distance two”, *Discrete Math.* **135**:1-3 (1994), 103–111. MR 96b:05150 Zbl 0811.05058
- [Gonçalves 2008] D. Gonçalves, “On the $L(p, 1)$ -labelling of graphs”, *Discrete Math.* **308**:8 (2008), 1405–1414. MR 2008k:05185 Zbl 1135.05065
- [Griggs and Král 2009] J. R. Griggs and D. Král, “Graph labellings with variable weights, a survey”, *Discrete Appl. Math.* **157**:12 (2009), 2646–2658. MR 2010m:05275 Zbl 1211.05145
- [Griggs and Yeh 1992] J. R. Griggs and R. K. Yeh, “Labelling graphs with a condition at distance 2”, *SIAM J. Discrete Math.* **5**:4 (1992), 586–595. MR 93h:05141 Zbl 0767.05080
- [Hale 1980] W. K. Hale, “Frequency assignment: Theory and applications”, *Proc. IEEE* **68**:12 (1980), 1497–1514.
- [Havet et al. 2012] F. Havet, B. Reed, and J.-S. Sereni, “Griggs and Yeh’s conjecture and $L(p, 1)$ -labelings”, *SIAM J. Discrete Math.* **26**:1 (2012), 145–168. MR 2902638 Zbl 1245.05110
- [Karst et al. 2015] N. Karst, J. Oehrlein, D. S. Troxell, and J. Zhu, “ $L(d, 1)$ -labelings of the edge-path-replacement by factorization of graphs”, *J. Comb. Opt.* **30**:1 (2015), 34–41. MR 3352872
- [Li and Zhou 2013] X. Li and S. Zhou, “Labeling outerplanar graphs with maximum degree three”, *Discrete Appl. Math.* **161**:1-2 (2013), 200–211. MR 2973362 Zbl 06109944
- [Lin and Dai 2015] W. Lin and B. Dai, “On (s, t) -relaxed $L(2, 1)$ -labelings of the triangular lattice”, *J. Comb. Optim.* **29**:3 (2015), 655–669. MR 3316710 Zbl 06435135
- [Lu and Zhou 2013] C. Lu and Q. Zhou, “Path covering number and $L(2, 1)$ -labeling number of graphs”, *Discrete Appl. Math.* **161**:13-14 (2013), 2062–2074. MR 3057011 Zbl 1286.05150
- [Shao and Solis-Oba 2013] Z. Shao and R. Solis-Oba, “ $L(2, 1)$ -labelings on the modular product of two graphs”, *Theoret. Comput. Sci.* **487** (2013), 74–81. MR 3049272 Zbl 1283.05246
- [Yeh 2006] R. K. Yeh, “A survey on labeling graphs with a condition at distance two”, *Discrete Math.* **306**:12 (2006), 1217–1231. MR 2007g:05167 Zbl 1094.05047

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nkarst@babson.edu

Mathematics and Sciences Division, Babson College,
Babson Park, MA 02457, United States

jessica.oehrlein@students.olin.edu

Franklin W. Olin College of Engineering, Olin Way,
Needham, MA 02492, United States

troxell@babson.edu

Mathematics and Sciences Division, Babson College,
Babson Park, MA 02457, United States

jjzhu@stanford.edu

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